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Primal and Dual Stability Criteria for Systems with Time-Varying Gains

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***Abstract*— Primal and dual stability criteria are derived for systems with uncertain or time-varying components that can be characterized using mixed multipliers. The constant part of the multiplier is used to model time-varying components while the frequency varying multiplier is used to model linear time-invariant uncertainties. It is shown that the dual criterion sometimes reduces to easy-to-use criteria that reveal the structure of the problem.**

I. INTRODUCTION

In this paper we present primal and dual stability criteria for analysis of systems consisting of linear time-invariant (LTI) dynamics interconnected over an uncertain network which either is modeled as a time-varying matrix or a LTI dynamics. The foundation of our analysis is to use quadratic relaxation techniques to describe uncertainty. This results in stability criteria that can be formulated as convex feasibility tests involving the nominal system dynamics and the parameters/multipliers used for the relaxation, see e.g. [10], [2], [9]. These primal stability criteria are in general infinite dimensional convex feasibility problems and restriction to a finite dimensional basis is necessary. The dual to these criteria is generally also an infinite dimensional test that must be tested numerically. There are, however, several reasons to introduce the dual. The dual formulation of the stability condition can benefit the user with 1) insight into essential structural properties of the primal criterion 2) tests for infeasibility or lower bounds on the stability margin and 3) alternative formulations that might be easier to use.

We illustrate the primal and dual criteria for a number of simple examples. The traditional application domain is robust control and our example indicate that multiplier based tests sometimes have more explicit and intuitively appealing dual counterparts. Another application domain is the use of multiplier techniques to characterize the network structure in large scale systems. We show by simple examples that the dual criterion can be reduced to simple criteria on the subsystem dynamics provided that appropriate multipliers are used.

The primary contributions of this paper is to generalize our recent primal-dual results in [9] to the case when time-varying gains are used in the network interconnection. The derivation is based on an earlier work, [6], where primal and dual formulations of multiplier optimization was considered. We consider the case when unstable subsystems are stabilized over a network. This leads us to use the integral quadratic constraints to characterize unstable systems that are pathwise connected in the ν-gap metric. The proof of primal stability follows from [8].

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A. Notation and Preliminaries

The real and complex numbers are denoted $\mathbf{R}$ and $\mathbf{C}$, respectively. The complex conjugate of $s \in \mathbf{C}$ is denoted $\bar{s}$ and the complex conjugate transpose of a matrix $M \in \mathbf{C}^{p\times m}$ is defined as $M^* = \bar{M}^T$. The largest and smallest singular values of M are denoted $\sigma_{\max}(M)$ and $\sigma_{\min}(M)$.

- Let X be a normed vector space. The dual of X is the normed space consisting of all bounded linear functionals on X and it is denoted by X^*. If $x \in X$ and $x^* \in X^*$, then $\langle x,\, x^*\rangle$ denotes the (real) value of the linear functional x^* at x. The vector spaces are in this paper defined over the real scalar field.
- The (Cartesian) product of two vector spaces X_1 and X_2 is denoted $X_1 \times X_2$ and it consists of all ordered pairs $x = (x_1, x_2)$, with $x_1 \in X_1$ and $x_2 \in X_2$.
- The dual of $X_1 \times X_2$ is given as $X_1^* \times X_2^*$, where X_1^* and X_2^* are the duals of X_1 and X_2 respectively. Given $x = (x_1, x_2) \in X_1 \times X_2$ and $x^* = (x_1^*, x_2^*) \in X_1^* \times X_2^*$, we define $\langle x,\, x^*\rangle = \langle x_1,\, x_1^*\rangle + \langle x_2,\, x_2^*\rangle$.
- X^N denotes the Cartesian product of N copies of X.
- $\times_{k=1}^N x_k \in X^N$ denotes a N tuple $x = (x_1, \ldots, x_N) \in X^N$.
- Let $H : X \to Y$ be a bounded linear operator. Then the adjoint operator $H^* : Y^* \to X^*$ is defined by the equation
$$\langle Hx,\, y^*\rangle = \langle x,\, H^* y^*\rangle\,,$$
for all $x \in X$ and $y^* \in Y^*$. We sometimes use the alternative notation $H^\times = H^*$.

The normed vector space X will in this paper be a Hilbert space, i.e. a vector space that possess an inner product. In this case $X^* = X$, the linear functional $\langle x,\, x^*\rangle$ is defined by the inner product and the norm is defined as $\|x\| = \langle x,\, x\rangle^{1/2}$.

We let $S_{\mathbf{C}}^{m\times m} = \{X \in \mathbf{C}^{m\times m} : X = X^*\}$ be the Hilbert space of Hermitian matrices equipped with the inner product $\langle X,\, Y\rangle = \mathrm{tr}(XY)$ and the corresponding norm $\|X\| = \mathrm{tr}(X^2)^{1/2}$ (the Frobenius norm). We use the standard notation $X \succ 0$ ($X \succeq 0$) to denote that the matrix $X \in S_{\mathbf{C}}^{m\times m}$ is positive definite (positive semidefinite).

Suppose $K \subset S_{\mathbf{C}}^{m\times m}$ is a convex cone. Then the negative polar cone is the closed convex cone defined as

$$K^\ominus = \left\{Y \in \mathcal{S}_{\mathbf{C}}^{m\times m} : \langle X,\, Y\rangle \le 0; \forall X \in K\right\}.$$

The definition and the properties of $\mathcal{S}_{\mathbf{R}} = \left\{X \in \mathbf{R}^{m\times m} : X = X^T\right\}$ is analogous to that of $S_{\mathbf{C}}^{m\times m}$.

Finally, we will use the convex hull $\mathrm{co}\{w_1,\ldots,w_n\} := \{\sum_{i=1}^n \alpha_i w_i : \alpha_i \geq 0; \sum_{i=1}^n \alpha_i = 1\}$, the convex conic hull $\mathrm{cone}\{w_1,\ldots,w_n\} := \{\sum_{i=1}^n \alpha_i w_i : \alpha_i \geq 0\}$, and the direct sum of matrices $\oplus_{i=1}^n M_i = \mathrm{diag}(M_1,\ldots,M_n)$.

B. Signals and Systems

Let the time axis $\mathbf{T}$ be either $\mathbf{R}_+ = [0,\infty)$, $\mathbf{R} = (-\infty,\infty)$. The space $\mathbf{L}_2^m(\mathbf{T})$ is the Hilbert space of square integrable $\mathbf{R}^m$ valued functions with inner product

$$\langle w,\, v\rangle \stackrel{\text{def}}{=} \langle w,\, v\rangle_{\mathbf{L}_2(\mathbf{T})} = \int_{\mathbf{T}} w(t)^T v(t)dt$$

and norm $\|w\|_{\mathbf{L}_2(\mathbf{T})} = \langle w,\, w\rangle^{1/2}$. By defining $v(t) = 0$, $t \leq 0$ for $v \in \mathbf{L}_2(\mathbf{R}_+)$ we get the useful subset inclusion $\mathbf{L}_2(\mathbf{R}_+) \subset \mathbf{L}_2(\mathbf{R})$.

