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Stabilization of Distributed Networked Control Systems with Minimal Communications Network

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Abstract— **In this paper we study general stability condition of distributed networked control systems with minimum number of necessary communication links. We use the Lyapunov direct method to find a general stability condition that guarantees the asymptotic stability of the entire networked control system and then use binary programming to design a communications network with minimum number of links that satisfies the general stability condition. Reducing the number of communication links implies minimization of the communications network's cost and energy consumption. The results apply to networks of linear time-invariant (LTI) systems.**

I. INTRODUCTION

A networked control system (NCS) consists of many coupled subsystems that are spatially distributed. Each subsystem comprises of a plant and a controller. The interaction of plants with each other forms the *dynamics network*. Control and feedback signals are exchanged using a *communications network* (information network) among controller components (Fig. 1). The main advantages of NCSs are reduced system wiring, scalability, simplicity of system diagnosis, maintenance and saving resources. Specific examples of NCS include electrical power grids, transportation networks, factory automation and tele-operations.

NCSs lie at the intersection of control and communication theories. Each NCS has a dynamics network including a set of subsystem that affects each other. It is obvious that even if each subsystem is asymptotically stable, the connected plants may be unstable due to the interactions between them. In such a scenario to stabilize the NCS, a communications network carrying distributed feedback information between different subsystems will be necessary.

In general, it is not feasible to control a large-scale networked system with a centralized approach. In a centralized approach, the control law uses the state information of all subsystems, which requires a very large and costly communications network for exchanging state information. This requirement limits the scalability of centralized approaches to networked control systems. To overcome this limitation, there are two major different approaches, namely decentralized and distributed control strategies [1]–[3].

In the decentralized control strategy, the control law uses only a subsystem's local state information to control the given subsystem. Such local controls can be very effective when the coupling between subsystems are weak. All decentralized and self triggering methods use this strategy to stabilize the whole system [4]–[6].

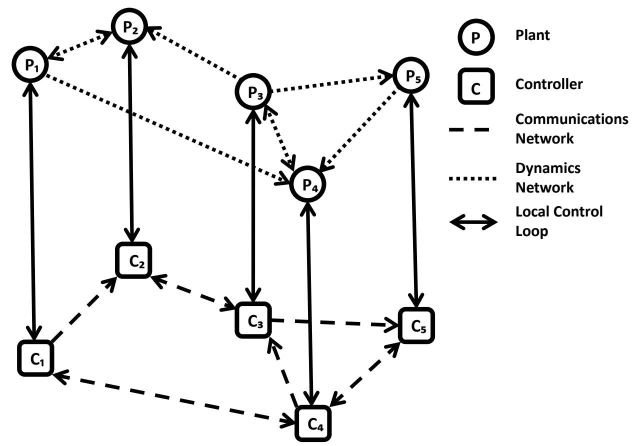

Fig. 1: A Networked Control System (NCS)

On the other hand, if subsystem's coupling is not weak, then we must use a distributed feedback control approach. In distributed networked control system, each given subsystem uses its state and the state of its neighbors that affects it. As this method uses feedback from affecting neighbor subsystems, distributed control can assure asymptotic stability with stronger subsystem coupling than decentralized control strategy [7] [8].

For both decentralized and distributed approaches, two tractable methods are provided in the literature: (i) using a condition, called "quadratic invariance", under which the above problem may be recast as a convex optimization problem [9] [10] and (ii) using the Lyapunov direct method [8] [11].

The question is, given a particular dynamics network, how can we stabilize it with minimum number of links in the communications network? In other words, how can we find a minimal communications network that guarantees the stability of the NCS? This problem is the focus of this paper.

We consider the problem of multiple coupled linear time-invariant (LTI) subsystems, each with its own controller. The controllers may communicate with each other through the communications network to exchange subsystem's state information. We find general stability condition by using the Lyapunov direct method that guarantees asymptotic stability of all coupled subsystems and extend the approach taken in [7] to include non-symmetric networked systems. We assume that the communication cost of the links are identical. Consequently, our problem reduces to minimize the number of links. Then, to design a communications network that has minimum number of links, we propose an algorithm that minimizes the number of links in the communications network while guaranteeing system stability.

The remainder of this paper is organized as follows. Section II describes the distributed networked control system under consideration. Section III derives a general stability condition that guarantees asymptotic stability of all coupled subsystems. In Section IV we propose an algorithm for designing necessary communication links that satisfy the general stability condition. In Section V we apply the results to a numerical example. Concluding remarks are given in Section VI.

