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Combining a backstepping controller with a local stabilizer

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Abstract— **We consider nonlinear control systems for which there exist some structural obstacles to the design of classical continuous stabilizing feedback laws. More precisely, it is studied systems for which the backstepping tool for the design of stabilizers can not be applied. On the contrary, it leads to feedback laws such that the origin of the closed-loop system is not globally asymptotically stable, but a suitable attractor (strictly containing the origin) is practically asymptotically stable. Then, a design method is suggested to build a hybrid feedback law combining a backstepping controller with a locally stabilizing controller. The results are illustrated for a nonlinear system which, due to the structure of the system, does not have *a priori* any globally stabilizing backstepping controller.**

I. INTRODUCTION

Over the years, research in control of nonlinear dynamical systems has lead to many different tools to design (globally) asymptotically stabilizing feedbacks, see e.g. [8], [18], [19]. Usually these techniques require to impose special structure on the control systems. Depending on the assumptions made on the model, the designer may use high-gain approaches (as in [13]), a backstepping technique (see [8], [20], [24]) or a forwarding approach (consider e.g., [17], [21], [31]), among others design methods. Unfortunately, in presence of unknown parameters or unstructured dynamics, these classical design methods may fail and some structural obstacles to large regions of attraction may exist. Examples of such systems are the partially linear cascades systems, considered e.g. in [5], [28] and [32], for which the local stabilization is linear but a perturbation may cause finite escape time, if some parts are not properly controlled. This phenomenon, so-called slow-peaking, has been studied (e.g. in [29], [30]) to design global stabilizers.

For such systems where the classical backstepping techniques can not be applied, the approach presented may solve the problem by combining a backstepping feedback law with a locally stabilizing controller. More precisely, it is designed a hybrid feedback law to blend both kinds of controllers. The backstepping controller renders a suitable compact set globally attractive, whereas the local one is assumed to have its basin of attraction containing the attractor of the system in closed-loop with the backstepping controller. The main result can thus be seen as a design techniques of hybrid feedback laws for systems, which *a priori* do not have classical nonlinear stabilizing controllers. The use of hybrid stabilizers for systems which do not have continuous stabilizers, is by now classical (see e.g., [14], [22], [25]). This approach has been particularly fruitful for control systems that do not satisfy the Brockett's condition [6] that is a necessary topological condition for the existence of a continuous stabilizing feedback (see in particular [9], [10], [15], [16], [26]). The considered class of hybrid feedback laws has the advantage to guarantee a robustness property with respect to measurement noise, actuators errors (see [27] and also [12] for related issues).

Best to our knowledge this is the first work suggesting a design method to adapt the backstepping technique to a given local controller in the context of hybrid feedback laws. Other works do exist in the context of continuous controllers (e.g., see [23] where a backstepping controller is blent with an LQ controller, and consider [1] where, using control Lyapunov functions, a globally stabilizing controller is combined with a local optimal controller). In contrast to these works, for the class of systems considered in this paper, *a priori* no continuous stabilizing controller does exist.

This paper is organized as follows. In Section II, we introduce precisely the problem under consideration in this paper and the class of controllers that will be used to solve this problem. In Section III the main result is stated, that is the existence of a hybrid feedback law combining a backstepping controller with a local stabilizer. In Section IV, the main result is illustrated on an example, and it is designed such a hybrid feedback law for a system for which the classical backstepping approach can not be applied. All technical proofs are collected in Section V, and Section VI contains some concluding remarks.

The proof of some results has been removed due to space limitation.

II. PROBLEM STATEMENT

Consider the nonlinear system

$$\left\{ \begin{array}{rcl} \dot{x}_1 & = & f_1(x_1,x_2)+h_1(x_1,x_2,u) \\ \dot{x}_2 & = & f_2(x_1,x_2)u+h_2(x_1,x_2,u), \end{array} \right. \tag{1}$$

where $(x_1,x_2)\in\mathbb{R}^{n-1}\times\mathbb{R}$ and $u\in\mathbb{R}$ is the input. The functions f_1, f_2, h_1 and h_2 are locally Lipschitz continuous. Furthermore, the functions satisfy $f_1(0,0)=h_1(0,0,0)=h_2(0,0,0)=0$ and $f_2(x_1,x_2)\neq 0$, $\forall (x_1,x_2)\in\mathbb{R}^n$.

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In a more compact notation, we denote system (1) by $\dot{x} = f_h(x,u)$. Furthermore, when $h_1 \equiv 0$ and $h_2 \equiv 0$ we write $\dot{x} = f(x,u)$.

A. Assumptions

The first assumption concerns the local stabilizability around the origin of system (1). More precisely,

Assumption 1: (Local stabilizability) *There exist a $\mathcal{C}^1$ positive definite and proper function $V_\ell : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, a continuous function $\varphi_\ell : \mathbb{R}^n \to \mathbb{R}$ and a positive constant v_ℓ such that,*

$$\partial_x V_\ell(x) \cdot f_h(x, \varphi_\ell(x)) < 0 \ , \ \forall x \in \{x : 0 < V_\ell(x) \leq v_\ell\}.$$

Note that, when the first order approximation of system (1) is controllable, Assumption 1 is trivially satisfied. Indeed, if the couple of matrices (A,B), with $A = \partial_x f_h(0,0)$ and $B = \partial_u f_h(0,0)$ is controllable, then there exist matrices $P > 0$ and K such that $V_\ell(x) = x^T P x$ and $\varphi_\ell(x) = Kx$. Thus Assumption 1 holds with a sufficiently small positive constant v_ℓ.

