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Triple Mode MPC or Laguerre MPC : a comparison

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Abstract— **This paper reviews triple mode predictive control for linear time invariant systems and considers the analogies with new approaches to conventional dual mode MPC algorithms deploying Laguerre polynomials. It is shown that there are strong analogies and moreover, that using the Laguerre insights within a Triple mode approach may significantly enlarge the feasibility region compared to recently proposed triple mode approaches. The improvements, with respect to an existing Triple mode algorithm, are demonstrated by examples.**

I. INTRODUCTION

Linear model predictive control (MPC) [2], [11], [12] is well established and widely used both in industry and academia, but there are still some theoretical and practical issues which have non-satisfactory answers. For instance, one well understood conflict is how to obtain a large feasible region, that is the operating region within which the closed loop input, output and state do not violate constraints, and at the same time retain optimum performance. The conundrum is that algorithms giving large feasibility regions often give suboptimal performance and vice versa. A simple example of this trade off is the observation that detuning a control law will typically result in smaller input variations but consequently inputs are less aggressive and thus likely to violate constraints.

Standard guidelines to ensure guarantees of feasibility and/or stability are commonly accepted, that is, many authors use the dual mode predictions paradigm [18] in conjunction with an infinite horizon. Within this paradigm and assuming therefore the use of infinite output horizons, the deployment of a terminal control law tuned to give high performance will often result in relatively small feasible region [10] unless one uses prohibitively large number of decision variables (or degrees of freedom, d.o.f.). There is a pragmatic limit to the number of d.o.f. for the global feasible region as this compromises the computational burden. A strategy with the same number of d.o.f. giving good feasibility will be achieved through detuning of terminal mode but may have relatively poor performance [13].

The designer has to get balance between the feasibility, computational load (implied by n_c) and the implied performance (affected by K and n_c). There are currently no systematic tools for achieving this balance. Authors [1], [14] have therefore looked at ways of maximizing the feasibility without sacrificing too much performance and while utilizing a computational inexpensive optimization. However, unsurprisingly, there is a hard limit on what can be achieved in this trade off when in essence, for a fixed n_c there is only one variable to play with, that is, the terminal control law K. Moreover changing K will change the shape as well as the volume of the feasible region and it can be hard to make precise judgements as to what is better.

One suggestion that is still relatively underexplored in the literature is the concept of triple mode control [15]. In this strategy one recognises that large feasible regions in conjunction with good performance often imply nonlinear or time varying (LTV) prediction dynamics [19]. In fact one could argue that the optimal law is piecewise affine, but that introduces directional dependence which is a further complication this paper wishes to avoid. Hence, a sensible objective is to find a suitable and fixed LTV control law which enlarges feasibility without too much detriment to performance.

The first triple mode controller [15] used the algorithm of [9] to specify the additional mode of the MPC control law. In [9] ellipsoidal feasible invariant sets were computed for a conventional dual mode MPC setup and the implied LTV law was extracted from these. Recently, the extension of these results in [3], [6] was used in [7] to specify a more flexible triple mode algorithm, but still for the nominal case. However, as the algorithm in [9], [3], [6] were originally developed for the robust case, later work [5] proposed a robust triple mode MPC algorithm; this is the base algorithm that will be used for comparisons in this paper.

Hereon the paper makes the assumption that the terminal mode is well tuned and considers how one can improve feasibility through a fixed, possibly LTV, control law for mode 2 dynamics. Specifically, it the intention is to consider the potential benefits of Laguerre based approaches that have been deployed recently within dual mode MPC [17] because it is known that in many cases changing the parametrization allows substantial improvements in feasibility with little or no detriment to performance. A dual mode MPC algorithm allows d.o.f. for the first n_c control movies and then assumes some fixed (terminal) control law thereafter. This paper first demonstrates that there are strong analogies between the dual mode Laguerre algorithms and the mode 2 dynamics proposed within existing Triple mode strategies, consequently it is worth investigating to what extent a Laguerre approach may be an effective alternative.