II. NOTATION AND SYSTEM DESCRIPTION

1) Notation: Matrices and vectors are denoted by capital and lower-case bold letters, respectively. The usual Euclidean (l_2) vector norm is represented by $\| \cdot \|$. When applied to a matrix $\| \cdot \|$ denotes the l_2 induced matrix norm, $\| \mathbf{A} \|^2 = \lambda_{max}(\mathbf{A}^T\mathbf{A})$. By $\lambda_{min}(\mathbf{A})$ and $\lambda_{max}(\mathbf{A})$ we denote the minimum and maximum eigenvalues of $\mathbf{A}$, respectively. We denote matrix $\mathbf{P} \in \mathbb{R}^{m\times m}$ being positive definite (p.d.) by $\mathbf{P} > \mathbf{0}$. We let $\mathcal{N}$ denote the set $\{1, 2, ..., N\}$. The cardinality of a set is denoted by $| \cdot |$.

2) System Description: The system under study is a collection of N coupled linear time-invariant subsystems. The local state of the ith plant that may be affected by all other subsystems is a function $\mathbf{x}_i(t) : \mathbb{R} \to \mathbb{R}^{n_i}$ where n_i is the local state space dimension and $i \in \mathcal{N}$ as follows

$$\dot{\mathbf{x}}_i(t) = \mathbf{A}_i\mathbf{x}_i(t) + \mathbf{B}_i\mathbf{u}_i(t) + \sum_{j\in\mathcal{N}-\{i\}} \mathbf{H}_{ij}\mathbf{x}_j(t) \quad (1)$$

$$\mathbf{x}_i(0) = \mathbf{x}_{i0}$$

where $\mathbf{x}_{i0} \in \mathbb{R}^{n_i}$ is the initial state.

The signal $\mathbf{u}_i(t) : \mathbb{R} \to \mathbb{R}^{m_i}$ is the local control signal generated by the ith controller where m_i is the dimension of the control set. $\mathbf{A}_i \in \mathbb{R}^{n_i\times n_i}$, $\mathbf{B}_i \in \mathbb{R}^{n_i\times m_i}$ and $\mathbf{H}_{ij} \in \mathbb{R}^{n_i\times n_j}$ are matrices with appropriate dimensions. Assume that for each $i \in \mathcal{N}$ the pair $(\mathbf{A}_i, \mathbf{B}_i)$ are fully controllable, which means that there exists $\mathbf{K}_i \in \mathbb{R}^{m_i\times n_i}$ for which the decoupled subsystem

$$\dot{\mathbf{x}}_i(t) = \mathbf{A}_i\mathbf{x}_i(t) + \mathbf{B}_i\mathbf{u}_i(t) \quad (2)$$

with state feedback law $\mathbf{u}_i(t) = \mathbf{K}_i\mathbf{x}_i(t)$ is asymptotically stable. In other words, there exists a control Lyapunov function $V_i(\mathbf{x}_i) = \mathbf{x}_i^T\mathbf{P}_i\mathbf{x}_i$ for system (2) where $\mathbf{P}_i \in \mathbb{R}^{n_i\times n_i}$ is unique, symmetric and p.d. solution of Lyapunov equation

$$(\mathbf{A}_i + \mathbf{B}_i\mathbf{K}_i)^T\mathbf{P}_i + \mathbf{P}_i(\mathbf{A}_i + \mathbf{B}_i\mathbf{K}_i) = -\mathbf{Q}_i \quad (3)$$

for any symmetric, p.d. $\mathbf{Q}_i \in \mathbb{R}^{n_i\times n_i}$.

We are looking for distributed control laws which means having feedback from (potentially) all other subsystems. Therefore, the modified state feedback law is

$$\mathbf{u}_i(t) = \mathbf{K}_i\mathbf{x}_i(t) + \sum_{j\in\mathcal{N}-\{i\}} \mathbf{L}_{ij}\mathbf{x}_j(t) \quad (4)$$

where $\mathbf{K}_i$ is the state feedback gain satisfying Lyapunov equation (3), $\mathbf{L}_{ij} \in \mathbb{R}^{m_i\times n_j}$ is a set of gains and $\mathbf{x}_j(t)$ is the state of subsystem j at time t. To design a minimal communications network, we seek a set of $\mathbf{L}_{ij}$ that guarantee stability, with minimum number of non-zero $\mathbf{L}_{ij}$'s.