The second hypothesis provides estimates on terms which prevents using the traditional backstepping method. More precisely, this assumption concerns the global stabilizability of the system

$$\dot{x}_1 = f_1(x_1, x_2) \tag{2}$$

with x_2 as an input and bounds of functions h_1 and h_2. This assumption will be also useful to state a global practical stability property of (1) (see Proposition 3.1 below).

Assumption 2: *There exist a $\mathcal{C}^1$ proper and positive definite function $V_1 : \mathbb{R}^{n-1} \to \mathbb{R}_{\geq 0}$, a $\mathcal{C}^1$ function $\varphi_1 : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\varphi_1(0) = 0$, a locally Lipschitz $\mathcal{K}^\infty$ function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, a continuous function $\Psi : \mathbb{R}^n \to \mathbb{R}$ and two positive constants $\varepsilon < 1$ and M such that the following properties hold.*

1. (Stabilizing controller φ_1 for (2)) $\forall x_1 \in \mathbb{R}^{n-1}$,
$$\partial_{x_1} V_1(x_1) \cdot f_1(x_1, \varphi_1(x_1)) \leq -\alpha(V_1(x_1)).$$
2. (Estimation on h_1) $\forall (x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$,
$$L_{h_1} V_1(x_1, \varphi_1(x_1), u) \leq (1-\varepsilon)\alpha(V_1(x_1)) + \varepsilon\alpha(M), \tag{3}$$
$$|h_1(x_1, x_2, u)| \leq \Psi(x_1, x_2) \tag{4}$$
3. (Estimation on $\partial_{x_2} h_1$) $\forall (x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$,
$$|\partial_{x_2} h_1(x_1, x_2, u)| \leq \Psi(x_1, x_2). \tag{5}$$
4. (Estimation on h_2) $\forall (x_1, x_2, u) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$,
$$|h_2(x_1, x_2, u)| \leq \Psi(x_1, x_2). \tag{6}$$

As we will see in this work, it is not necessary that φ_1 be $\mathcal{C}^1$ in a neighborhood of the origin because, in such a region, we use the local controller φ_ℓ.

Before introducing the third assumption, let us denote $\mathbf{A}$ the subset of $\mathbb{R}^n$ defined by

$$\mathbf{A} = \{(x_1, x_2) \in \mathbb{R}^n : V_1(x_1) \leq M, \ x_2 = \varphi_1(x_1)\}. \tag{7}$$

Since, by Assumption 2, the function V_1 is proper, this set is compact. Moreover, it will be proven below (see Proposition 3.1) that with the other items of Assumption 2 a controller to (1) can be designed such that $\mathbf{A}$ is globally practically stable to the system in closed-loop with this controller.

The last assumption describes that $\mathbf{A}$ is included in the basin of attraction of the controller φ_ℓ.

Assumption 3: (Inclusion assumption)

$$\max_{x \in \mathbf{A}} V_\ell(x) < v_\ell \ . \tag{8}$$

The problem under consideration in this paper is the design of a controller such that the origin is globally asymptotically stable for (1). Due to the presence of the functions h_1 and h_2 and their dependence with respect to u, a classical backstepping can not be achieved to compute a global stabilizer.[1]

However we succeed to design a controller rendering a compact set globally asymptotically stable to (1) in closed-loop. Then a natural approach is to combine this controller with a local feedback law given by Assumption 1. Global asymptotical stabilization of the origin of $\mathbb{R}^n$ can be achieved by considering a hybrid controller which blends the different controllers according to each basin of attraction. The strategy is similar to that one developed in [25], namely, we divide the continuous state space in two open sets introducing a region with hysteresis. This asks to make precise the class of controllers under consideration in this paper.

B. Class of controllers

Definition 2.1: *A* hybrid feedback law *to* (1)*, denoted by* $\mathbb{K}$*, consists of*

- *a totally ordered countable set Q;*
- *for each $q \in Q$,*
  - *closed sets $C_q \subset \mathbb{R}^n$ and $D_q \subset \mathbb{R}^n$ such that $C_q \cup D_q = \mathbb{R}^n$;*
  - *a continuous function $\varphi_q : C_q \to \mathbb{R}$;*