The main contributions of this paper are twofold: Firstly to show the analogies between Triple mode strategies and the recent work on Laguerre Optimal MPC. Secondly, using the insight gained from this, to give a new avenue to the use of

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Laguerre polynomials to obtain large feasible regions without sacrificing too much local optimality. Section II will give the necessary background about dual mode MPC, conflicts for optimal MPC, Laguerre polynomials and Triple mode MPC. Section III gives essential detail on existing Triple mode MPC algorithms based on ellipsoidal sets from which a novel Laguerre Triple mode MPC algorithms is developed in section IV. Analytical comparisons between existing Triple mode and the new proposed algorithm are given in section V followed by some numerical examples in section VI. Finally conclusions and future work are in section VII.

Remark 1.1: It is straightforward to show, with conventional arguments, that all algorithms have guaranteed stability and recursive feasibility in nominal case.

II. Background

This section summarises a conventional dual mode MPC algorithm followed by a simple triple mode variant for the LTI case. Both are formulated using polyhedral feasible invariant sets, and hence online optimization is based on quadratic programming (QP).

A. Dual mode or Optimal MPC

Consider a discrete-time state-space model of the form:

$$x_{k+1} = Ax_k + Bu_k; \qquad x_k \in R^{n_x}, \;\; u_k \in R^{n_u}. \quad (1)$$

Assumes the following inputs and states constraints:

$$\underline{\Delta u} \le \Delta u_k \le \overline{\Delta u}, \quad \underline{u} \le u_k \le \overline{u}, \quad \underline{x} \le x_k \le \overline{x} \quad (2)$$

The performance index [18] to be minimized, at each sample instant, with respect to $u_k, u_{k+1}...$ is

$$J = \sum_{i=0}^{\infty} (x_{k+i+1})^T Q (x_{k+i+1}) + (u_{k+i})^T R (u_{k+i})$$
$$s.t. \begin{cases} (1),(2) & \forall k \ge 0, \\ u_k = -Kx_k & \forall k \ge n_c \end{cases} \quad (3)$$

with Q and R positive definite state and input cost weighting matrices. Where K is the optimal feedback gain minimizing J in the absence of constraints (2). Practical limitations imply that only a finite number, that is n_c, of free control moves can be used [12]. For these cases, $u_k = -Kx_k$ is implemented [6] by composing that the state n_c must be contained in a polytopic control invariant set (that is MAS):

$$S_0 = \{x_0 \in \mathbb{R}^{n_x} : \underline{x} \le x_k \le \overline{x},$$
$$\underline{u} \le -Kx_k \le \overline{u}, x_{k+1} = Ax_k + Bu_k, \forall k \ge 0\} \quad (4)$$

For simplicity of notation, the MAS is described in the form $S_0 = \{x : Mx \le b\}$ for suitable M and b and the d.o.f. can be reformulated in terms of a new variable c_k:

$$u_k = -Kx_k + c_k, \qquad k = 0, ..., n_c - 1,$$
$$u_k = -Kx_k, \qquad k \ge n_c, \quad (5)$$

The MCAS (maximal controlled admissible set) is given as

$$S_D = \{x_k : \exists\, C \,, Mx_k + NC \;\le\; b\} \quad (6)$$

where $C = [c_k^T, \ldots, c_{k+n_c-1}^T]^T$ and hence the equivalent optimization to (3) is:

$$\min_{C} \; C^T SC \; s.t. \; Mx_k + NC \le b; \quad (7)$$

Details of how to compute positive definite matrix S, matrices N, M and vector b are omitted as by now well known in the literature [4], [11], [12]. The optimal MPC (OMPC) algorithm is given by solving the QP optimization (7) at every sampling instant then implementing the first component of C, that is c_k in the control law of (5). When the unconstrained control law is not predicted to violate constraints (i.e. $x_k \in S_0$), the optimizing C is zero so the control law is $u_k = -Kx_k$. The optimization of (7) can require a large n_c d.o.f. to obtain both good performance and a large feasible region.

B. Conflicts for Optimal MPC

The major conflict is between maximizing the feasibility and the achievable closed loop performance.

- If n_c is large enough [18], the MCAS is the largest feasible space possible and moreover the control law is the global optimum.
- In general, for computational (and sometime robustness) reasons, n_c is chosen small.
- If n_c is small, then the volume of the MCAS maybe dominated by the implied state feedback K within (5), hence a highly tuned K could give rise to small MCAS and a lesser tuned K could give much larger feasible regions.
- Conversely, if K is poorly tuned, then the cost function could be dominated by poorly performing predictions and hence the closed loop control will be suboptimal.

The designer has to get a balance between the feasibility, computational load and the implied performance.