The corresponding frequency domain spaces are denoted $\mathbf{L}_2^m(j\mathbf{R})$ and $\mathcal{H}_2^m$ and consists of Fourier transforms of signals in $\mathbf{L}_2^m(\mathbf{R})$ and $\mathbf{L}_2^m(\mathbf{R}_+)$, respectively. The time and frequency domain spaces are isometrically isomorphic, i.e. $\|v\|_{\mathbf{L}_2(\mathbf{R})} = \|\hat{v}\|_{\mathbf{L}_2(j\mathbf{R})}$, where $\hat{v}$ denotes the Fourier transform of v. We sometimes suppress the spatial dimension m from the notation. Finally, we let

$$\mathbf{L}_\infty^{m\times m}(j\mathbf{R}) = \{\Pi : j\mathbf{R} \to \mathbf{C}^{m\times m} : \|\Pi\| < \infty\},$$

where $\|\Pi\| = \sup_{\omega\in\mathbf{R}} \sigma_{\max}(\Pi(j\omega))$. We use frequency weighted quadratic forms defined by $\Pi \in \mathbf{L}_\infty^{m\times m}$ as

$$\langle \hat{v},\, \Pi\hat{v}\rangle_{\mathbf{L}_2(j\mathbf{R})} = \int_{-\infty}^{\infty} \hat{v}(j\omega)^*\Pi(j\omega)\hat{v}(j\omega)d\omega.$$

It satisfies the bound $\langle \hat{v},\, \Pi\hat{v}\rangle_{\mathbf{L}_2(j\mathbf{R})} \leq \|\Pi\| \cdot \|\hat{v}\|^2_{\mathbf{L}_2(j\mathbf{R})}$.

We let $\mathcal{A}^{p\times m}(\beta)$ be algebra of transfer functions obtained as the Laplace transforms of the impulse response functions (see [4], [3])

$$h(t) = h_c(t)\theta(t) + \sum_{k=0}^{\infty} h_k\delta(t - t_k),$$

where $e^{-\beta t}h_c(t) \in \mathbf{L}_1^{p\times m}[0,\infty)$, $h_k \in \mathbf{R}^{p\times m}$, $t_0 = 0$, $t_k > 0$, $k \geq 1$, $\sum_{k=0}^{\infty} e^{-\beta t_k}|h_k| < \infty$, and where $\theta(\cdot)$ is the unit step function and $\delta(\cdot)$ is the dirac distribution. To each $H \in \mathcal{A}^{p\times m}$ there is an associated causal time-domain operator $\mathbf{H} : \mathbf{L}_2^m(\mathbf{R}_+) \to \mathbf{L}_2^p(\mathbf{R}_+)$ defined as

$$(\mathbf{H}v)(t) = \int_0^t h_c(t-\tau)v(\tau) + \sum_{k=0}^{t} h_k v(t - t_k)$$

with induced norm $\|\mathbf{H}\| = \sup_{\omega\in\mathbf{R}} \sigma_{\max}(H(j\omega))$. The extension $\mathbf{H} : \mathbf{L}_2^m(\mathbf{R}) \to \mathbf{L}_2^p(\mathbf{R})$ to the doubly infinite time axis is defined in the same way except that the lower bound of the integral is ∞.

We let $\mathcal{A}^{p\times m} = \mathcal{A}^{p\times m}(0)$ and $\mathcal{A}_-^{p\times m} = \{\mathcal{A}^{\times m}(\beta) : \beta < 0\}$. The subsets of constantly proper transfer functions $\mathcal{A}_{cp}^{p\times m}$ and $\mathcal{A}_{cp,-}^{p\times m}$ has $h_k = 0$ for $k \geq 1$ which implies that the transfer functions are continuous at infinity.

The Callier-Desoer class $\mathcal{B}^{p\times m}$ consists of transfer functions where each element belongs to the quotient algebra $\mathcal{A}_-[\mathcal{A}_\infty]$, where $\mathcal{A}_\infty = \{G \in \mathcal{A}_- : \lim_{|s|\to\infty} \sigma_{\min}(G(s)) > 0\}$. Each transfer function from $\mathcal{B}^{p\times m}$ have (see [1], [2]) normalized right and left coprime factorizations $H = UV^{-1} = \widetilde{V}^{-1}\widetilde{U}$, where $U, \widetilde{U} \in \mathcal{A}_-^{p\times m}$, $V \in \mathcal{A}_{-,\infty}^{m\times m}$, $\widetilde{V} \in \mathcal{A}_{-,\infty}^{p\times p}$ are such that

$$\begin{aligned} U(j\omega)^*U(j\omega) + V(j\omega)^*V(j\omega) &= I, \\ \widetilde{U}(j\omega)\widetilde{U}(j\omega)^* + \widetilde{V}(j\omega)\widetilde{V}(j\omega)^* &= I. \end{aligned}$$

To $H \in \mathcal{B}^{p\times m}$ there is an associated causal time-domain operator $\mathbf{H} : \mathrm{dom}(\mathbf{H}) \subset \mathbf{L}_2^m(\mathbf{R}_+) \to \mathbf{L}_2^p(\mathbf{R}_+)$ defined as

$$(\mathbf{H}v)(t) = \int_0^t h_c(t-\tau)v(\tau)d\tau + \sum_{k=0}^{\infty} h_k v(t - t_k),$$

where $h_c(t) + \sum_{k=0}^{\infty} h_k\delta(t - t_k) = \mathcal{L}^{-1}H(s)$ (inverse one-sided Laplace transform) and the explicit expression for the domain is

$$\mathrm{dom}(\mathbf{H}) = \{v : \hat{v} = V\hat{w};\, \hat{w} \in \mathcal{H}_2\}.$$

Note that $\mathbf{H}$ is unbounded outside this domain of definition and is therefore regarded as an unstable system.

In this paper we will make use of a nu-gap distance of the form introduced in [11]. In particular, its generalization and interpretation in the context of time-varying system in [8] is used. For this purpose we restrict attention to the subclass $\mathcal{B}_{cp}^{p\times m}$ for which the ν-gap always is well defined. Each matrix element of $\mathcal{B}_{cp}^{p\times m}$ belongs to the quotient algebra $\mathcal{A}_{cp,-}[\mathcal{A}_{cp,\infty}]$, where $\mathcal{A}_{cp,\infty} = \{G \in \mathcal{A}_{cp,-} : \lim_{|s|\to\infty} \sigma_{\min}(G(s)) > 0\}$. Any two $H_1, H_2 \in \mathcal{B}_{cp}^{p\times m}$ have normalized left and right coprime factorizations $H_k = U_kV_k^{-1} = \widetilde{V}_k^{-1}\widetilde{U}_k$, $k = 1,2$ from which we can define the so-called right and left graph symbols

$$\begin{aligned} G_{H_k} &= \begin{bmatrix} V_k \\ U_k \end{bmatrix} \in \mathcal{A}_{cp,-}^{(p+m)\times m}, \\ \widetilde{G}_{H_k} &= \begin{bmatrix} -\widetilde{U}_k & \widetilde{V}_k \end{bmatrix} \in \mathcal{A}_{cp,-}^{p\times(p\times m)}. \end{aligned}$$