[1] More precisely, following the classical basckstepping approach, let us assume that item 1 of Assumption 2 holds and let us consider the Lyapunov function candidate $V(x_1, x_2) = V_1(x_1) + \frac{1}{2}(x_2 - \varphi_1(x_1))^2$. We compute along the solutions of (1), for all (x_1, x_2, u) in $\mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned} \dot{V} \quad \leq \quad & -\alpha(V_1(x_1)) + [x_2 - \varphi_1(x_1)]\,[f_2(x_1,x_2)u + h_2(x_1,x_2,u) \\ & - \tfrac{\partial \varphi_1}{\partial x_1}(x_1) \cdot (f_1(x_1,x_2) + h_1(x_1,x_2,u)) \\ & + \tfrac{\partial V_1}{\partial x_1}(x_1) \cdot \textstyle\int_0^1 f_1(x_1, sx_2 - (1-s)\varphi_1(x_1))ds\Big] \\ & + \tfrac{\partial V_1}{\partial x_1}(x_1) \cdot h_1(x_1,x_2,u) \ . \end{aligned}$$

And thus to get an term $(x_2 - \varphi_1(x_1))^2$ in the right-hand side of this inequality, it is natural to look for a control $u = u(x_1, x_2)$ satisfying the following identity, for all (x_1, x_2) in $\mathbb{R}^{n-1} \times \mathbb{R}$,

$$\begin{aligned} & f_2(x_1,x_2)u + h_2(x_1,x_2,u) - \tfrac{\partial \varphi_1}{\partial x_1}(x_1) \cdot (f_1(x_1,x_2) + h_1(x_1,x_2,u)) \\ & + \tfrac{\partial V_1}{\partial x_1}(x_1) \cdot \textstyle\int_0^1 f_1(x_1, sx_2 - (1-s)\varphi_1(x_1))ds = -k\,(x_2 - \varphi_1(x_1)) \end{aligned}$$

for some positive value k. However this equation is implicit in the variable u due to dependance of h_1 and of h_2 with respect to u. Therefore it seems to us that the classical backstepping cannot be achieved to compute a stabilizer for (1).

– *an outer semi-continuous*[2]*, and locally bounded*[3]*, uniformly in q, set-valued mapping $G_q : D_q \rightrightarrows Q$ with non-empty images,*

such that the family $\{C_q\}_{q\in Q}$ is locally finite and covers $\mathbb{R}^n$.

System (1) in closed loop with $\mathbb{K}$ lies in the class of hybrid systems as considered in e.g., [3]. It is defined as the hybrid system

$$\mathbb{H} : \left\{ \begin{array}{rcll} \dot{x} & = & f_h(x, \varphi_q(x)), & x \in C_q \\ q^+ & \in & G_q(x), & x \in D_q. \end{array} \right. \tag{9}$$

Note that the state space of $\mathbb{H}$ is $\mathbb{R}^n \times Q$.

Definition 2.2: *A* hybrid time domain $S \subset \mathbb{R}_{\geq 0}\times\mathbb{N}$*, is the union of finitely of infinitely many time intervals $[t_j, t_{j+1}] \times \{j\}$, where the sequence $\{t_j\}_{j\geq 0}$ in nondecreasing, with the last interval, if it exists, possibly of the form $[t, T)$ with T finite or $T = \infty$.*

Definition 2.3: *A* solution *to* $\mathbb{H}$ *with initial condition $(x(0,0), q(0,0)) = (x_0, q_0)$ consists of*

- *A hybrid time domain $S \neq \emptyset$;*
- *A function $x : S \to \mathbb{R}^n$, where $t \mapsto x(t, j)$ is absolutely continuous, for a fixed j, and constant in j for a fixed t over $(t, j) \in S$;*
- *A function $q : S \to Q$ such that $q(t, j)$ is constant in t, for a fixed j over $(t, j) \in S$;*

meeting the conditions

S_1) $x(0,0) \in C_{q(0,0)} \cup D_{q(0,0)}$;

S_2) $\forall j \in \mathbb{N}$ and æ t such that $(t, j) \in S$,

$$\dot{q}(t, j) = 0,\ \dot{x}(t, j) \in F_{q(t,j)}(x(t, j)),\ x(t, j) \in C_{q(t,j)};$$

S_3) $\forall (t, j) \in S$ such that $(t, j + 1) \in S$

$$x(t, j + 1) = x(t, j),\ q(t, j + 1) \in G_{q(t,j)}(x(t, j)),\ x(t, j) \in D_{q(t,j)}.$$

From now on, we will refer to the domain of a solution (x, q) to $\mathbb{H}$ as $\mathtt{dom}(x, q)$. A solution (x, q) to $\mathbb{H}$ is called *maximal* if it cannot be extended, i.e., does not exists any solution defined on a larger domain of definition and equal to (x, q) on $\mathtt{dom}(x, q)$. A solution is *complete* if its domain is unbounded.

During flows, x evolves according to the differential equation $\dot{x} = f_h(x, \varphi_q(x))$, $x \in C_q$ while q remains constant. During jumps, q evolves according to the difference inclusion $q^+ \in G_q(x)$, $x \in D_q$ while x remains constant.

Remark 2.4: Note that a sufficient condition for the existence of a hybrid stabilizer for (1) is the global asymptotic controllability (see [27], Theorem 3.7 for more details).