C. Laguerre OMPC(LOMPC)

A fundamental weakness of the OMPC algorithm when K is well tuned is relatively poor feasibility, that is small MCAS, when n_c is small. The weakness can be overcome by increasing the n_c to allow more steps for reaching the MAS, but obviously this is at the expense of an increased computational burden. Another way of increasing the feasibility is by detuning the terminal law K which may compromise performance. However, an alternative highlighted in [17] is to parameterise the d.o.f. differently so that the impact of the perturbation sequence C on the input predictions (5) is over a longer horizon, thus relaxing the time requirement for entering MAS. This section summarises an algorithm which uses Laguerre polynomials for this parameterisation; readers may be interested that alternatives do exist but constitute ongoing work.

1) Laguerre Polynomials: Laguerre polynomials are defined as follows:

$$L_i(z) = \sqrt{(1-a^2)} \frac{(z^{-1} - a)^{i-1}}{(1 - az^{-1})^i}; \;\; 0 \le a < 1 \quad (8)$$

These are orthonormal with time constant of a and hence span the input prediction space effectively. Laguerre polynomials allow the perturbation signals c_k to evolve over a slower time scale than single perturbations as in the conventional MPC algorithm, consequently the associated feasible region or MCAS can be bigger [16], [17]. The Laguerre sequences can be computed using the following state-space model.

$$L(k+1) = \underbrace{\begin{pmatrix} a & 0 & 0 & 0 & \dots \\ \beta & a & 0 & 0 & \dots \\ -a\beta & \beta & a & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}}_{A_L} L(k); \qquad (9)$$

$$L(0) = \sqrt{1-a^2}[1, -a, a^2, \dots]^T; \; \beta = 1 - a^2$$

where $L_i(z) = e_i^T[L(0), L(1), L(2), \dots][1, z^{-1}, z^{-2}, \dots]^T$ and e_i^T is the i-th standard basis vector. Note that if $a = 0$ than the matrix A_L is a shift matrix, that is ones on the lower diagonal. The dimension of the state-space predictor (9) can be taken as large (or small) as needed to capture the desired polynomial sequences. A combination of these sequences could be computed as:

$$C = \begin{pmatrix} c_k \\ \vdots \\ c_{k+n_c-1} \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} L(0)^T \\ L(1)^T \\ \vdots \end{pmatrix} \eta = H_L \eta \qquad (10)$$

where η is the n_L dimension decision variable when one uses the first n_L column of H_L.

2) LOMPC: Laguerre polynomial and MPC: Laguerre OMPC (LOMPC) is a dual mode MPC algorithm [17] where the input perturbations c_k are parameterised in term of Laguerre polynomials (10). First define the input perturbation sequence C as in (10) as opposed to the finite form of OMPC. The key difference here is that the H_L matrix has a large number of rows, technically infinite (it is better to capture the asymptotic behavior with Lyapunov equations). The prediction cost can be represented in term of η as:

$$J = \eta^T [\sum_{i=0}^{\infty} A_L^i L(0) S L(0)^T (A_L^i)^T] \eta = \eta^T S_L \eta \qquad (11)$$

with $c_{k+i} = L(i)^T \eta$ and $L(i) = A_L L(i-1)$. Constraints represented by the MCAS can also be rewritten in the form

$$M x_k + N H_L \eta \leq l \qquad (12)$$

for appropriate M, N. The LOMPC algorithm is defined by minimising J of (11) w.r.t. η and subject to (12). Define the optimum $c_k = L(0)^T \eta$ and implement the control law in (5).

D. Basic Triple mode MPC algorithm

A different suggestion for overcoming the conflict between performance and feasibility is to allow more complex terminal control laws [15], [19]. For example, a LTV law may be closer to the true piecewise affine (PWA) optimal law, and also allows much larger feasibility regions and moreover gives an exact quadratic cost. So, instead of the dual mode prediction structure of (5), some authors have proposed terminal controls such as:

$$\begin{aligned} u_k &= -Kx_k + c_k, & k &= 0, \dots, n_c - 1 (MODE1), \\ u_k &= -Kx_k + d_{k-n_c}, & k &= n_c, \dots, n_c + n_d - 1 (MODE2), \\ u_k &= -Kx_k, & k &\geq n_c + n_d (MODE3). \end{aligned} \qquad (13)$$