III. GENERAL STABILITY CONDITION

In this section we derive a condition that assures the entire connected subsystems is asymptotically stable. To do this, we will employ a methodology similar to that of [7]. Consider the Lyapunov function for each subsystem i as

$$V_i(\mathbf{x}_i) = \mathbf{x}_i^T\mathbf{P}_i\mathbf{x}_i \quad (5)$$

where $\mathbf{P}_i$ satisfies the Lyapunov equation (3). Then, a good candidate Lyapunov function for the entire system, $V : \mathbb{R}^{\sum_i n_i} \to \mathbb{R}$, is

$$V(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_N) = \sum_{i\in\mathcal{N}} V_i(\mathbf{x}_i) = \sum_{i\in\mathcal{N}} \mathbf{x}_i^T\mathbf{P}_i\mathbf{x}_i \quad (6)$$

We use this Lyapunov function in Theorem 1 to establish a general stability condition.

Theorem 1: Consider the system in (1) where

1) the control input $\mathbf{u}_i$ is the distributed control in (4)
2) $\mathbf{P}_i, \mathbf{K}_i$ and $\mathbf{Q}_i$ satisfy the Lyapunov equation (3)

Then, the networked system (1) is asymptotically stable, if there exists positive real constants δ_{ji} such that

$$\sum_{j\in\mathcal{N}-\{i\}} \frac{\| \mathbf{P}_j(\mathbf{B}_j\mathbf{L}_{ji} + \mathbf{H}_{ji}) \|^2}{\delta_{ji}} + \sum_{j\in\mathcal{N}-\{i\}} \delta_{ij} < \lambda_{min}(\mathbf{Q}_i) \quad (7)$$

for all $i \in \mathcal{N}$.

Proof: Substituting control input from (4) into (1), the closed loop system is

$$\dot{\mathbf{x}}_i(t) = (\mathbf{A}_i + \mathbf{B}_i\mathbf{K}_i)\mathbf{x}_i(t) + \sum_{j\in\mathcal{N}-\{i\}} (\mathbf{B}_i\mathbf{L}_{ij} + \mathbf{H}_{ij})\mathbf{x}_j(t) \quad (8)$$

taking derivative of the Lyapunov function (5) along the trajectories of system (8) and defining $\mathbf{A}_{\mathbf{K}_i} \triangleq \mathbf{A}_i + \mathbf{B}_i\mathbf{K}_i$, we have

$$\begin{aligned}\dot{V}_i(\mathbf{x}_i) &= \frac{\partial V_i}{\partial \mathbf{x}_i}\dot{\mathbf{x}}_i = \dot{\mathbf{x}}_i^T\mathbf{P}_i\mathbf{x}_i + \mathbf{x}_i^T\mathbf{P}_i\dot{\mathbf{x}}_i \\ &= \mathbf{x}_i^T \underbrace{(\mathbf{A}_{\mathbf{K}_i}^T\mathbf{P}_i + \mathbf{P}_i\mathbf{A}_{\mathbf{K}_i})}_{-\mathbf{Q}_i} \mathbf{x}_i \\ &\quad + 2\sum_{j\in\mathcal{N}-\{i\}} \mathbf{x}_i^T\mathbf{P}_i(\mathbf{B}_i\mathbf{L}_{ij} + \mathbf{H}_{ij})\mathbf{x}_j \end{aligned} \quad (9)$$

using the following inequality [7]

$$\| \delta\mathbf{z} - \mathbf{R}\mathbf{y} \|^2 \geq 0 \Rightarrow 2\mathbf{z}^T\mathbf{R}\mathbf{y} \leq \delta \| \mathbf{z} \|^2 + \frac{\| \mathbf{R}\mathbf{y} \|^2}{\delta} \quad (10)$$

where $\mathbf{z} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{R} \in \mathbb{R}^{n\times m}$ and δ is any positive real constant, then (9) can be upper-bounded as

$$\begin{aligned}\dot{V}_i(\mathbf{x}_i) &\le -\mathbf{x}_i^T\mathbf{Q}_i\mathbf{x}_i \\ &+ \sum_{j\in\mathcal{N}-\{i\}}\left[\delta_{ij}\parallel \mathbf{x}_i\parallel^2 + \frac{\parallel \mathbf{P}_i(\mathbf{B}_i\mathbf{L}_{ij}+\mathbf{H}_{ij})\mathbf{x}_j\parallel^2}{\delta_{ij}}\right] \\ &\le -\left(\lambda_{min}(\mathbf{Q}_i) - \sum_{j\in\mathcal{N}-\{i\}}\delta_{ij}\right)\parallel \mathbf{x}_i\parallel^2 \\ &+ \sum_{j\in\mathcal{N}-\{i\}}\frac{\parallel \mathbf{P}_i(\mathbf{B}_i\mathbf{L}_{ij}+\mathbf{H}_{ij})\parallel^2\parallel \mathbf{x}_j\parallel^2}{\delta_{ij}} \qquad (11)\end{aligned}$$