Together with locally Lipschitz continuity assumption, we consider the Filippov regularization of (1) which assures existence, uniqueness and bounded dependence on the initial condition for solutions of $\mathbb{H}$. Moreover, $\mathbb{H}$ is robust and its solution behaves as follows: it is either complete or blows in a finite hybrid domain time or eventually jumps out of $C_q \cup D_q$, $q \in Q$. For further information, see [2], [4], [7], [11] and [12].

We can now define the notion of stability needed to design the controller for the hybrid closed loop system.

Definition 2.5:

- *A set $\mathbf{A} \subset \mathbb{R}^n$ is* stable *for $\mathbb{H}$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that any solution (x, q) to* (9) *with $|x_0|_{\mathbf{A}} \leq \delta$ satisfies $|x(t, j)|_{\mathbf{A}} \leq \varepsilon$, for all $(t, j) \in \mathtt{dom}(x, q)$;*
- *A set $\mathbf{A} \subset \mathbb{R}^n$ is* attractive *for $\mathbb{H}$ if there exists $\delta > 0$ such that*
  - *for all $(\bar{x}, \bar{q}) \in \mathbb{R}^n \times Q$ with $|\bar{x}|_{\mathbf{A}} \leq \delta$ there exists a solution to $\mathbb{H}$ with $(x, q)(0, 0) = (\bar{x}, \bar{q})$;*
  - *for any maximal solution (x, q) to $\mathbb{H}$ with $|x(0, 0)|_{\mathbf{A}} \leq \delta$ we have $|x(t, j)|_{\mathbf{A}} \to 0$ as $t \to \sup_t(\mathtt{dom}(x, q))$.*
- *The set $\mathbf{A} \subset \mathbb{R}^n$ is* asymptotically stable *if it is stable and attractive;*
- *The* basin of attraction*, denoted by $\mathbb{B}_{\mathbb{H}}(\mathbf{A})$, is the set of all $\bar{x} \in \mathbb{R}^n$ such that for all $\bar{q} \in Q$, there exists a solution to $\mathbb{H}$ with $x(0, 0) = \bar{x}$, $q(0, 0) = \bar{q}$ and any such solution that is also maximal satisfies $|x(t, j)|_{\mathbf{A}} \to 0$ as $t \to \sup_t \mathtt{dom}(x, q)$;*
- *The set $\mathbf{A} \subset \mathbb{R}^n$ is* globally asymptotically stable *if $\mathbb{B}_{\mathbb{H}}(\mathbf{A}) = \mathbb{R}^n$.*

III. Main result

Let us denote the unit closed ball in $\mathbb{R}^n$ by $\mathbf{B}$. Before stating our main result, let us first solve a preliminary design problem by adapting the backstepping technique:

Proposition 3.1: *Under Assumption 2, the set $\mathbf{A}$ defined by (7) is globally practically stabilizable, i.e. for each $a > 0$ there exists a continuous controller φ_g such that the set $\mathbf{A} + a\mathbf{B}$ contains a set that is globally asymptotically stable for system (1) in closed-loop with φ_g.*

We are now in position to state our main result.

Theorem 1: *Let v_ℓ and $\widetilde{v}_\ell$ be two positive constants satisfying $0 < \widetilde{v}_\ell < v_\ell$. Under Assumptions 1, 2 and 3 there exists $a > 0$ such that the hybrid controller $\mathbb{K}$ defined by $Q = \{1, 2\}$, subsets*

$$\begin{array}{rcl} C_1 & = & \{(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R} : V_\ell(x_1, x_2) \leq v_\ell\}, \\ C_2 & = & \{(x_1, x_2) \in \mathbb{R}^{n-1} \times \mathbb{R} : V_\ell(x_1, x_2) \geq \widetilde{v}_\ell\}, \\ D_q & = & \overline{(\mathbb{R}^{n-1} \times \mathbb{R}) \setminus C_q},\ \forall q = 1, 2, \end{array}$$

controllers $C_1 \ni (x_1, x_2) \mapsto \varphi_1(x_1, x_2) = \varphi_\ell(x_1, x_2) \in \mathbb{R}$ and $C_2 \ni (x_1, x_2) \mapsto \varphi_2(x_1, x_2) = \varphi_g(x_1, x_2, a) \in \mathbb{R}$ and set-valued mapping $D_q \ni (x_1, x_2) \mapsto G_q(x_1, x_2) = \{3-q\}$, $q \in Q$, renders the origin globally asymptotically stable for (1) *in closed-loop with $\mathbb{K}$.*

Let us emphasize that this result is more than an existence result since its proof allows to design a suitable hybrid feedback law. Let us sketch the proof of Theorem 1. First, we use Assumption 2, and Proposition 3.1 is applied to design a controller, denoted φ_g, such that the set $\mathbf{A}$ is globally practically stable for the system (1) in closed-loop with φ_g.

[2]a set-valued mapping $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is said to be outer semicontinuous if each sequence (x_i, f_i) in $\mathbb{R}^m \times \mathbb{R}^n$ that satisfies $f_i \in F(x_i)$ for each i, and converges to a point (x, f) in $\mathbb{R}^m \times \mathbb{R}^n$ has the property that $f \in F(x)$.