where the notable change is the introduction of terms d_i, $i = 0, \dots, n_d - 1$ and hence the addition of a 3rd mode into the predicted control law. Here, c_i are the only d.o.f. to be optimized online, whereas, ideally, the d_i could be inferred online based on offline or previous optimizations (it is mentioned in [5] that occasionally this needs to be reseeded). For example, in [15], [19], the second model mode control moves are defined as

$$d = [d_0^T, \dots, d_{n_d-1}^T] = H x_{n_c}, \qquad (14)$$

that is, the d_i values depend only on the predicted state at the commencement of mode 2. Then, with trivial algebra (simulating the model (1) with (13) and (14)), one can show that the Mode 2 predictions take the form of LTV feedback,

$$u_k = -K_{k-n_c} x_k, \qquad k = n_c, \dots, n_c + n_d - 1, \qquad (15)$$

where K_i depend on K, H, A and B.

The cost function J for the triple mode predicted feedback structure can be written $J = C^T S C + d^T W d + \acute{p}$ (W defined analogously to S). As $d = H x_{n_c}$ and with $\Phi = A - BK$, $x_{n_c} = \Phi^{n_c} x_0 + [\Phi^{n_c-1} B, \dots, B] C$, we can write

$$J = C^T W_T C + C^T V_T x_0 + p_T, \qquad (16)$$

for suitable W_T, V_T and p_T [15] with constraint of the form

$$M_T x + N_T C \leq b_T \qquad (17)$$

The matrix H is chosen such that it implies, in some sense, a maximal feasible invariant set. The next section will discuss earlier ellipsoidal based algorithms developed for selecting the best H to use in triple mode MPC.

III. Triple mode MPC using Ellipsoidal invariant sets

Early Triple mode algorithms were motivated by the robust case and thus began with the work of [8] and ellipsoidal invariant sets, e.g.:

$$S_E = \{x : x^T P x \leq 1\}; \qquad P > 0 \qquad (18)$$

where $\Phi^T P \Phi \leq P$, $\Phi = A - BK$ and P small enough such that within S_E, constraints (2) are always satisfied with the control law $u = -Kx$. However, ellipsoidal invariant sets are conservative in volume and thus give artificially tight limits on feasibility; points outside the set may still be feasible. Within Triple mode algorithms, the ellipsoidal sets are used as an systematic but interim step to finding a suitable H and are not deployed in the final algorithm.

Dual mode control is so effective because one is able to make implicit assumption the terminal mode and hence only compute the initial mode explicitly using polytopic constraints. Similarly, to form an efficient triple mode algorithm, it is necessary to make implicit assumptions for the terminal mode and mode 2 while selecting the initial Mode explicitly using polytopic constraints.

In order to find an implicit choice for Mode 2, different type of choices based on ellipsoidal sets with only few parameters were proposed in [15] and [9]. The idea of defining an augmented system model incorporating the mode 2, 3 d.o.f. was proposed in [9] to handle the feasibility maximisation offline by optimizing the size of an invariant ellipsoidal subject to constraints. This offline problem is convex and known as an Efficient Robust Predictive Control (ERPC) and an equivalent convex semidefinite programing (SDP) based problem was also proposed in [6] known as generalized ERPC (GERPC). The rest of this section will specify the GERPC offline problem (with ERPC as a special case) for triple mode MPC.

A. ERPC

Define an augmented system model which incorporates the 'd.o.f.' d_k as follows:

$$z_{k+1} = \underbrace{\begin{bmatrix} A-BK & BE \\ 0 & G \end{bmatrix}}_{\Psi} z_k; z_k = \begin{bmatrix} x \\ d \end{bmatrix} \quad (19)$$

where E and G are variables that are used to optimize size and shape of the associated feasible invariant ellipsoid. ERPC used $G = I_L$, where

$$E = [I, 0, \ldots, 0], \quad I_L = \begin{bmatrix} 0 & I & 0 & \ldots & 0 \\ 0 & 0 & I & \ldots & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \ldots & 0 & I \end{bmatrix}.$$