We use the following version of the ν-gap metric

$$\delta_\nu(H_1,H_2) = \begin{cases} \overline{\gamma}(\widetilde{G}_{H_1}G_{H_2}), & \underline{\gamma}(\widetilde{G}_{H_1}\widetilde{G}_{H_2}^*) > 0 \text{ \& } \\ & \mathrm{wno}(\widetilde{G}_{H_1}\widetilde{G}_{H_2}^*) = 0 \\ 1, & \text{otherwise}, \end{cases} \quad (1)$$

where

$$\begin{aligned} \overline{\gamma}(\widetilde{G}_{H_1}G_{H_2}) &= \sup_{\omega\in\mathbf{R}} \sigma_{\max}(\widetilde{G}_{H_1}G_{H_2})(j\omega), \\ \underline{\gamma}(\widetilde{G}_{H_1}\widetilde{G}_{H_2}^*) &= \inf_{\omega\in\mathbf{R}} \sigma_{\min}(\widetilde{G}_{H_1}\widetilde{G}_{H_2}^*)(j\omega), \end{aligned}$$

and where the winding is defined as

$$\mathrm{wno}(G) = \lim_{\omega\to\infty} \frac{\arg(G(j\omega)) - \arg(G(-j\omega))}{2\pi}.$$

Finally, let

$$\mathbf{K} = \{K : \mathbf{R} \to \mathbf{R}^{m\times p} : K(\cdot) \text{ is piecewise continuous and } \sup_{t\in\mathbf{R}} \sigma_{\max}(K(t)) < \infty\}.$$

II. Primal and Dual Stability Criteria

We consider the interconnection $[\mathbf{\Gamma}, \mathbf{H}]$ defined as

$$\begin{aligned} e_1 &= \mathbf{\Gamma} e_2 + r_1, \\ e_2 &= \mathbf{H} e_1 + r_2, \end{aligned} \tag{2}$$

where $\mathbf{H} : \mathrm{dom}(\mathbf{H}) \subset \mathbf{L}_2^m(\mathbf{R}_+) \to \mathbf{L}_2^p(\mathbf{R}_+)$ and $\mathbf{\Gamma} : \mathrm{dom}(\mathbf{\Gamma}) \subset \mathbf{L}_2^p(\mathbf{R}_+) \to \mathbf{L}_2^m(\mathbf{R}_+)$ are linear causal operators. This interconnection is called stable if 1) the mapping $r = (r_1, r_2) \to (e_1, e_2)$ is causal and 2) there exists $c > 0$ such that $\|e\|_{\mathbf{L}_2(\mathbf{R}_+)} \le c\|r\|_{\mathbf{L}_2(\mathbf{R}_+)}$, for all $r \in \mathbf{L}_2(\mathbf{R}_+)$.

The following cases will be considered in this paper:

1) $\mathbf{H}$ is defined by a transfer function from either $\mathcal{A}_{cp}^{p\times m}$ or the Callier Desoer class $\mathcal{B}_{cp}^{p\times m}$ defined above.
2) $\mathbf{\Gamma}$ is either defined by a transfer function from $\mathcal{A}^{m\times p}$ or from $\mathcal{B}^{m\times p}$, or is a time-varying matrix gain, i.e. $(\mathbf{\Gamma} w)(t) = \Gamma(t) w(t)$, where $\Gamma \in \mathbf{K}^{m\times p}$.

Our main analysis criterion is a frequency-wise criterion on the transfer function H that defines $\mathbf{H}$. For this purpose we define the operator $M_H : \mathcal{S}_{\mathbf{C}}^{(p+m)\times(p+m)} \to \mathcal{S}_{\mathbf{C}}^{m\times m}$ and its adjoint $M_H^\times : \mathcal{S}_{\mathbf{C}}^{m\times m} \to \mathcal{S}_{\mathbf{C}}^{(p+m)\times(p+m)}$ as[1]

$$M_H \Pi = G_H^* \Pi G_H, \quad \text{and} \quad M_H^\times Z = G_H Z G_H^*,$$

where $G_H = \begin{bmatrix} V \\ U \end{bmatrix} \in \mathbf{C}^{(p+m)\times m}$ is a matrix that will represent a frequency evaluation of a right (inverse graph) symbol of the transfer function $H(s) = U(s)V^{-1}(s)$.

Our stability criteria will be formulated in terms of integral quadratic constraints (IQC) defined by multipliers. For any $\Pi \in \mathbf{L}_\infty^{(p+m)\times(p+m)}(j\mathbf{R})$ such that $\Pi(j\omega) = \Pi(j\omega)^*$ we say $\mathbf{\Gamma} \in \mathrm{IQC}^c(\Pi)$ if and only if

$$\langle \hat{w}, \Pi \hat{w} \rangle_{\mathbf{L}_2(j\mathbf{R})} \le 0, \quad \forall w \in \mathcal{G}_{\mathbf{\Gamma}},$$

where $\mathcal{G}_{\mathbf{\Gamma}} = \{w = (w_1, w_2) : w_2 \in \mathbf{L}_2^p(\mathbf{R}_+); w_1 = \mathbf{\Gamma} w_2 \in \mathbf{L}_2^m(\mathbf{R}_+)\}$. In this paper, a combination of constant and frequency varying multipliers are used to characterize structural properties of Γ. The multipliers will be defined in terms of a convex cone on the form

$$\begin{aligned} \Pi_\Gamma = \{\Phi + \Psi : \Phi \in \mathbf{L}_\infty^{(p+m)\times(p+m)}(j\mathbf{R}); \\ \Phi(j\omega) \in \Phi_\Gamma,\ \forall \omega \in \mathbf{R};\ \Psi \in \Psi_\Gamma\}, \end{aligned} \tag{3}$$

where $\Phi_\Gamma \subset \mathcal{S}_{\mathbf{C}}^{(p+m)\times(p+m)}$ and $\Psi_\Gamma \subset \mathcal{S}_{\mathbf{R}}^{(p+m)\times(p+m)}$ are closed convex cones.

Assumption 1 (Assumptions under known $\mathbf{\Gamma}$):

(a) there exists a set Π_Γ of multipliers of the form (3), where $\Phi_\Gamma \subset \{\Phi \in \mathcal{S}_{\mathbf{C}}^{(p+m)\times(p+m)} : \Phi_{22} \preceq 0\}$ and $\Psi_\Gamma \subset \{\Psi \in \mathcal{S}_{\mathbf{R}}^{(p+m)\times(p+m)} : \Psi_{22} \preceq 0\}$ are closed convex cones such that $\mathbf{\Gamma} \in \mathrm{IQC}^c(\Pi)$, $\forall \Pi \in \Pi_\Gamma$.

(b) there exists a causal LTI operator $\mathbf{H}_0$ such that $[\mathbf{\Gamma}, \mathbf{H}_0]$ is stable and moreover such that $\mathbf{H}$ and $\mathbf{H}_0$ are defined by transfer functions that has right and left coprime factorizations on the form

$$\begin{aligned} H &= UV^{-1} = \widetilde{V}^{-1}\widetilde{U} \in \mathcal{B}_{cp}^{p\times m} \\ H_0 &= U_0 V^{-1} = \widetilde{V}^{-1}\widetilde{U}_0 \in \mathcal{B}_{cp}^{p\times m}. \end{aligned}$$

Remark 1: The assumption implies that H and H_0 have the same unstable poles. Note that the coprime factorizations do not need to be normalized.