and for the entire system we have

$$\begin{aligned}&\dot{V}(\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_N) = \sum_{i\in\mathcal{N}}\dot{V}_i(\mathbf{x}_i) \le \\ &-\sum_{i\in\mathcal{N}}\left(\lambda_{min}(\mathbf{Q}_i) - \sum_{j\in\mathcal{N}-\{i\}}\delta_{ij}\right)\parallel \mathbf{x}_i\parallel^2 \\ &+\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}-\{i\}}\frac{\parallel \mathbf{P}_i(\mathbf{B}_i\mathbf{L}_{ij}+\mathbf{H}_{ij})\parallel^2\parallel \mathbf{x}_j\parallel^2}{\delta_{ij}} \qquad (12)\end{aligned}$$

using the following reorganization of the terms

$$\begin{aligned}&\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}-\{i\}}\frac{\parallel \mathbf{P}_i(\mathbf{B}_i\mathbf{L}_{ij}+\mathbf{H}_{ij})\parallel^2\parallel \mathbf{x}_j\parallel^2}{\delta_{ij}} = \\ &\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}-\{i\}}\frac{\parallel \mathbf{P}_j(\mathbf{B}_j\mathbf{L}_{ji}+\mathbf{H}_{ji})\parallel^2\parallel \mathbf{x}_i\parallel^2}{\delta_{ji}} \qquad (13)\end{aligned}$$

(12) can be written as

$$\begin{aligned}&\dot{V}(\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_N) \le \sum_{i\in\mathcal{N}}\parallel \mathbf{x}_i\parallel^2 \times \\ &\left[\lambda_{min}(\mathbf{Q}_i) - \sum_{j\in\mathcal{N}-\{i\}}\delta_{ij} - \sum_{j\in\mathcal{N}-\{i\}}\frac{\parallel \mathbf{P}_j(\mathbf{B}_j\mathbf{L}_{ji}+\mathbf{H}_{ji})\parallel^2}{\delta_{ji}}\right] \\ &\qquad (14)\end{aligned}$$

and $\dot{V} < 0$ will be forced, if we have

$$\sum_{j\in\mathcal{N}-\{i\}}\frac{\parallel \mathbf{P}_j(\mathbf{B}_j\mathbf{L}_{ji}+\mathbf{H}_{ji})\parallel^2}{\delta_{ji}} + \sum_{j\in\mathcal{N}-\{i\}}\delta_{ij} < \lambda_{min}(\mathbf{Q}_i) \qquad (15)$$

for all $i \in \mathcal{N}$. This is a sufficient condition that guarantees asymptotic stability of equilibrium point of (1). ■

Remark 1: To satisfy (15), we should try to minimize Euclidian norm on the left side of this equation, by designing $\mathbf{L}_{ji}$. This suggests that our aim is to *decouple* each link $\mathbf{H}_{ji}$ using the link $\mathbf{L}_{ji}$, as much as possible (in Euclidian norm sense). Note that if the matching condition $\mathbf{B}_j\mathbf{L}_{ji}+\mathbf{H}_{ji} = \mathbf{0}$ can be satisfied, the link is completely decoupled.

Remark 2: One trivial case is when $\mathbf{H}_{ji} = \mathbf{0}$. In this case by choosing $\mathbf{L}_{ji} = \mathbf{0}$ the matching condition will be satisfied. Note that $\mathbf{H}_{ji} = \mathbf{0}$ means that there is no coupling dynamics link. Thus, we do not need the corresponding communication link. This means that the communications network will always be a subset of the dynamics network. By defining $\mathcal{N}_i \subset \mathcal{N} - \{i\}$ as the set of neighbors that affect subsystem i ($\mathbf{H}_{ij} \ne \mathbf{0}$, $j \in \mathcal{N}_i$) and $\mathcal{N}'_i \subset \mathcal{N} - \{i\}$ as the set of neighbors that are affected by subsystem i ($\mathbf{H}_{ji} \ne \mathbf{0}$, $j \in \mathcal{N}'_i$), we can reformulate the general stability condition (15) as

$$\sum_{j\in\mathcal{N}'_i}\frac{\parallel \mathbf{P}_j(\mathbf{B}_j\mathbf{L}_{ji}+\mathbf{H}_{ji})\parallel^2}{\delta_{ji}} + \sum_{j\in\mathcal{N}_i}\delta_{ij} < \lambda_{min}(\mathbf{Q}_i) \quad (16)$$

Note that in general, $\mathcal{N}_i \ne \mathcal{N}'_i$, which means that neighborhoods is not necessarily a symmetric relation.