Define control perturbations through dynamics

$$u_k = -Kx_k + Ed_k$$
$$d_{k+1} = Gd_k$$

The existence of an ellipsoidal $\varepsilon_z = \{z : z^T Q_z^{-1} z \leq 1\}$ ensure feasibility if there exist E, G, Q_z and W such that

$$\Psi^T Q_z^{-1} \Psi - Q_z^{-1} < 0,$$
$$\begin{bmatrix} W & [M \quad ND] \\ [M \quad ND]^T & Q_z^{-1} \end{bmatrix} > 0, \; W \leq b^2. \quad (20)$$

The size of the projection of ε_z to the x-space i.e. ε_x which is proportional to the $ln \; det(TQ_zT^T)^{-1}$, where $x = Tz$. Hence maximization of ε_x is obtained by

$$\min_{Q_z, D, G} \; ln \; det(TQ_zT^T)^{-1} \;\; s.t. \; (20) \quad (21)$$

B. GERPC

GERPC improved on ERPC by allowing dynamics in the update of d_k (that is $G \neq I_L$) rather than just the shift operator deployed in ERPC. Hence the augmented model for full triple mode (that is with the initial mode also added) becomes:

$$z_{k+1} = \underbrace{\begin{bmatrix} A-BK & BD & BE \\ 0 & G & 0 \\ 0 & 0 & I_L \end{bmatrix}}_{\Psi} z_k; z_k = \begin{bmatrix} x \\ d \\ C \end{bmatrix} \quad (22)$$

with modified triple mode control law

$$\begin{aligned} u_k &= -Kx_k + Dd_k + c_k, & k &= 0, \ldots, n_c - 1, \\ u_k &= -Kx_k + Dd_k, & k &= n_c, \ldots, n_c + n_d - 1, \\ u_k &= -Kx_k, & k &\geq n_c + n_d. \end{aligned} \quad (23)$$

where c_k is d.o.f. to be optimized online, while d_k is defined from the GERPC offline solution as $d_{k+1} = Gd_k$ (d_k will need to be seeded). Whereas ERPC use a default choice of $d = Hx$ for a suitable H, in fact the initial value of d can be also be considered as a d.o.f., so the total d.o.f. in triple mode could be $n_c + n_d$. Corresponding inequalities to ensure constraint satisfaction take the form:

$$Mx + N_1 d + N_2 c \leq b \quad (24)$$

and the cost $J(x, d, C)$ to be minimised, subject to (24) can be constructed as

$$J(x, d, C) = [x \; d \; C]^T P_T [x \; d \; C], \quad (25)$$

for suitable P_T which is simple to find in the nominal LTI case. The reader should note that there are some minor conditions on P_T in the uncertain case which require SDP solver, but those are outside the remit of this paper. *Algorithm A (Triple mode MPC):*

1) Given design parameters n_c, W and R, calculate D ,Q_z, G and P_T from (21).
2) If $x_0 \in S_0$ implement terminal mode control law. Otherwise,
   a) Solve the optimization problem $min_{C,d} \;\; J(x_k, d_k, C) \;\; s.t. \; to \;\; Mx_k + N_1 d_k + N_2 C \leq b$.
   b) Implement $u_k = -Kx_k + Dd_k + Ec_k$ to the plant.
3) Set $k = k + 1$, go to step 1.

IV. USING LAGUERRE FUNCTIONS IN TRIPLE MODE MPC

The main weakness in triple mode MPC is linked to the efficiency of the middle mode; can this be computed implicitly or explicitly and also is the offline optimisation for identifying a suitable dynamic G overly complex? This section explores a more intuitive technique based on predefined dynamics in the middle mode; for this paper Laguerre polynomials are proposed as these have been shown to be effective elsewhere in improving feasibility without detriment to performance [16], [17]. This section shows how Laguerre functions are analogous to the mode 2 of GERPC based Triple mode and thus can be deployed in the middle mode for a triple mode MPC algorithm. Later sections use examples to demonstrate the benefits.