Theorem 1: Under Assumption 1, the system in (2) is stable if either of the following equivalent conditions are satisfied

(a) **Primal condition:** There exists $\Pi \in \Pi_\Gamma$ such that for every $\omega \in \mathbf{R} \cup \{\infty\}$

$$(M_H \Pi)(j\omega) \succ 0 \quad \text{and} \quad (M_{H_0}\Pi)(j\omega) \succ 0. \tag{4}$$

(b) **Dual condition:** For every grid $\Omega_N = \{\omega_1, \ldots, \omega_N\}$ of $N = \dim(\Psi_\Gamma) + 1$ frequencies it hold (if $0 \ne \mathrm{ri}\,\Psi_\Gamma$ then $N = \dim(\Psi_\Gamma)$)

$$\begin{aligned} &(M_H^\times Z_{1,k})(j\omega_k) + (M_{H_0}^\times Z_{2,k})(j\omega_k) \notin \Phi_\Gamma^\ominus,\ \forall \omega_k \in \Omega_N \\ &\sum_{k=1}^N \mathrm{Re}\left[(M_H^\times Z_{1,k})(j\omega_k) + (M_{H_0}^\times Z_{2,k})(j\omega_k)\right] \notin \Psi_\Gamma^\ominus \end{aligned} \tag{5}$$

for all 2N-tuples $Z = \times_{k=1}^N (Z_{1,k}, Z_{2,k}) \in \mathcal{Z}$, where

$$\begin{aligned} \mathcal{Z} = \{Z \in (\mathcal{S}_{\mathbf{C}}^{m\times m} \times \mathcal{S}_{\mathbf{C}}^{m\times m})^N : Z_{1,k}, Z_{2,k} \succeq 0; \\ \sum_{k=1}^N \mathrm{tr}(Z_{1,k}) + \mathrm{tr}(Z_{2,k}) = 1\}. \end{aligned} \tag{6}$$

Proof: A proof can be found in the appendix. ∎

It is often the case that the network interconnection is not exactly specified or known. Assume that $\mathbf{\Gamma} \in \mathbf{S}_{\mathbf{\Gamma}}$, where $\mathbf{S}_{\mathbf{\Gamma}}$ is a set of operators defined either by transfer functions from $\mathcal{S}_\Gamma \subset \mathcal{B}_{cp}^{m\times p}$ or a time-varying matrix gains from $\mathcal{S}_\Gamma \subset \mathbf{K}^{m\times p}$ such that the following assumption holds

Assumption 2 (Assumptions under uncertain $\mathbf{\Gamma} \in \mathbf{S}_{\mathbf{\Gamma}}$):

(a) there exists a set Π_Γ of multipliers of the form (3) where $\Phi_\Gamma \subset \mathcal{S}_{\mathbf{C}}^{(p+m)\times(p+m)}$ and $\Psi_\Gamma \subset \mathcal{S}_{\mathbf{R}}^{(p+m)\times(p+m)}$ are closed convex cones such that every $\mathbf{\Gamma} \in \mathbf{S}_{\mathbf{\Gamma}}$ satisfies $\mathbf{\Gamma} \in \mathrm{IQC}^c(\Pi)$, $\forall \Pi \in \Pi_\Gamma$. Note that the constraints $\Phi_{22} \preceq 0$ and $\Psi_{22} \preceq 0$ are no longer necessary to include.

(b) the set $\mathbf{S}_{\mathbf{\Gamma}}$ is pathwise connected (in the topology defined by the ν-gap distance if $\mathbf{\Gamma}$ is defined by a transfer function from $\mathcal{B}_{cp}^{m\times p}$).

(c) there exists $\mathbf{\Gamma}_0 \in \mathbf{S}_{\mathbf{\Gamma}}$ such that the interconnection $[\mathbf{\Gamma}_0, \mathbf{H}]$ is stable.

Given these assumptions we get the alternative result

Corollary 1: Under Assumption 2, the system in (2) is stable if either of the following equivalent conditions are satisfied

(a) **Primal condition:** There exists $\Pi \in \Pi_\Gamma$ such that for every $\omega \in \mathbf{R} \cup \{\infty\}$

$$(M_H \Pi)(j\omega) \succ 0. \tag{7}$$

[1] If the transfer function H has no poles on the imaginary axis then we can equally well define the operators M_H and $M_H^\times$ as

$$M_H \Pi = \begin{bmatrix} I \\ H \end{bmatrix}^* \Pi \begin{bmatrix} I \\ H \end{bmatrix}, \quad \text{and} \quad M_H^\times Z = \begin{bmatrix} I \\ H \end{bmatrix} Z \begin{bmatrix} I \\ H \end{bmatrix}^*.$$

where $H = H(j\omega)$.

(*b*) **Dual condition:** For every grid $\Omega_N = \{\omega_1, \dots, \omega_N\}$ of $N = \dim(\Psi_\Gamma) + 1$ frequencies it hold (if $0 \neq \mathrm{ri}\,\Psi_\Gamma$ then $N = \dim(\Psi_\Gamma)$)

$$\begin{aligned} &(M_H^\times Z_k)(j\omega_k) \notin \Phi_\Gamma^\ominus,\ \forall \omega_k \in \Omega_N \\ &\sum_{k=1}^{N} \mathrm{Re}\,(M_H^\times Z_k)(j\omega_k) \notin \Psi_\Gamma^\ominus \end{aligned} \tag{8}$$

for all N-tuples $Z = \times_{k=1}^N Z_k \in \mathcal{Z}$, where

$$\mathcal{Z} = \{Z \in (\mathcal{S}_{\mathbf{C}}^{m\times m})^N : Z_k \succeq 0; \sum_{k=1}^{N} \mathrm{tr}(Z_k) = 1\}.$$

III. Applications

In this section we provide some examples illustrating the primal and dual criterion. The purpose is to derive the dual conditions and to show that they often can be reduced to attractive and easy-to-use criteria.

Example 1: Consider the interconnection $[\mathbf{\Gamma}, \mathbf{H}]$ in the case when $\mathbf{H}$ is defined in terms of a transfer function $H \in \mathcal{B}_{cp}^{p\times m}$ and $\mathbf{\Gamma}$ is defined by a transfer function $\Gamma \in \mathcal{S}_\Gamma$, where $\mathcal{S}_\Gamma$ is a pathwise connected subset (in the topology defined by the ν-gap metric) of

$$\{\Gamma \in \mathcal{B}^{m\times p} : \Gamma(j\omega) \in \mathbf{\Delta}\}$$

where $\mathbf{\Delta} \subset \mathbf{C}^{m\times p}$ is a bounded convex set. We assume there exists $\Gamma_0 \in \mathcal{S}_\Gamma$ such $[\mathbf{\Gamma}_0, \mathbf{H}]$ is stable.