[3]a set-valued mapping $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is said to be locally bounded if, for each compact set $K_1 \subset \mathbb{R}^n$, there exists a compact set $K_2 \subset \mathbb{R}^n$ such that $F(K_1) := \bigcup_{x\in K_1} \subset K_2$. The boundedness is said to be uniform with respect to a parameter if the set K_2 can be selected uniformly with respect to this parameter.

Using Assumptions 1 and 3, this set is shown to be included in the basin of attraction of the system (1) in closed-loop with φ_ℓ. Then we design a hybrid feedback law based on an hysteresis of both controllers φ_ℓ and φ_g on appropriate sets. This latter construction is adapted from other works like [11] or [25]. The complete proof of Theorem 1 is in Section V below.

IV. Illustration

Before applying the main result of this paper, let us first consider the following example in $\mathbb{R}^2$

$$\left\{\begin{array}{rcl}\dot{x}_1 &=& x_1+\theta x_1^2+x_2\\ \dot{x}_2 &=& u\end{array}\right. , \qquad (10)$$

where θ is a positive constant.

This system is in backstepping form and many references on how to design a global stabilizer are presented in the literature, for instance, the reader may see [8], [18], and [19]. Following this approach, in a first step, we consider the two smooth functions $\varphi_1(x_1)=-(1+c_1)x_1-\theta x_1^2$ and $V_1(x_1)=\frac{1}{2}x_1^2$ where c_1 is a positive constant. It can be checked that this function is such that, for all x_1 in $\mathbb{R}$,

$$\partial_{x_1}V_1(x_1)\left[x_1+\theta x_1^2+\varphi_1(x_1)\right]=-2c_1V_1(x_1)\ . \qquad (11)$$

This gives the control law, for all (x_1,x_2) in $\mathbb{R}^2$,

$$\begin{array}{rcl}\varphi_b(x_1,x_2) &=& -(1+c_1+2\theta x_1)(x_1+\theta x_1^2+x_2)\\ && -x_1-c_2\left(x_2+(1+c_1)x_1+\theta x_1^2\right)\end{array}$$

which is such that along the solutions of (10),

$$\dot{V}_b(x_1,x_2)=-c_1x_1^2-c_2\left(x_2+(1+c_1)x_1+\theta x_1^2\right)^2$$

where $V_b(x_1,x_2)=V_1(x_1)+\frac{1}{2}(x_2+(1+c_1)x_1+\theta x_1^2)^2$.

However the backstepping technique cannot be applied to the following system:

$$\left\{\begin{array}{rcl}\dot{x}_1 &=& x_1+x_2+\theta[x_1^2+(1+x_1)\mathtt{sin}\,(u)]\\ \dot{x}_2 &=& u\end{array}\right. \qquad (12)$$

due to the presence of the term $(1+x_1)\mathtt{sin}\,(u)$ in the time-derivative of x_1 (recall the discussion in Footnote 1). Therefore, it is necessary to revise the controller design for (1) and to apply Theorem 1. With obvious definitions of the functions f_1, f_2, h_1 and h_2, system (12) may be rewritten as system (1) and system (10) may be rewritten as $\dot{x}=f(x,u)$. There exists $\theta>0$ sufficiently small such that we may apply Theorem 1. Indeed we have the following result.

Lemma 4.1: *Let θ be a positive constant. If θ is sufficiently small, then Assumptions 1, 2, and 3 hold for system* (12).

The proof has been removed due to space limitation.

Combining this result with Theorem 1, we may design a hybrid feedback law $\mathbb{K}$ such that the origin is globally asymptotically stable to (12) in closed-loop with $\mathbb{K}$.

Let us consider the following parameters $\theta=10^{-3}$, $\rho=2$, $c_1=\frac{(2+\rho)\theta}{2}+1=1.0020$, $a=10$ and $c=10$. Item 1 of Assumption 2 is satisfied with $\alpha(s)=2c_1s$, $\forall s\geq 0$. Item 2 is satisfied with positive constants $\varepsilon=1-\theta\frac{2+\rho}{2c_1}=0.998$ and $M=\frac{\theta}{2\rho(2c_1-\theta(2+\rho))}=1.25\times 10^{-4}$. Items 3 and 4 are satisfied with $\Psi(x_1,x_2)=\theta(1+|x_1|)$.

Since the pair of matrices $(A,B)=(\partial_x f_h(0,0),\partial_u f_h(0,0))$ is controllable, Assumption 1 holds with $\varphi_\ell(x)=k_1x_1+k_2x_2$, where $k_1=-5-\theta$ and $k_2=-3+3\theta+\theta^2$, $V_\ell(x)=\frac{1}{2}(x_1-\theta x_2)^2+\frac{1}{2}(2x_1+(1-2\theta)x_2)^2$ and $v_\ell=\left(\frac{2}{\theta p(\theta)}\right)^2=0.1042$. Moreover, in the set defined by

$$\mathbf{A}=\{(x_1,x_2)\in\mathbb{R}^2:|x_1|\leq\sqrt{2.5\times 10^{-4}},\ x_2=\varphi_1(x_1)\}.$$