A. Laguerre triple mode MPC

It was noted earlier (e.g. eqn.(9)) that using Laguerre polynomials, one can define the input perturbations as $c_{k+i} = L(i)^T\eta$ where $L(i) = A_L L(i-1)$. Unpacking this into a different format one gets:

$$\eta_{k+1} = A_L^T \eta_k, \quad d_k = L_0^T \eta_k, \tag{26}$$

$$u_k = -Kx_k + d_k = -Kx_k + L_0^T \eta_k \tag{27}$$

It is clear therefore that this is equivalent to Triple mode where the choices are $G = A_L^T, D = L_0^T$. Consequently, an equivalent augmented GERPC system is:

$$z_{k+1} = \underbrace{\begin{bmatrix} A - BK & BL_0^T & BE \\ 0 & A_L^T & 0 \\ 0 & 0 & I_L \end{bmatrix}}_{\Psi} \begin{bmatrix} x_k \\ \eta_k \\ c_k \end{bmatrix} \tag{28}$$

where $E = [I, 0, \ldots, 0]$. These dynamics should fulfill the constraints given by

$$Mx + N_1 H_L \eta + N_2 C \le b. \tag{29}$$

To ensure the proper synergy with the triple mode MPC and to allow strict statements about recursive feasibility and convergence, we will demonstrate how easily the corresponding cost J can be computed which includes the entire implied dynamic. The predicted cost can be represented in terms of perturbations defined as $d_{k+i} = L(i)^T\eta$ as:

$$J(C,\eta) = C^T S_c C + \eta^T S_\eta \eta, \tag{30}$$
$$s.t. \;\; Mx + N_1 H_L \eta + N_2 C \le b,$$

with

$$S_\eta = H_L^T S_c H_L; \qquad S_c = B^T \Sigma B + R;$$
$$\Sigma - \Phi^T \Sigma \Phi = Q + KRK; \qquad \Phi = A - BK \tag{31}$$

Algorithm B (Laguerre triple mode MPC)
Given design parameters n_c, n_d, Q and R and calculate L_0^T and A_L from (9). Calculate H_L form (10) and finally calculate S_c and S_η from (31).

1) K=0; if $x_0 \in S_0$ implement terminal mode control law i.e. $u = -Kx_0$ else
2) Solve the optimization problem $\min_{C,\eta} \; J(C,\eta) \;\; subject\ to$ $Mx_k + N_1 H_L \eta_k + N_2 C \le b.$
3) Implement $u_k = -Kx_k + L_0^T \eta_k + Ec_k$ to the plant.
4) Set $k = k+1$, go to step 2.

V. Comparison between Triple mode MPC and Triple Laguerre MPC

The previous section has shown that Laguerre polynomials [17] are an alternative to GERPC [5] for generating mode 2 dynamics in triple mode MPC algorithms using polytopic constraints. Triple mode MPC algorithm takes dual mode predictions as a base and increases the region of attraction by adding a third mode (in act what is denoted as mode 2 is the additional mode). The motivation is to improve feasibility without detriment to performance and preferably with little impact on the computational burden.

Triple mode is known to be effective in improving feasibility and without detriment to performance so the key question to discern is whether the strategy is better than just increasing the number of d.o.f. available to a standard OMPC algorithm. Secondly, there is interest in whether the proposed Laguerre approach has benefits over the earlier GERPC based approaches. It should be noted that all cases lead to a quadratic programming (QP) problem - investigations into the structure of these QPs and their exploitation is ongoing work.

In terms of offline computations, the proposed approach is a significant improvement on the GERPC based approach: (i) GERPC requires a challenging SDP in order to determine the dynamic G whereas (ii) the Laguerre algorithm requires only the choice of a where in general a larger a inproves feasibility but slows predicted responses and thus this is an intuitive design parameter and a pragmatic choice. Nevertheless GERPC is more systematic, but because it is based on ellipsoidal rather than polytopic sets, it is unreasonable to make this case too strongly.

In terms of online computations, Triple mode has $n_c + n_d$ d.o.f. which is an increase on OMPC - although it is implicit in choosing Triple mode that OMPC with n_c d.o.f. has too small a feasible region so more d.o.f. are required and the question is more, how best to use these? Should we use a Triple mode approach or add n_d d.o.f. to a conventional OMPC approach? It is already known that Triple is more effective in general and so in this paper focus will given on comparing the efficacy of the GERPC approach and the Laguerre approach.

VI. Numerical Examples

The purpose of this section is to demonstrate by example the differences in GERPC and Laguerre based Triple mode algorithms. Two examples are given and plots of the feasibility regions are given for: (i) MAS; (ii) the ellipsoidal invariant set for augmented system (for information as GERPC is based on this); (iii) the feasible regions for the two Triple mode approaches. For the purposes of visualization, examples are restricted to second order systems for which it is possible to plot regions of attraction.