In this case we may use the frequency varying multipliers

$$\begin{aligned} \Phi_\Gamma &= \left\{ \Phi \in \mathbf{C}^{(p+m)\times(p+m)} : \begin{bmatrix} \Delta \\ I \end{bmatrix}^* \Phi \begin{bmatrix} \Delta \\ I \end{bmatrix} \preceq 0;\ \forall \Delta \in \mathbf{\Delta} \right\} \\ &= \Bigg\{ \Phi \in \mathbf{C}^{(p+m)\times(p+m)} : \left\langle \Phi, \begin{bmatrix} \Delta \\ I \end{bmatrix} X \begin{bmatrix} \Delta \\ I \end{bmatrix}^* \right\rangle \le 0; \\ &\qquad\qquad \forall X \in S_{\mathbf{C}}^{p\times p};\ X \succeq 0; \Delta \in \mathbf{\Delta} \Bigg\}. \end{aligned}$$

The polar cone can be formulated as

$$\begin{aligned} \Phi_\Gamma^\ominus = \mathrm{cl\,cone} \Bigg\{ \begin{bmatrix} \Delta \\ I \end{bmatrix} X \begin{bmatrix} \Delta \\ I \end{bmatrix}^* : \\ \forall X \in S_{\mathbf{C}}^{p\times p};\ X \succeq 0; \Delta \in \mathbf{\Delta} \Bigg\}. \end{aligned}$$

We will next see that the dual stability criterion in Corollary 1 may take very concrete forms. For example, let $m = 1$ and $\mathbf{\Delta} \subset \mathbf{C}^{1\times m}$. It is easy to see that the dual is violated at ω if and only if there exists $X \succeq 0$ and $\delta \in \mathbf{\Delta}$ such that

$$\begin{bmatrix} I \\ H(j\omega) \end{bmatrix} \begin{bmatrix} I \\ H(j\omega) \end{bmatrix}^* = \begin{bmatrix} \delta \\ I \end{bmatrix} X \begin{bmatrix} \delta \\ I \end{bmatrix}^*.$$

It is thus required that $X = H(j\omega)H(j\omega)^*$. It follows that the dual stability criterion holds if and only if for every $\omega \in \mathbf{R} \cup \{\infty\}$, $\delta H(j\omega) \neq I$, i.e. if $\frac{1}{|H(j\omega)|^2} H(j\omega)^* \notin \mathbf{\Delta}$.

Example 2: In this example we derive the dual criterion in an example where unstable SISO LTI systems are stabilized over constant respectively time-varying interconnections. The obtained criteria will be of the form obtained in [7].

We consider the system equations in (2) with $\mathbf{H}$ defined by a diagonal transfer function $H = \oplus_{k=1}^n H_k$, where $H_k \in \mathcal{B}_{cp}$. We assume that either

1) $\mathbf{\Gamma}$ is defined by multiplication by a constant symmetric matrix $\Gamma = \Gamma^T \in \mathbf{R}^{n\times n}$ with $\mathrm{eig}(\Gamma) \in [\alpha, \beta]$,
2) $\mathbf{\Gamma}$ is defined by multiplication by a time-varying symmetric matrix $\Gamma(t) \in \mathbf{K}^{n\times n}$ with $\mathrm{eig}(\Gamma(t)) \in [\alpha, \beta]$,

where $\alpha < \beta < 0$. Finally, we assume that the interconnection $[\gamma I, \mathbf{H}]$, is stable for some $\alpha \le \gamma \le \beta$.

Let $\mathbf{S_\Gamma} = \{(1-\theta)\gamma I_n + \theta \mathbf{\Gamma} : \theta \in [0,1]\}$, which obviously is pathwise connected. It then follows that Assumption 2 holds if for the case 1) we use the frequency wise multipliers from

$$\Phi_\Gamma = \left\{ \begin{bmatrix} 2xI & -(\alpha+\beta)xI + jyI \\ -(\alpha+\beta)xI + jyI & 2\alpha\beta xI \end{bmatrix} : \begin{array}{l} x \ge 0 \\ y \in \mathbf{R} \end{array} \right\}$$

while for case 2) we use the constant multipliers from

$$\Psi_\Gamma = \left\{ \begin{bmatrix} 2xI & -(\alpha+\beta)xI \\ -(\alpha+\beta)xI & 2\alpha\beta xI \end{bmatrix} : x \ge 0 \right\}.$$

We may apply Corollary 1 for the stability analysis.

Claim 1: If we apply Corollary 1, then the dual criterion for the two cases considered above hold if and only if $\forall \omega \in \mathbf{R} \cup \{\infty\}$

1) $\mathrm{co}\{(H_1, |H_1|^2), \dots, (H_n, |H_n|^2)\}(j\omega)(j\omega) \cap \Omega_1 \neq \emptyset$,
2) $\mathrm{co}\{(H_1, |H_1|^2), \dots, (H_n, |H_n|^2)\}(j\omega)(j\omega) \cap \Omega_2 \neq \emptyset$,

where

$$\begin{aligned} \Omega_1 &= \left\{ (\lambda, r) : \mathrm{Im}\lambda = 0;\ r \le \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \mathrm{Re}\,\lambda - \frac{1}{\alpha\beta} \right\} \\ \Omega_2 &= \left\{ (\lambda, r) :\ r \le \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \mathrm{Re}\,\lambda - \frac{1}{\alpha\beta} \right\} \end{aligned}$$

Proof: The polar cones becomes ($W = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^* & W_{22} \end{bmatrix}$)

$$\begin{aligned} \Phi_\Gamma^\ominus = \{&W \in \mathcal{S}_{\mathbf{C}}^{2n\times 2n} : \mathrm{Im}\,\mathrm{tr}(W_{12}) = 0; \\ &\mathrm{tr}(W_{11}) - (\alpha+\beta)\mathrm{Re}\,\mathrm{tr}(W_{12}) + \alpha\beta\mathrm{tr}(W_{22}) \le 0\} \end{aligned}$$

and

$$\begin{aligned} \Psi_\Gamma^\ominus = \{&W \in \mathcal{S}_{\mathbf{R}}^{2n\times 2n} : \\ &\mathrm{tr}(W_{11}) - (\alpha+\beta)\mathrm{Re}\,\mathrm{tr}(W_{12}) + \alpha\beta\mathrm{tr}(W_{22}) \le 0\} \end{aligned}$$

The dual condition $(M_H^\times Z)(j\omega) \notin \Phi_\Gamma^\ominus$ simplifies since it is no restriction to make $Z \in \mathcal{Z}$ diagonal. The dual condition then reduces to

$$\mathrm{co}\{(H_1, |H_1|^2), \dots, (H_n, |H_n|^2)\}(j\omega) \cap \Omega_1 = \emptyset$$

where Ω_1 is defined in the statement of the claim.

Since $\dim(\Psi_\Gamma) = 1$ and $0 \notin \mathrm{ri}\,\Psi_\Gamma$ it is sufficient with one frequency in the grid. The derivation is therefore analogous to the first case. ■

Example 3: Consider the feedback interconnection in Figure 1 with two SISO LTI systems defined by transfer functions $G, F \in \mathcal{A}_{cp}$, which are interconnected through uncertainties δ_1 and δ_2. The system can be represented as an interconnection between $H = G \oplus F$ and

$$\Gamma \in \mathcal{S}_\Gamma = \left\{ \begin{bmatrix} 0 & \delta_2 \\ -\delta_1 & 0 \end{bmatrix} : \delta_1, \delta_2 \text{ satisfies 1), 2), or 3)} \right\},$$

where either of the following three cases is considered

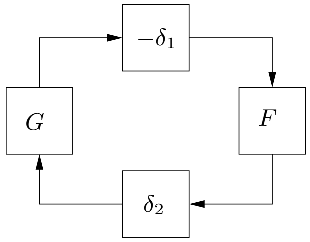

Fig. 1. Feedback interconnection with two uncertainties.