Remark 3: If it is possible to choose the decoupling gains $\mathbf{L}_{ji}$ to satisfy the matching condition $\mathbf{B}_j\mathbf{L}_{ji} = -\mathbf{H}_{ji}$ for all links, our general stability condition (16) reduces to

$$\sum_{j\in\mathcal{N}_i}\delta_{ij} < \lambda_{min}(\mathbf{Q}_i) \qquad (17)$$

Subsequently, we can choose all δ_{ij} equal to δ_i and find the δ_i for each subsystem as

$$\delta_i < \frac{\lambda_{min}(\mathbf{Q}_i)}{\mid \mathcal{N}_i \mid} \qquad (18)$$

which means that a communications network identical to the dynamics network will stabilize the system, though it may not be minimal.

IV. Minimal Communications Network Design

In this section we design a communications network with minimum number of necessary links that satisfies general stability condition (16). Assuming that the communication cost for each link is the same, the problem reduces to one of minimizing the number of links.

One way to interpret (16) is choosing control gains $\mathbf{K}_i$ such that the decoupled subsystem (2) has desired closed loop eigenvalues or use linear optimal control (LQR) with specific cost function then, solve Lyapunov equation (3) with $\mathbf{Q}_i = \mathbf{I}$ ($\lambda_{min}(\mathbf{Q}_i) = 1$) where $\mathbf{I} \in \mathbb{R}^{n_i\times n_i}$ is the identity matrix. This gives the largest convergence rate estimate for each subsystem i (Ch. 3 in [12]). By defining coupling coefficients as

$$c_{ji} \triangleq \parallel \mathbf{P}_j(\mathbf{B}_j\mathbf{L}_{ji}+\mathbf{H}_{ji})\parallel^2 \qquad (19)$$

general stability condition (16) reduces to

$$\sum_{j\in\mathcal{N}'_i}\frac{c_{ji}}{\delta_{ji}} + \sum_{j\in\mathcal{N}_i}\delta_{ij} < 1 \qquad (20)$$

note that the coupling coefficients c_{ji} are functions of decoupled gains $\mathbf{L}_{ji}$. Then, if we set $\delta_{ji} = \sqrt{c_{ji}}$, (20) is reduced to

$$\sum_{j\in\mathcal{N}'_i}\sqrt{c_{ji}} + \sum_{j\in\mathcal{N}_i}\sqrt{c_{ij}} < 1 \qquad (21)$$

Define

$$c^o_{ji} \triangleq \parallel \mathbf{P}_j\mathbf{H}_{ji}\parallel^2 \qquad (22)$$

and

$$c^c_{ji} \triangleq \min_{\mathbf{L}_{ji}} \; c_{ji} = \parallel \mathbf{P}_j(\mathbf{B}_j\mathbf{L}_{ji}+\mathbf{H}_{ji})\parallel^2 \qquad (23)$$

Coupling coefficient with superscript "o" represents the case where we do not use a communication link from subsystem j to i and coupling coefficient with superscript "c" represents the case where we do use a communication link from subsystem j to subsystem i, that minimizes c_{ji}. The minimization problem (23) is a matrix spectral norm (maximum singular value) minimization problem. To solve it, let us define matrix $\mathbf{M}(l_{mn,ji}) \triangleq \mathbf{P}_j(\mathbf{B}_j\mathbf{L}_{ji} + \mathbf{H}_{ji})$ where $\mathbf{M}(\cdot) \in \mathbb{R}^{n_j \times n_i}$ and $l_{mn,ji}$ are elements of matrix $\mathbf{L}_{ji}$. Let us write matrix $\mathbf{B}_j$ as its column,

$$\mathbf{B}_j^{n_j \times m_j} = \begin{bmatrix} \mathbf{b}_{1,j}^{n_j \times 1} & \mathbf{b}_{2,j}^{n_j \times 1} & \cdots & \mathbf{b}_{m_j,j}^{n_j \times 1} \end{bmatrix} \tag{24}$$

and the matrix $\mathbf{L}_{ji}$ as its elements, $l_{mn,ji}$,

$$\mathbf{L}_{ji}^{m_j \times n_i} = \begin{bmatrix} l_{11,ji} & \cdots & l_{1n_i,ji} \\ \vdots & \ddots & \vdots \\ l_{m_j 1,ji} & \cdots & l_{m_j n_i,ji} \end{bmatrix} \tag{25}$$

then, we can write $\mathbf{M}(l_{mn,ji})$ as

$$\mathbf{M}(l_{mn,ji}) = \mathbf{M}_{0,ji} + \sum_{m=1}^{m_j}\sum_{n=1}^{n_j} l_{mn,ji}\mathbf{M}_{mn,ji} \tag{26}$$