The region of attraction for examples 1 and 2 are plotted in Fig.1 and Fig.2 respectively. It is clear from both figures that Laguerre triple mode MPC has a larger feasible region than triple mode MPC for the same number of d.o.f.. For completeness it is also noted that these feasible regions are far larger than that for OMPC with a comparable number of d.o.f.. It is self-evident from Table I that Laguerre MPC is expected to give the best dynamic performance.

1) Example 1:

$$A = \begin{bmatrix} 0.6 & -0.4 \\ 1 & 1.4 \end{bmatrix}; \; B = \begin{bmatrix} 0.2 \\ 0.05 \end{bmatrix}; \; C = \begin{bmatrix} 1 & -2.2 \end{bmatrix} \tag{32}$$

with constraints

$$-1.5 \le u_k \le 1.5; \quad -5 \le x_k \le 5; -5 \le y_k \le 5 \tag{33}$$

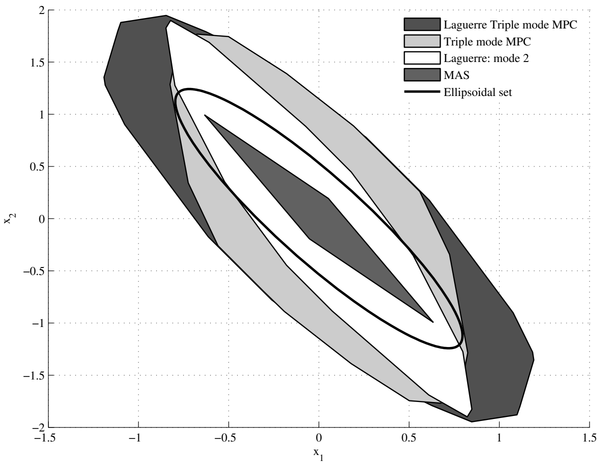

Fig. 1: Comparison MCAS for $n_c = n_d = 2$ - Example 1

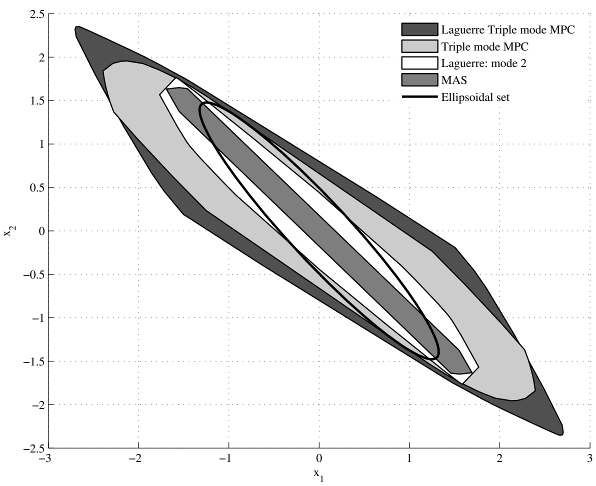

Fig. 2: Comparison MCAS for $n_c = n_d = 2$ - Example 2

Using $Q = I$, $R = 0.5$, $a = 0.8$ and $n_c = 2$, $n_d = 2$.

2) Example 2:

$$A = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix};\ B = \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix};\ C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{34}$$

with constraints (33). Using $Q = C^T C$, $R = 0.1$, $a = 0.8$ and $n_c = 2$, $n_d = 2$.

VII. CONCLUSIONS AND FUTURE WORKS

The main contribution of this paper was to demonstrate analogies between earlier triple mode MPC algorithms and recently published MPC algorithms deploying Laguerre polynomials. Following on from this it was shown how Laguerre polynomials can be embedded within a Triple mode approach. Examples demonstrate that there are many cases where such an approach is an improvement on earlier work and thus this avenue of research is worth pursuing further. However, whereas Laguerre is a pragmatic choice, there is a need to investigate in parallel issues such as: what alternatives are there to Laguerre and what choices lead to a QP structure which lends itself to efficient online optimisation? Finally, this paper has focussed on the LTI case whereas algorithms such as GERPC were originally posed for the robust case and thus there is interest in considering how best to extend results to the robust case.

TABLE I: Runtime costs for example 1 and 2.

Example	x_0	Optimal	Triple Mode MPC	Laguerre MPC
1	0.1,0.05	0.4147	0.4577	0.4304
2	-2.0,1.8	8.0473	8.0473	8.0473

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