1) $\delta_k = \delta \in \mathcal{A}$, where $|\delta(j\omega)| \leq 1$, $k = 1, 2$,
2) $\delta_k \in \mathcal{A}$ with $|\delta_k(j\omega)| \leq 1$, $k = 1, 2$
3) $\delta_k \in \mathbf{K}$ with $\delta_k(t) \in [-1, 1]$ for all t, $k = 1, 2$.

Let Γ_k denote the interconnection matrix for the three respective cases. Since H is assumed stable it follows that $[\Gamma_0, H]$ is stable for $\Gamma_0 = 0 \in \mathcal{S}_\Gamma$ and we may apply Corollary 1.

For $k = 1, 2$ we may use the frequency varying multipliers

$$\Phi_\Gamma = \left\{ \begin{bmatrix} x_1 & 0 & y_1 & 0 \\ 0 & x_2 & 0 & y_2 \\ \bar{y}_1 & 0 & -x_2 & 0 \\ 0 & \bar{y}_2 & 0 & -x_1 \end{bmatrix} : \begin{array}{l} x_1, x_2 \geq 0, y_1 = e^{2j\phi} y,\ y_2 = \bar{y};\ y \in \mathbf{C}, k = 1 \\ x_1, x_2 \geq 0, y_1 = y_2 = 0,\ k = 2 \end{array} \right\},$$

where at each frequency $\phi = \arg(\delta(j\omega))$. For $k = 3$ we may use the constant multipliers in

$$\Psi_\Gamma = \{\mathrm{diag}(x_1, x_2, -x_2, -x_1) : x_1, x_2 \geq 0\}$$

Claim 2: If we apply Corollary 1, the dual reduces to the following criteria for the three cases considered above:

1) $e^{-2j\arg(\delta(j\omega))} G(j\omega) F(j\omega) \notin (-\infty, -1]$, $\forall \omega \in \mathbf{R} \cup \{\infty\}$
2) $|G(j\omega) F(j\omega)| < 1$, $\forall \omega \in \mathbf{R} \cup \{\infty\}$
3) $\|G\| \|F\| < 1$

Proof: To prove the first two cases we compute the polar cone

$$\Phi_\Gamma^\ominus = \left\{ \begin{bmatrix} w_{11} & w_{12} & w_{13} & w_{14} \\ \bar{w}_{12} & w_{22} & w_{23} & w_{24} \\ \bar{w}_{13} & \bar{w}_{23} & w_{33} & w_{34} \\ \bar{w}_{14} & \bar{w}_{24} & \bar{w}_{34} & w_{44} \end{bmatrix} : \begin{array}{l} w_{33} \geq w_{22}; w_{44} \geq w_{11};\ k = 1 \text{ and } k = 2 \\ \qquad e^{2j\phi} \bar{w}_{13} + w_{24} = 0 \text{ if } k = 1 \end{array} \right\}.$$

In case $k = 1, 2$ $\Psi_\Gamma = 0$, we need to consider only one frequency in the grid. This gives the dual

$$M_{H(j\omega)}^* Z \notin \Phi_\Gamma^\ominus$$

for all ω and all $Z \in \mathcal{Z}$. It is no restriction to assume diagonal variables $Z = z_G \oplus z_F$ and thus the dual simplifies to the condition that the following system

$$\begin{aligned} z_G |G(j\omega)|^2 &\geq z_F \\ z_F |F(j\omega)|^2 &\geq z_G \\ z_G \bar{G}(j\omega) e^{2j\phi} + z_F F(j\omega) &= 0 \quad (\text{if } k = 1) \\ z_G + z_F = 1,\ z_G, z_F &\geq 0 \end{aligned}$$

must not have a solution for $\omega \in \mathbf{R} \cup \{\infty\}$. An equivalent formulation is that the system

$$\begin{gathered} |G(j\omega) F(j\omega)| \geq 1, \\ \arg(e^{-2j\phi} G(j\omega)) + \arg(F(j\omega)) = (2p + 1)\pi, \quad (\text{if } k = 1) \end{gathered}$$

must not have a solution for $\omega \in \mathbf{R} \cup \{\infty\}$ and for any integer p. This proves 1) and 2).

Finally, we will prove 3). The polar cone becomes

$$\Psi_\Gamma^\ominus = \left\{ W \in \mathcal{S}_{\mathbf{R}}^{4 \times 4} : w_{11} - w_{33} \geq 0, w_{22} - w_{44} \geq 0 \right\}.$$

Since $N = \dim(\Psi_\Gamma) = 2$ and $0 \notin \mathrm{ri}\, \Psi_\Gamma$ it is sufficient with two frequencies in the dual. In analogy to case 2) the dual criterion reduces to the condition that the following system

$$\begin{gathered} z_{1,G} |G(j\omega_1)|^2 + z_{2,G} |G(j\omega_2)|^2 \geq z_{1,F} + z_{2,F} \\ z_{1,F} |F(j\omega_1)|^2 + z_{2,F} |F(j\omega_2)|^2 \geq z_{1,G} + z_{2,G} \\ z_{1,G} + z_{2,G} + z_{1,F} + z_{2,F} = 1 \\ z_{1,G}, z_{2,G}, z_{1,F}, z_{2,F} \geq 0 \end{gathered}$$

must not have a solution for any $\omega_1, \omega_2 \in \mathbf{R} \cup \{\infty\}$. One can show that this is equivalent to 3). ■

IV. Appendix: Proof of the Main Results

The proofs of Theorem 1 and Corollary 1 consists of two steps

1) Use convex duality results to show that the primal and dual stability criteria are equivalent. The proof of this equivalence between (a) and (b) in Theorem 1 can be found in Subsection IV-A.
2) Use the stability result in [8] to prove that the primal condition implies stability of the interconnection $[\mathbf{\Gamma}, \mathbf{H}]$. The proofs can be found in Subsection IV-B.

A. Proof of Equivalence of the Primal and Dual Criterion

To prove that (a) and (b) are equivalent in Theorem 1, we first derive a criterion for (a) being violated. We will use the next lemma, which follows from the results in [6].

Lemma 1: For given $\omega \in \mathbf{R} \cup \{\infty\}$ define

$$C_\omega = \{\Psi \in \Psi_\Gamma : \exists \Phi \in \Phi_\Gamma,\ \text{s.t. } M_H(j\omega)(\Phi + \Psi) \succ 0\}.$$

The condition $\bigcap_{\omega \in \mathbf{R} \cup \{\infty\}} C_\omega = \emptyset$ holds if and only if there exists at most $N = \dim(\Psi_\Gamma) + 1$ frequencies $\omega_1, \ldots, \omega_N \in \mathbf{R} \cup \{\infty\}$ such that $\cap_{k=1}^N C_{\omega_k} = \emptyset$. Furthermore, if $0 \notin \mathrm{ri}\, \Psi_\Gamma$, then $N \leq \dim(\Psi_\Gamma)$.