where

$$\begin{aligned}
\mathbf{M}_{0,ji}^{n_j \times n_i} &= \mathbf{P}_j^{n_j \times n_j}\mathbf{H}_{ji}^{n_j \times n_i} \\
\mathbf{M}_{11,ji}^{n_j \times n_i} &= \mathbf{P}_j^{n_j \times n_j}\begin{bmatrix} \mathbf{b}_{1,j}^{n_j \times 1} & \mathbf{0}^{n_j \times 1} & \cdots & \mathbf{0}^{n_j \times 1} \end{bmatrix}^{n_j \times n_i} \\
\mathbf{M}_{12,ji}^{n_j \times n_i} &= \mathbf{P}_j^{n_j \times n_j}\begin{bmatrix} \mathbf{0}^{n_j \times 1} & \mathbf{b}_{1,j}^{n_j \times 1} & \cdots & \mathbf{0}^{n_j \times 1} \end{bmatrix}^{n_j \times n_i} \\
&\vdots \\
\mathbf{M}_{1n_i,ji}^{n_j \times n_i} &= \mathbf{P}_j^{n_j \times n_j}\begin{bmatrix} \mathbf{0}^{n_j \times 1} & \mathbf{0}^{n_j \times 1} & \cdots & \mathbf{b}_{1,j}^{n_j \times 1} \end{bmatrix}^{n_j \times n_i} \\
&\vdots \\
\mathbf{M}_{m_j n_i,ji}^{n_j \times n_i} &= \mathbf{P}_j^{n_j \times n_j}\begin{bmatrix} \mathbf{0}^{n_j \times 1} & \mathbf{0}^{n_j \times 1} & \cdots & \mathbf{b}_{m_j,j}^{n_j \times 1} \end{bmatrix}^{n_j \times n_i}
\end{aligned} \tag{27}$$

since $\| \mathbf{M}(l_{mn,ji}) \|$ is an affine function of $l_{mn,ji}$ (26), this is a convex optimization problem. Using the fact that $\| \mathbf{M}(\cdot) \| < s$ if and only if $\mathbf{M}(\cdot)^T\mathbf{M}(\cdot) < s^2\mathbf{I}$ (and $s \geq 0$), we can express the problem in the form

$$\begin{aligned} &\min \quad s \\ &\text{subject to} \quad \mathbf{M}^T(l_{mn,ji})\mathbf{M}(l_{mn,ji}) \leq s^2\mathbf{I} \end{aligned} \tag{28}$$

with variables $l_{mn,ji}$ and s.

We can also formulate the problem using a single linear matrix inequality of size $(n_j+n_i) \times (n_j+n_i)$ using the fact that

$$\mathbf{M}^T\mathbf{M} \leq t^2\mathbf{I} \text{ (and } t \geq 0) \Leftrightarrow \begin{bmatrix} t\mathbf{I} & \mathbf{M} \\ \mathbf{M}^T & t\mathbf{I} \end{bmatrix} \geq \mathbf{0} \tag{29}$$

this results in the semi-definite program (SDP)

$$\begin{aligned} &\min \quad t \\ &\text{subject to} \quad \begin{bmatrix} t\mathbf{I} & \mathbf{M}(l_{mn,ji}) \\ \mathbf{M}^T(l_{mn,ji}) & t\mathbf{I} \end{bmatrix} \geq \mathbf{0} \end{aligned} \tag{30}$$

with variables $l_{mn,ji}$ and t (see Ch. 4 in [13]).

By rearranging the constraint in (30) as

$$\begin{aligned}
\mathbf{F}(l_{mn,ji}, t) \triangleq &\begin{bmatrix} t\mathbf{I} & \mathbf{M}(l_{mn,ji}) \\ \mathbf{M}^T(l_{mn,ji}) & t\mathbf{I} \end{bmatrix} \\
&= t\mathbf{I} + \begin{bmatrix} \mathbf{0} & \mathbf{M}_{0,ji} \\ \mathbf{M}_{0,ji}^T & \mathbf{0} \end{bmatrix} \\
&+ \sum_{m=1}^{m_j}\sum_{n=1}^{n_j} l_{mn,ji}\begin{bmatrix} \mathbf{0} & \mathbf{M}_{mn,ji} \\ \mathbf{M}_{mn,ji}^T & \mathbf{0} \end{bmatrix}
\end{aligned} \tag{31}$$

we can transform (30) to the following eigenvalue problem (EVP)

$$\begin{aligned} &\min \quad t \\ &\text{subject to} \quad \mathbf{F}(l_{mn,ji}, t) \geq \mathbf{0} \end{aligned} \tag{32}$$

where $\mathbf{F}(\cdot,\cdot) \in \mathbb{R}^{(n_j+n_i)\times(n_j+n_i)}$ is affine. This is a standard and tractable feasibility problem. One simple and efficient approaches to solve (32) is the ellipsoid algorithm which converges in polynomial-time (see Ch. 2 in [14]).