we may check that

$$\mathtt{max}_{\,x\in\mathbf{A}}V_\ell(x_1,x_2)=0.0001<v_\ell,$$

and thus Assumption 3 holds. Following Proposition 3.1 and Theorem 1, we may define a hybrid controller. More precisely, computing $k=2\frac{M+a}{a^2}=0.2$, we define the global controller

$$\varphi_g(x_1,x_2)=\frac{\widetilde{u}}{k}-(1+c_1+2\theta x_1)(x_1+\theta x_1^2+x_2)+\frac{x_1}{2k},$$

where $\widetilde{u}=(x_1-\varphi_1(x_1))\left[-c-\frac{c}{4}\Delta(x_1,x_2)^2\right]$ and

$$\Delta(x_1,x_2)=|x_1|\theta(1+|x_1|)+\theta(1+|x_1|)\cdot k(1+|(1+c_1)x_1+\theta x_1^2|)$$

Then, letting $\widetilde{v}_\ell=0.05$, the origin is globally asymptotically stable for (12) in closed-loop with the hybrid controller $\mathbb{K}$ defined as in Theorem 1.

Let us check this property on numerical simulations. To do that, we consider the initial condition $x_1(0,0)=0.5$, $x_2(0,0)=0.1$ and $q(0,0)=1$. See Fig. 1 for the time evolution of the x_1, x_2 and q components of the solution of (12) in closed-loop with $\mathbb{K}$. First the system (12) is in closed-loop with the controller φ_g (for continuous time between 0 and 0.5314). Then the system (12) is in closed-loop with the controller φ_ℓ, and the solution converges to the origin.

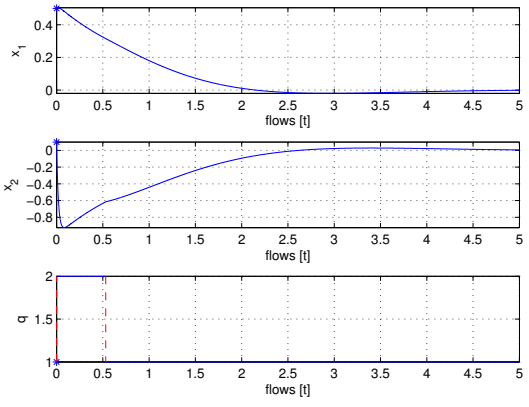

Fig. 1. At top, time evolution of x_1, at middle, time evolution of x_2 and, at bottom, time evolution of q.

V. Proof of Theorem 1

A. Proof of Proposition 3.1

Proof: Let a be a positive value. We wish to show that there exists a continuous controller φ_g such that $\mathbf{A}+a\mathbf{B}$

contains a set that is globally and asymptotically stable.

First of all, note that if we introduce the function $r_1(x_1,x_2,u) = f_1(x_1,x_2) + h_1(x_1,x_2,u)$, we get with Item 1 and Item 2 of Assumption 2 that along the solutions of (1), we have for all (x_1,x_2) in $\mathbb{R}^n$ and u in $\mathbb{R}$,

$$\begin{array}{rcl} \dot{V}_1(x_1) & \leq & \varepsilon[\alpha(M) - \alpha(V_1(x_1))] \\ & & +\partial_{x_1} V_1(x_1) \cdot [r_1(x_1,x_2,u) - r_1(x_1,\varphi_1(x_1),u)] \end{array} \quad (13)$$

Moreover, with the $\mathcal{C}^1$ function $\eta_{x_1,x_2}(s) = sx_2 + (1-s)\varphi_1(x_1)$, it yields

$$\partial_s r_1(x_1,\eta_{x_1,x_2}(s),u) = \partial_{x_2} r_1(x_1,\eta_{x_1,x_2}(s),u)(x_2 - \varphi_1(x_1)) \ ,$$

which implies

$$\begin{array}{rcl} r_1(x_1,x_2,u) - r_1(x_1,\varphi_1(x_1),u) & = & (x_2 - \varphi_1(x_1)) \\ & & \int_0^1 \partial_{x_2} r_1(x_1,\eta_{x_1,x_2}(s),u)\, ds. \end{array}$$

Hence, Equation (13) becomes,

$$\begin{array}{l} \dot{V}_1(x_1) \leq \varepsilon[\alpha(M) - \alpha(V_1(x_1))] \\ +(x_2 - \varphi_1(x_1))\partial_{x_1} V_1(x_1) \cdot \int_0^1 \partial_{x_2} r_1(x_1,\eta_{x_1,x_2}(s),u)\, ds. \end{array}$$

Let $V(x) = V_1(x_1) + \frac{k}{2}(x_2 - \varphi_1(x_1))^2$ for all (x_1,x_2) in $\mathbb{R}^n$ with $k = 2\frac{M+a}{a^2}$. Let a' be a positive value such that $V_1(x_1) \leq a'$ implies $x_1 \in \{x_1' : V_1(x_1') \leq a'\} + a\mathbf{B}$, in other words, a' is such that

$$V_1(x_1) \leq a' \Rightarrow \exists x_1' \text{ s.t. } V_1(x_1') \leq a' \text{ and } |x_1 - x_1'| \leq a \ .$$

Such positive value a' exists since V_1 is assumed to be a proper function. Let $\widetilde{a} = \mathtt{min}\{a, a'\}$. With these definitions of k and a', we get

$$\{x : V(x) \leq M + \widetilde{a}\} \subset \mathbf{A} + a\mathbf{B} \quad (14)$$

Consider now the control φ_g defined for all $\widetilde{u}$ in $\mathbb{R}$ as in Proposition 3.1.