Hence, the criterion in (a) is violated if and only if there exist at most N frequencies $\Omega_N = \{\omega_1, \ldots, \omega_N\}$ such that the following convex sets are disjoint

$$\begin{aligned} C_1 &= \{M_H^{\Omega_N}(\Phi_1, \ldots, \Phi_k, \Psi) : \Phi_k \in \Phi_\Gamma,\ \Psi \in \Psi_\Gamma\}, \\ C_2 &= \left\{ X \in \mathcal{S}_{\mathbf{C}}^{m \times m} : X \succ 0 \right\}^{2N}, \end{aligned}$$

where $M_H^{\Omega_N} : (\mathcal{S}_{\mathbf{C}}^{(p+m)\times(p+m)})^N \times \mathcal{S}_{\mathbf{R}}^{(p+m)\times(p+m)} \to \mathcal{S}_{\mathbf{C}}^{m\times m}$ is defined as (here we let $M_H^{\omega_k} = M_{H(\omega_k)}$)

$$\begin{aligned} M_H^{\Omega_N}(\Phi_1, \ldots, \Phi_k, \Psi) \\ &= \times_{k=1}^N (M_H^{\omega_k}(\Phi_k + \Psi), M_{H_0}^{\omega_k}(\Phi_k + \Psi)). \end{aligned}$$

By the separating hyperplane theorem [Theorem 11.3. in [10]] there exists a nonzero tuple $Z = \times_{k=1}^N (Z_{1,k}, Z_{2,k})$, $Z_{1,k}, Z_{2,k} \succeq 0$ such that (the argument ω is suppressed)

$$\begin{aligned} &\left\langle M_H^{\Omega_N}(\Phi_1, \ldots, \Phi_k, \Psi), Z \right\rangle \leq 0, \ \forall \Phi_k \in \Phi_\Gamma, \ \Psi \in \Psi_\Gamma \\ \Leftrightarrow &\left\langle (\Phi_1, \ldots, \Phi_k, \Psi), (M_H^{\Omega_N})^\times Z \right\rangle \leq 0, \ \forall \Phi_k \in \Phi_\Gamma, \ \Psi \in \Psi_\Gamma \end{aligned}$$

where the dual $(M_H^{\Omega_N})^\times : \mathcal{S}_{\mathbf{C}}^{m\times m} \to (\mathcal{S}_{\mathbf{C}}^{(p+m)\times(p+m)})^N \times \mathcal{S}_{\mathbf{R}}^{(p+m)\times(p+m)}$ is defined as

$$\begin{aligned} (M_H^{\Omega_N})^\times Z =& (\times_{k=1}^N ((M_H^{\omega_k})^\times Z_{1,k} + (M_{H_0}^{\omega_k})^\times Z_{2,k}), \\ &\mathrm{Re}\left(\sum_{k=1}^N (M_H^{\omega_k})^\times Z_{1,k} + \sum_{k=1}^N (M_{H_0}^{\omega_k})^\times Z_{2,k}\right)). \end{aligned}$$

This implies

$$\begin{aligned} &(M_H^{\omega_k})^\times Z_{1,k} + (M_{H_0}^{\omega_k})^\times Z_{2,k} \in \Phi_\Gamma^\ominus, \ k = 1, \ldots, n, \\ &\mathrm{Re}\left(\sum_{k=1}^N (M_H^{\omega_k})^\times Z_{1,k} + \sum_{k=1}^N (M_{H_0}^{\omega_k})^\times Z_{2,k}\right) \in \Psi_\Gamma^\ominus. \end{aligned} \tag{9}$$

Note that we may normalize the dual variables such that $\sum_{k=1}^N \mathrm{tr}(Z_{1,k}) + \mathrm{tr}(Z_{2,k}) = 1$, i.e., $Z \in \mathcal{Z}$, where $\mathcal{Z}$ defined in (6) is a compact convex set.

In order to prove the theorem we need to establish the reverse direction of the above duality result. Hence, if (9) fails, i.e. (5) holds, then for every grid $\Omega_N = \{\omega_1, \ldots, \omega_N\}$ the two convex sets

$$\begin{aligned} C_3 &= \left\{ (M_H^{\Omega_N})^\times Z : Z \in \mathcal{Z} \right\}, \\ C_4 &= (\Phi_\Gamma^\ominus)^N \times \Psi_\Gamma^\ominus \end{aligned}$$

are disjoint. Since

$$((\Phi_\Gamma^\ominus)^N \times \Psi_\Gamma^\ominus)^\ominus = (\Phi_\Gamma)^N \times \Psi_\Gamma,$$

where we used that Φ_Γ and Ψ_Γ are closed convex cones in the topology defined by the Frobenius norm and therefore $\Phi_\Gamma^{\ominus\ominus} = \Phi_\Gamma$ and analogously for Ψ_Γ. Hence, since C_3 is convex and compact and C_4 is closed and convex it follows that there exists a hyperplane that separates the two sets strongly [Corollary 11.4.2 in [10]], i.e. there exists a nonzero $\Upsilon = (\Phi_1, \ldots, \Phi_N, \Psi) \in (\Phi_\Gamma)^N \times \Psi_\Gamma$ such that

$$\begin{aligned} &\left\langle \Upsilon, (M_H^{\Omega_N})^\times Z \right\rangle > 0, \ \forall Z \in \mathcal{Z} \\ \Leftrightarrow &\left\langle M_H^{\Omega_N} \Upsilon, Z \right\rangle > 0, \ \forall Z \in \mathcal{Z} \\ \Leftrightarrow &M_H^{\omega_k}(\Phi_k + \Psi) \succ 0 \quad \text{and} \quad M_{H_0}^{\omega_k}(\Phi_k + \Psi) \succ 0. \end{aligned}$$

for all $\omega_k \in \Omega_N$. Hence, for any arbitrary grid $\Omega_N = \{\omega_1, \ldots, \omega_N\}$ we have $\cap_{k=1}^N C_{\omega_k} \neq \emptyset$, which by[2] Lemma 1 implies that $\bigcap_{\omega \in \mathbf{R} \cup \{\infty\}} C_\omega \neq \emptyset$. It follows that the primal statement holds, i.e. for each $\omega \in \mathbf{R} \cup \{\infty\}$ there exists $\Pi \in \Pi_\Gamma$ such that $M_{H(j\omega)}\Pi \succ 0$ and $M_{H_0(j\omega)}\Pi \succ 0$.

[2]The lemma is applied with system $\mathrm{diag}(H, H_0)$ and multipliers $\mathrm{daug}(\Phi, \Phi)$, and $\mathrm{daug}(\Psi, \Psi)$.

B. Proof of Primal Stability Conclusion

The stability conclusions of Theorem 1 and Corollary 1 are proven using the main result in [8].

1) Proof of Primal Stability in Theorem 1: For the purpose of using the result in [8] we recall that

1) $\mathbf{H}$ is defined by a transfer function from $\mathcal{B}_{cp}^{p\times m}$. In this case the ν-gap distance may be computed as in (1).
2) $\mathbf{\Gamma}$ is either defined by a normalized coprime factorization from $\mathcal{B}^{m\times p}$ or a time-varying matrix gain. In the later case we use normalized coprime factorizations

$$\begin{aligned} \mathbf{G}_{\mathbf{\Gamma}} &= \begin{bmatrix} \mathbf{\Gamma} \\ I \end{bmatrix} (I + \mathbf{\Gamma}^*\mathbf{\Gamma})^{-1/2}, \\ \widetilde{\mathbf{G}}_{\mathbf{\Gamma}} &= (I + \mathbf{\Gamma}\mathbf{\Gamma}^*)^{-1/2} \begin{bmatrix} I - \mathbf{\Gamma} \end{bmatrix}. \end{aligned}$$

Corollary 2: Consider system (2) under Assumption 1. Then the system is stable if either of the equivalent conditions (a) and (b) in Theorem 1 hold.