Now, given c_{ji}^o and c_{ji}^c, we are looking to find the minimum number of communication links such that general stability condition (21) holds. In other words, to design a communications network, we need to know which links are necessary to satisfy equation (21). To find a solution for this problem, first we define binary parameter $\alpha_{ji} \in \{0, 1\}$. Then (21) can be written as

$$\begin{aligned} &\sum_{j\in\mathcal{N}_i'}[\alpha_{ji}\sqrt{c_{ji}^o} + (1-\alpha_{ji})\sqrt{c_{ji}^c}] + \\ &\sum_{j\in\mathcal{N}_i}[\alpha_{ij}\sqrt{c_{ij}^o} + (1-\alpha_{ij})\sqrt{c_{ij}^c}] < 1 \end{aligned} \tag{33}$$

If $\alpha_{ji} = 1$ we only have coupling coefficients with superscript "o" in (33) which implies that we do not need a communication link from subsystem j to i and if $\alpha_{ji} = 0$ we have just coupling coefficients with superscript "c" in (33) which suggests that we need a communication link from subsystem j to subsystem i. By defining $\Delta c_{ji} \triangleq \sqrt{c_{ji}^o} - \sqrt{c_{ji}^c} \geq 0$ we can rewrite general stability condition (21) as

$$\sum_{j\in\mathcal{N}_i'}\Delta c_{ji}\alpha_{ji} + \sum_{j\in\mathcal{N}_i}\Delta c_{ij}\alpha_{ij} < 1 - \sum_{j\in\mathcal{N}_i'}\sqrt{c_{ji}^c} - \sum_{j\in\mathcal{N}_i}\sqrt{c_{ij}^c} \tag{34}$$

for all $i \in \mathcal{N}$. Minimizing the number of communication links is equivalent to maximizing the number of $\alpha_{ji} = 1$, or in other words, maximizing the summation of all α_{ji}. For each $i \in \mathcal{N}$ the number of α_{ji} in each inequality (34) is

$$\sum_{j\in\mathcal{N}_i'}\alpha_{ji} + \sum_{j\in\mathcal{N}_i}\alpha_{ij} \tag{35}$$

and for all subsystems we have

$$\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}_i'}\alpha_{ji} + \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}_i}\alpha_{ij} = 2\sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}_i'}\alpha_{ji} \tag{36}$$

Finally, our problem reduces to the following binary program

$$\begin{aligned} \max \quad & \sum_{i\in\mathcal{N}}\sum_{j\in\mathcal{N}_i'} \alpha_{ji} \\ \text{subject to} \quad & \sum_{j\in\mathcal{N}_i'} \Delta c_{ji}\alpha_{ji} + \sum_{j\in\mathcal{N}_i} \Delta c_{ij}\alpha_{ij} < \\ & 1 - \sum_{j\in\mathcal{N}_i'} \sqrt{c_{ji}^c} - \sum_{j\in\mathcal{N}_i} \sqrt{c_{ij}^c} \end{aligned} \tag{37}$$

To solve problem (37), we use the branch and bound technique which is based on dividing the problem into a number of smaller problems. In the worst case, we have a complete binary tree to depth n where n is the number of binary parameter α_{ji}, which requires 2^n iterations. When the tree is large, one can also use sub-optimal methods (see Ch. 12 in [15]).

V. Numerical Results

This section presents numerical results demonstrating distributed networked control systems under minimum communication link. The system under study is a collection of three subsystems that are coupled together with the following interpretation

$$\begin{aligned} \mathbf{A}_1 = \mathbf{A}_3 &= \begin{bmatrix} 0 & 1 \\ \frac{15}{4} & 0 \end{bmatrix} \\ \mathbf{A}_2 &= \begin{bmatrix} 0 & 1 \\ \frac{10}{4} & 0 \end{bmatrix} \\ \mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}_3 &= \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} \end{aligned} \tag{38}$$

$\mathbf{H}_{ij}$, are given by

$$\begin{aligned} \mathbf{H}_{12} = \mathbf{H}_{21} &= \begin{bmatrix} 0 & \frac{1}{6} \\ \frac{5}{6} & 0 \end{bmatrix} \\ \mathbf{H}_{23} = \mathbf{H}_{32} &= \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{5}{3} & 0 \end{bmatrix} \\ \mathbf{H}_{13} = \mathbf{H}_{31} &= \mathbf{0} \end{aligned} \tag{39}$$