Along the solutions of (1) with $u = \varphi_g(x_1,x_2,\widetilde{u})$, it yields for all (x_1,x_2) in $\mathbb{R}^n$ and $\widetilde{u}$ in $\mathbb{R}$,

$$\begin{array}{l} \dot{V}(x) \leq \varepsilon[\alpha(M) - \alpha(V_1(x_1))] + (x_2 - \varphi_1(x_1))[\widetilde{u} \\ \qquad +\Upsilon(x_1,x_2,u)], \end{array}$$

where

$$\begin{array}{l} \Upsilon(x_1,x_2,u) = \partial_{x_1} V_1(x_1) \cdot \int_0^1 \partial_{x_2} h_1(x_1,\eta_{x_1,x_2}(s),u)\, ds \\ \qquad +kh_2(x_1,x_2,u) - k\partial_{x_1}\varphi_1(x_1) h_1(x_1,x_2,u). \end{array}$$

With Item 2, 3 and 4 of Assumption 2, the function Υ satisfies $|\Upsilon(x_1,x_2,u)| \leq \Delta(x_1,x_2)$ with

$$\begin{array}{l} \Delta(x_1,x_2) = |\partial_{x_1} V_1(x_1)| \int_0^1 \Psi(x_1,\eta_{x_1,x_2}(s))\, ds \\ \qquad +\Psi(x_1,x_2)k(1 + |\partial_{x_1}\varphi_1(x_1)|) \end{array} \quad (15)$$

Using a particular case of the Cauchy-Schwartz inequality (i.e. $\alpha \leq \frac{1}{c} + \frac{c}{4}\alpha^2$), we get, for all $c > 0$

$$\begin{array}{l} (x_2 - \varphi_1(x_1))\Upsilon(x_1,x_2,u) \leq \frac{1}{c} \\ \qquad +\frac{c}{4}(x_2 - \varphi_1(x_1))^2 \Delta(x_1,x_2)^2. \end{array}$$

Consequently, it implies, that by taking

$$\widetilde{u} = (x_2 - \varphi_1(x_1)) \left[-c - \frac{c}{4}\Delta(x_1,x_2)^2 \right] \ , \quad (16)$$

it yields along the solutions of

$$\dot{x} = f(x, \varphi_g(x_1,x_2,\widetilde{u})) \ . \quad (17)$$

and for all (x_1,x_2) in $\mathbb{R}^n$,

$$\dot{V}(x) \leq \varepsilon[\alpha(M) - \alpha(V_1(x_1))] + \tfrac{1}{c} - c(x_2 - \varphi_1(x_1))^2 \ . \quad (18)$$

Note that for all $c \geq 1$, it gives,

$$\dot{V}(x) \leq \varepsilon[\alpha(M) - \alpha(V_1(x_1))] + 1 - (x_2 - \varphi_1(x_1))^2 \ .$$

The function V_1 being proper, the set $\mathbf{A}_1 \subset \mathbb{R}^n$ defined by

$$\mathbf{A}_1 = \left\{ x : \varepsilon\alpha(V_1(x_1)) + (x_2 - \varphi_1(x_1))^2 \leq \varepsilon\alpha(M) + 1 \right\},$$

is compact. Moreover, selecting $c > 1$, we get, along the solutions of (17), $\dot{V}(x) < 0$, for all x such that $V(x) \geq \zeta$, where ζ is the positive value defined as $\zeta = \mathtt{max}_{x \in \mathbf{A}_1}\{V(x)\}$. Consequently, for all $c > 1$, the set $\{x, V(x) \leq \zeta\}$ is globally asymptotically stable for (17).

The function α being locally Lipschitz, we can define K_α its Lipschitz constant in the compact set $\{x, V(x) \leq \zeta\}$. Hence, for all x in $\{x, V(x) \leq \zeta\}$, it yields,

$$|\alpha(V_1(x_1)) - \alpha(V(x))| \leq \tfrac{kK_\alpha}{2}(x_2 - \varphi_1(x_1))^2 \ .$$

Consequently, with (18) and $c > 1$, we get along the solutions of (17), for all x such that $V(x) \leq \zeta$,

$$\begin{array}{l} \dot{V}(x) \leq \varepsilon[\alpha(M) - \alpha(V(x))] + \frac{1}{c} \\ \qquad -\left(c - \varepsilon\frac{kK_\alpha}{2}\right)(x_2 - \varphi_1(x_1))^2 \ . \end{array}$$

Finally, taking $c > c_g$ where

$$c_g = \mathtt{max}\left\{ \tfrac{1}{\varepsilon[\alpha(M+\widetilde{a}) - \alpha(M)]}, \varepsilon\tfrac{kK_\alpha}{2}, 1 \right\},$$

it gives, along the trajectories of (17), for all x such that $V(x) \leq \zeta$, $\dot{V}(x) \leq \varepsilon\left[\alpha(M + \widetilde{a}) - \alpha(V(x))\right]$.