Proof: The proof follows by showing that the primal criterion imply that the conditions in [8] hold. Indeed, by Assumption 1 (a) and (b)

1) there exist $\Pi \in \mathbf{L}_\infty^{(p+m)\times(p+m)}(j\mathbf{R})$ such that $\mathbf{\Gamma} \in \mathrm{IQC}^c(\Pi)$,
2) there exists $\mathbf{H}_0$ such that $[\mathbf{\Gamma}, \mathbf{H}_0]$ is stable.

We will prove that there exists a parametrization $\mathbf{H}_\theta$ that is continuous in the ν-gap with $\mathbf{H}_1 = \mathbf{H}$ and such that $\mathbf{H}_\theta \in \mathrm{SIQC}(\Pi)$, $\forall \theta \in [0, 1]$, where SIQC denotes strict IQC. Then the result follows from [8] .

We will construct normalized coprime factorizations such that for any $\epsilon \in (0, 1)$, we have $\delta_\nu(H_a, H_b) \leq \epsilon$ whenever $|a - b|$ is sufficiently small.

Let $\widetilde{G}_{H_\theta} = S_\theta^{-1}\widetilde{G}_\theta$, where $\widetilde{G}_\theta$ is defined as

$$\widetilde{G}_\theta = \begin{bmatrix} -(1-\theta)\widetilde{U}_0 - \theta\widetilde{U} & \widetilde{V} \end{bmatrix}$$

and S_θ is a spectral factor defined by a transfer function satisfying $S_\theta, S_\theta^{-1} \in \mathcal{A}_{cp,-}$ and such that

$$\widetilde{G}_\theta \widetilde{G}_\theta^* = S_\theta S_\theta^*.$$

Note

$$\begin{aligned} \widetilde{G}_{H_\theta}\widetilde{G}_{H_\theta}^* &= S_\theta^{-1}\widetilde{G}_\theta\widetilde{G}_\theta^*(S_\theta^{-1})^* \\ &= S_\theta^{-1}S_\theta S_\theta^*(S_\theta^*)^{-1} = I \end{aligned}$$

Since the spectral factor is continuous in θ, see [5], it follows that $\widetilde{G}_{H_\theta}$ is continuous as a function of θ. Hence, if $|a - b|$ is sufficiently small such that $\overline{\gamma}(\widetilde{G}_{H_a}(\widetilde{G}_{H_b} - \widetilde{G}_{H_a})^*) < \epsilon$, then the identity

$$\widetilde{G}_{H_a}\widetilde{G}_{H_b}^* = I + \widetilde{G}_{H_a}(\widetilde{G}_{H_b} - \widetilde{G}_{H_a})^*$$

implies that

$$\underline{\gamma}(\widetilde{G}_{H_a}\widetilde{G}_{H_b}^*) \geq 1 - \overline{\gamma}(\widetilde{G}_{H_a}(\widetilde{G}_{H_b} - \widetilde{G}_{H_a})^*) \geq 1 - \epsilon > 0.$$

This implies that $\underline{\gamma}(I) = 1 > \overline{\gamma}(\widetilde{G}_{H_a}(\widetilde{G}_{H_b} - \widetilde{G}_{H_a})^*)$ and thus it follows that $\mathrm{wno}(\widetilde{G}_{H_a}\widetilde{G}^*_{H_b}) = \mathrm{wno}(I) = 0$. Moreover, it can be shown that

$$\overline{\gamma}(\widetilde{G}_{H_a} G_{H_b})^2 = 1 - \underline{\gamma}(\widetilde{G}_{H_a}\widetilde{G}^*_{H_b})^2 \leq \epsilon(2-\epsilon).$$

Hence H_θ is continuous in the ν-gap distance.

To prove $\mathbf{H}_\theta \in \mathrm{SIQC}(\Pi)$ we let $G_{H_\theta} = G_\theta S_\theta^{-1}$ where

$$G_\theta = \begin{bmatrix} V \\ (1-\theta)U_0 + \theta U \end{bmatrix},$$

and where $S_\theta \in \mathcal{A}_{cp,-}$ is the spectral factor satisfying $S_\theta^{-1} \in \mathcal{A}_{cp,-}$ and $G_\theta^* G_\theta = S_\theta^* S_\theta$. Hence,

$$(M_{H_\theta}\Pi)(j\omega) = G_{H_\theta}(j\omega)^* \Pi(j\omega) G_{H_\theta}(j\omega) \succ 0,$$

$\forall \theta \in [0,1]$, which follows by convexity since $\Pi_{22}(j\omega) = \Phi_{22}(j\omega) + \Psi_{22} \preceq 0$. ■

2) Proof of Primal Stability in Corollary 1: We recall that

1) $\mathbf{H}$ is defined by a transfer function from $\mathcal{B}_{cp}^{p\times m}$.
2) $\mathbf{\Gamma}$ is either defined by a normalized coprime factorization from $\mathcal{B}_{cp}^{m\times p}$ or a time-varying matrix gain. In the later case the ν-gap distance simplifies to

$$\delta_\nu(\Gamma_a, \Gamma_b) = \begin{cases} \sup_t \sigma_{\max}(\Psi(t)), & \ker(\Theta(t)) \neq 0,\ \forall t \\ 1, & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} \Psi(t) =& (I + \Gamma_a(t)\Gamma_a(t)^T)^{-1/2}(\Gamma_b(t) - \Gamma_a(t)) \\ & \times (I + \Gamma_b(t)\Gamma_b(t)^T)^{-1/2}, \\ \Theta(t) =& (I + \Gamma_a(t)\Gamma_b(t)^T). \end{aligned}$$

Pathwise connectedness follows since the parametrization of Γ is continuous.

Proposition 1: Consider system (2) under Assumption 2, where the set $\mathcal{S}_\mathbf{\Gamma}$ now is assumed pathwise connected in the ν-gap distance. Then the system is stable if either of the equivalent conditions (a) and (b) in Corollary 1 hold.

Proof: Let us rewrite the system equations as

$$\begin{aligned} \check{v} &= \check{\mathbf{\Gamma}}\check{w} + \check{r}_1, \\ \check{w} &= \check{\mathbf{H}}\check{v} + \check{r}_2, \end{aligned}$$

where $\check{\mathbf{H}} = \mathbf{\Gamma}$ and $\check{\mathbf{\Gamma}} = \mathbf{H}$. Since $\mathbf{S}_\mathbf{\Gamma}$ is pathwise connected in the ν-gap distance there exists a continuous parametrization $\check{\mathbf{H}}_\theta = \mathbf{\Gamma}_\theta$ with $\check{\mathbf{H}}_1 = \mathbf{\Gamma}_1 = \mathbf{\Gamma}$ and where $[\check{\mathbf{\Gamma}}, \check{\mathbf{H}}_0] = [\mathbf{H}, \mathbf{\Gamma}_0]$ is stable by assumption. If we define

$$J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

and $\check{\Pi} = -J^T \Pi J + \epsilon I$ then it follows that $\check{\mathbf{H}}_\theta \in \mathrm{SIQC}(\check{\Pi})$, $\theta \in [0,1]$ and $\check{\mathbf{\Gamma}} \in \mathrm{IQC}^c(\check{\Pi})$ for some suitable small positive number. Stability now follows from [8].

■

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