The control gains $\mathbf{K}_i$ are chosen to obtain the decoupled subsystems' poles at -1 and -2. This results in

$$\begin{aligned} \mathbf{K}_1 = \mathbf{K}_3 &= \begin{bmatrix} -23 & -12 \end{bmatrix} \\ \mathbf{K}_2 &= \begin{bmatrix} -18 & -12 \end{bmatrix} \end{aligned} \tag{40}$$

Now we solve the Lyapunov equation (3): Setting $\mathbf{Q}_i = \mathbf{I}$ where $\mathbf{I}$ is 2×2 identity matrix, for all i we get

$$\mathbf{P}_i = \frac{1}{4}\begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix} \tag{41}$$

The matching condition cannot be satisfied since there is no

$$\mathbf{L}_{ji} = \begin{bmatrix} l_1 & l_2 \end{bmatrix} \tag{42}$$

such that $\mathbf{B}_j\mathbf{L}_{ji} + \mathbf{H}_{ji} = \mathbf{0}$. Based on the coupling structure, this is a symmetric network with

$$\begin{aligned} \mathcal{N}_1 &= \mathcal{N}_1' = \{2\} \\ \mathcal{N}_2 &= \mathcal{N}_2' = \{1, 3\} \\ \mathcal{N}_3 &= \mathcal{N}_3' = \{2\} \end{aligned} \tag{43}$$

The coupling coefficients without communication links are

$$\begin{aligned} c_{12}^o &= c_{21}^o = 0.12207 \\ c_{23}^o &= c_{32}^o = 0.48827 \end{aligned} \tag{44}$$

To find the coupling coefficients with communication links, we should solve matrix Euclidian norm minimization (23) to find $\mathbf{L}_{ji}$. Since in this example $\mathbf{P}_j(\mathbf{B}_j\mathbf{L}_{ji} + \mathbf{H}_{ji})$ is 2×2 matrix, we can find the solution for this matrix Euclidian norm minimization analytically, by taking partial derivative of

$$\lambda_{max}[(\mathbf{B}_j\mathbf{L}_{ji} + \mathbf{H}_{ji})^T\mathbf{P}_j^2(\mathbf{B}_j\mathbf{L}_{ji} + \mathbf{H}_{ji})] \tag{45}$$

respect to l_1 and l_2, and setting them equal to zero, which yields

$$\begin{aligned} \mathbf{L}_{12} = \mathbf{L}_{21} &= \begin{bmatrix} \frac{-10}{3} & -2 \end{bmatrix} \\ \mathbf{L}_{23} = \mathbf{L}_{32} &= \begin{bmatrix} \frac{-20}{3} & -4 \end{bmatrix} \end{aligned} \tag{46}$$

and

$$\begin{aligned} c_{12}^c &= c_{21}^c = \frac{1}{72} \\ c_{23}^c &= c_{32}^c = \frac{1}{18} \end{aligned} \tag{47}$$

To find the minimum necessary links, we solve the following binary program

$$\begin{aligned} \max \quad & \alpha_{12} + \alpha_{21} + \alpha_{23} + \alpha_{32} \\ \text{subject to} \quad & \alpha_{12} + \alpha_{21} < 3.3011 \\ & \alpha_{12} + \alpha_{21} + 2\alpha_{23} + 2\alpha_{32} < 1.2650 \\ & \alpha_{23} + \alpha_{32} < 1.1415 \end{aligned} \tag{48}$$

The solution to above binary program is either $\alpha_{23} = \alpha_{32} = \alpha_{12} = 0$ and $\alpha_{21} = 1$ or $\alpha_{23} = \alpha_{32} = \alpha_{21} = 0$ and $\alpha_{12} = 1$. This means that we need three links: $\mathbf{L}_{23}$, $\mathbf{L}_{32}$ and one of $\mathbf{L}_{12}$ or $\mathbf{L}_{21}$, with values given in (46).

VI. Conclusion

We have presented a general stability condition of distributed networked control systems including multiple coupled LTI subsystems under minimum number of necessary communication links from immediate neighbors. First using the Lyapunov direct method we have found a general stability condition that guarantees the asymptotic stability of the entire networked control system. We showed that in the case where matching condition cannot be satisfied, designing communications network is challenging as there is no analytical solution for matrix spectral norm minimization. We thus formulate it as a convex optimization problem into standard LMI format, which reduces to a standard SDP, and use numerical algorithm to minimize the coupling coefficients that is the key step of designing communications network. Finally, we formulate the communications network design problem with minimum number of communication links as a binary program.

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