Therefore, with $c > c_g$, for all x such that $\zeta \geq V(x) > M + \widetilde{a}$, we get along the solutions of (17), $\dot{V}(x) < 0$. Since $c_g > 1$ the same control gives also $\dot{V}(x) < 0$ for all x such that $V(x) \geq \zeta$. Therefore the set $\{x,\ V(x) \leq M + \widetilde{a}\}$ in an attractor for system (1) in closed-loop with $u = \varphi_g(x_1,x_2,\widetilde{u})$. Consequently, with (14), the set $\mathbf{A} + a\mathbf{B}$ contains a set that is globally and asymptotically stabilizable with the control law $\varphi_g(x_1,x_2) = \varphi_g(x_1,x_2,\widetilde{u})$ where $\widetilde{u}$ is defined in (16) and $c > c_g$. This concludes the proof of Proposition 3.1. ■

B. Proof of Theorem 1

Proof: Since Assumption 2 holds, Proposition 3.1 applies. Let us choose the positive real number $0 < a$ such that

$$\mathtt{max}_{x \in \mathbf{A} + a\mathbf{B}} V_\ell(x) < \widetilde{v}_\ell \ . \quad (19)$$

Such values exist since Assumption 3 holds, and since V_ℓ is a proper function.

Let us consider the controller φ_g given by Proposition 3.1 with this value of a.

Let us design a hybrid feedback law $\mathbb{K}$ defining it as in Theorem 1, i.e., building an hysteresis of φ_ℓ and φ_g on appropriate domains (see also [11] or [25] for similar concepts applied to different control problems).

Consider an initial condition $(x(0,0), q(0,0))$ in $\mathbb{R}^n \times Q$, and a maximal solution (x, q) of (1) in closed-loop with the hybrid feedback law $\mathbb{K} = (Q, (C_q, D_q, \varphi_q)_{q=1,2})$. Let us assume, for the time-being, the following

Lemma 5.1: *There exists a hybrid time $(\bar{t}, \bar{j})$ in $\mathtt{dom}(x, q)$ such that $q(\bar{t}, \bar{j}) = 1$ and $x(\bar{t}, \bar{j})$ in C_1.*

Now, recalling (19) and using Assumption 1, the sets C_1 is forward invariant for system (1) in closed-loop with φ_ℓ. Thus with Lemma 5.1, we get that (1) in closed-loop with the hybrid feedback law $\mathbb{K}$ is globally asymptotically stable (since system (1) in closed-loop with φ_ℓ is locally asymptotically stable).

Therefore to conclude the proof of Theorem 1, it remains to prove Lemma 5.1. Let us prove this result by assuming the converse and exhibiting a contradiction. More precisely, let us assume that, for all (t, j) in $\mathtt{dom}(x, q)$,

$$x(t,j) \notin C_1 \quad \text{or} \quad q(\bar{t}, \bar{j}) = 2 \ . \tag{20}$$

Thus, due to the expression of D_2, for all (t, j) in $\mathtt{dom}(x, q)$, we have

$$x(t,j) \in D_2 \setminus C_1 \quad \text{or} \quad q(\bar{t}, \bar{j}) = 2 \ . \tag{21}$$

If there is a time such that $x(\bar{t}, \bar{j}) \in D_2 \setminus C_1$ and $q(\bar{t}, \bar{j}) = 1$, then a jump occurs for the q-variable and, due to the expression of G_1, $x(\bar{t}, \bar{j}+1) \in C_1$ and $q(\bar{t}, \bar{j}+1) = 2$, which is a contradiction with (20). Therefore, if $x(\bar{t}, \bar{j}) \in D_2 \setminus C_1$, then $q(\bar{t}, \bar{j}) = 2$. Thus we get with (21), for all (t, j) in $\mathtt{dom}(x, q)$, $x(t, j) \in D_2$ and $q(\bar{t}, \bar{j}) = 2$. Therefore the x-component is a solution of (1) in closed-loop with φ_g with does not enter C_1. Since, with (19), C_1 strictly contains the set $\mathbf{A}$, we get the existence of a solution of (1) in closed-loop with φ_g which does not converge to $\mathbf{A} + a\mathbf{B}$. This is a contradiction with the choice of the controller φ_g satisfying the conclusion of Proposition 3.1.

This concludes the proof of Theorem 1. ∎

VI. Conclusion

A new design method has been suggested in this paper to combine a backstepping controller with a local feedback law. The class of designed controllers lies in the set of hybrid feedback laws. It allows us to define a stabilizing control law for nonlinear control systems for which there exist some structural obstacles to the existence of classical continuous stabilizing feedback laws. More precisely, it is studied systems for which the backstepping tool for the design of stabilizers can not be applied.

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