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Constructing a Bimodal Switched Lyapunov Function for Non-Uniformly Sampled-Data Feedback Systems

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Abstract— **Stability analysis of non-uniformly sampled-data feedback control systems is considered. An algorithm is proposed based on the property that the exponential stability is implied by the existence of a switched Lyapunov function for the associate discrete-time systems. In order to reduce the computational complexity, the algorithm is proposed taking account of the dimensions of LMIs to be solved. It is shown that the proposed algorithm constructs a bimodal switched Lyapunov function in a finite step if one exists.**

I. INTRODUCTION

Motivated by networked control systems with packet loss (See, e.g., [10]) and embedded control systems with incomplete real-time property (See, e.g., [13]), analysis and design problems for sampled-data systems with uncertainly time-varying sampling intervals have been studied in these days [1], [2], [4]–[9], [11], [12], [14]–[28]. It is obvious that the problems are much harder than those for the standard periodic sampling case, and that robustness plays a crucial role in the study.

Since quadratic stability of the associate discrete-time system implies exponential stability of the sampled-data system [28], one can discuss the stability in the discrete-time domain, as in most of recent studies. Then the main difficulty to be solved is the fact that the discrete-time system is time-varying and uncertain. Improving the approximation by gridding in earlier studies [1], [21], [28], the recent studies take the robustness into account to guarantee the stability. In particular, (i) a robust control approach based on a linear fractional transformation (LFT) uncertainty modeling [6], [8], [24], (ii) a robust control approach based on a polytopic uncertainty modeling [1], [2], [11], [12], and (iii) a robust linear matrix inequality (LMI) approach [19], [20] have been proposed. We note that numerical experiences in the references suggest that these discrete-time approaches are less conservative than other approaches constructing continuous-time Lyapunov functions in, e.g., [4], [5], [7], [14]–[18], [23], [25]–[27]. On the other hand, since these discrete-time approaches are based on the sufficient condition of the quadratic stability of the discrete-time system, they cannot avoid the common conservatism.

It is pointed out recently in [9] that the existence of a switched Lyapunov function [3], which was proposed for stability analysis of switched systems, for the associate discrete-time system implies exponential stability of the sampled-data system. This fact suggests that less conservative analysis could be achieved by modeling the associate discrete-time system as a switched system so that each mode has smaller uncertainty. Indeed it has been shown by using numerical examples in [9] that the stability analysis based on the switched Lyapunov function reduces the conservatism in the analysis based on the quadratic stability. We emphasize the fundamental difference between [9] and other discrete-time approaches in terms of the stability criteria. Hence it is desirable to develop a concrete method for stability analysis based on the switched Lyapunov function for the associate discrete-time system.

The reference [9], however, only suggests the usefulness of switched Lyapunov functions and no concrete algorithm is provided. In order to develop a concrete algorithm, we need a method to model the associate discrete-time system as a switched system. It is, however, not trivial. It might be natural to chop the range of sampling intervals into pieces and determine each mode of the switched system by each piece, but it is not obvious how we should chop. Moreover we have to take care the total size of LMIs to be solved. One could extend algorithms developed for quadratic stability analysis. Indeed a direct extension of the algorithm in [6] has been suggested in [9], however, such a naive extension can result an intractable total size of LMIs. We will see this later more in detail.

The purpose of this paper is to provide a concrete algorithm for stability analysis of the non-uniformly sampled-data feedback control systems based on the existence of a switched Lyapunov function for the associate discrete-time system. The proposed algorithm constructs a bimodal switched Lyapunov function for the associate discrete-time system in a finite step, if one exists, by solving LMIs of reasonable dimensions in compare to the LMIs in the algorithm of [6].

This paper is organized as follows: The problem is formulated in Section II. A stability analysis algorithm is proposed in Section III and its usefulness is demonstrated in Section IV.

II. PROBLEM SETUP

Consider the sampled-data feedback system T depicted in Fig. 1. The plant is given by the state-space representation

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where u and x respectively denote the control input and the states. The sampled-data $x(\tau_k)$ $(k = 0, 1, 2, \ldots)$ of x are obtained by the sampler S, where τ_k denotes the k-th sampling instant. The control input u is determined as

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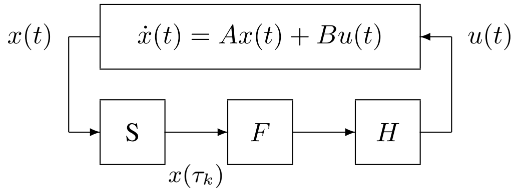

Fig. 1. Non-uniformly sampled data system T

a sampled-data feedback by using a static gain F and a zero order hold H synchronized with the sampler:

$$u(t) = Fx(\tau_k), \quad \forall t \in [\tau_k, \tau_{k+1}).$$

The set of sampling periods $\{\tau_k\}_{k=0}^{\infty}$ is uncertain but assumed to satisfy $\tau_0 = 0$ and

$$h_\ell \le \tau_{k+1} - \tau_k \le h_u, \quad \forall k \in \{0, 1, 2, \dots\}$$

for a given $0 < h_\ell < h_u < \infty$. The purpose of this paper is to verify the stability of T.

The evolution of x at the sampling instants is described by the associate discrete-time system T_d:

$$\xi[k+1] = \Phi_{\tau_{k+1}-\tau_k}\xi[k], \quad \Phi_h := \mathrm{e}^{Ah} + \int_0^h \mathrm{e}^{A\tau}\mathrm{d}\tau\, BF.$$

Here $\xi[k] := x(\tau_k)$. Since it has been pointed out that quadratic stability of T_d implies exponential stability of T [28], several methods are proposed for analysis and design for T based on the fact [1], [2], [6], [8], [11], [12], [19], [20], [24]. The following property shown in [9], however, suggests the conservatism of the approach based on quadratic stability:

Property 1: The following two conditions are equivalent:

(i) T is exponentially stable.

(ii) T_d is exponentially stable.

Namely we could introduce more flexibility to verify the stability of T_d. Along this line, let us consider the following condition related to a switched Lyapunov function [3] for T_d:

Condition 1: There exist $h_s \in (h_\ell, h_u)$ and positive definite matrices $X_1 = X_1^T > 0$, $X_2 = X_2^T > 0$ such that

$$\Phi_h X_i \Phi_h^T - X_j < 0, \quad \forall h \in \mathcal{H}_i \tag{1}$$

for all $i, j \in \{1, 2\}$, where

$$\mathcal{H}_1 := [h_\ell, h_s], \quad \mathcal{H}_2 := [h_s, h_u].$$

The following lemma with Condition 1 is used in [9] for stability analysis of T:

Lemma 1: T is exponentially stable if Condition 1 holds.

Lemma 1 is coincident with the stability criterion based on quadratic stability of T_d [28] when $X_1 = X_2$, and hence is expected to be less conservative. In fact, the reduction of the conservatism is shown in [9] by using numerical examples. Thus this article considers the following problem:

Problem 1: For a given T, verify if there exists a triplet (X_1, X_2, h_s) satisfying Condition 1.

Note that Problem 1 contains at least two difficulties of (i) how to find an h_s and (ii) how to verify the condition for the infinitely many h. In particular the difficulty (i) has not been discussed in [9].

Remark 1: A multi-modal version of Lemma 1 is used in [9]. This article focuses on the bimodal case.

III. MAIN RESULTS

This section provides the main results of this paper. We first discuss how to solve the difficulties in Problem 1 in Section III-A. Then we develop a method to solve the difficulties by using LMIs taking account of computational complexity. Finally we propose a stability analysis algorithm in Section III-C.

A. Extending the LFT Modeling Approach

References [6], [24] cast stability analysis of T to a robust quadratic stability analysis of the feedback connection of Σ:

$$\Sigma : \begin{cases} \xi[k+1] = \Phi_{h_0}\xi[k] + w[k] \\ z[k] = \Psi_{h_0} x(\tau_k) \end{cases}$$

and $\Delta(\theta_k)$ for a given $h_0 > 0$, according to the fact that T_d is a feedback connection of Σ and $\Delta(\theta_k)$. Here $\xi[k] = x(\tau_k)$, $\theta_k := \tau_{k+1} - \tau_k - h_0$, and

$$\Psi_h := A\Phi_h + BF, \quad \Delta(\theta_k) := \int_0^{\theta_k} \mathrm{e}^{At}\mathrm{d}t. \tag{2}$$

The method is extended in [9] to solve the difficulty (ii) of Problem 1. The following lemma can be implied from [9, Theorem 1]. The derivation from [9, Theorem 1] is straightforward so it is omitted.

Lemma 2: Suppose that there exist positive definite matrices $X_1 = X_1^T > 0$, $X_2 = X_2^T > 0$,

$$\mathcal{G} := \{h_1, h_2, \dots, h_N\} \subset [h_\ell, h_u], \quad h_i < h_{i+1}, \tag{3}$$

$k \in \{1, 2, \dots, N-1\}$, $\gamma > 0$, and $\sigma > 0$ satisfying that

$$h_{i+1} - h_i \le \theta_L(\gamma) + \theta_U(\gamma), \quad \forall i \in \{1, 2, \dots, N-1\}, \tag{4}$$

$$h_1 - h_\ell \le \theta_L(\gamma), \quad h_u - h_N \le \theta_U(\gamma) \tag{5}$$

and

$$M_h(X_i, X_j, \gamma, \sigma) < 0, \quad \forall h \in \mathcal{G}_i, \tag{6}$$

for all $i, j \in \{1, 2\}$, where

$$\mathcal{G}_1 := \{h_1, \dots, h_k\}, \quad \mathcal{G}_2 := \{h_{k+1}, \dots, h_N\} \tag{7}$$

and

$$M_h(X, Y, \gamma, \sigma) := \begin{bmatrix} \Phi_h & \sigma I \\ \frac{1}{\sigma}\Psi_h & 0 \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi_h & \sigma I \\ \frac{1}{\sigma}\Psi_h & 0 \end{bmatrix}^T - \begin{bmatrix} Y & 0 \\ 0 & \gamma^2 I \end{bmatrix}.$$

Here $\theta_L(\gamma)$, $\theta_U(\gamma)$ are defined by

L1) if $\mu(-A) = 0$ then $\theta_L(\gamma) := \gamma^{-1}$,

L2) else if $\mu(-A) \le -\gamma$ then $\theta_L(\gamma) := \infty$,

L3) else

$$\theta_L(\gamma) := \frac{1}{\mu(-A)} \log\left(1 + \gamma^{-1}\mu(-A)\right).$$

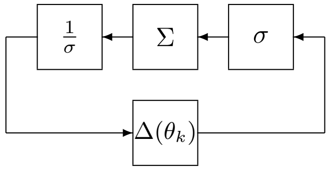

Fig. 2. Alternative representation of T_d

U1) if $\mu(A) = 0$ then $\theta_U(\gamma) := \gamma^{-1}$,
U2) else if $\mu(A) \le -\gamma$ then $\theta_U(\gamma) := \infty$,
U3) else

$$\theta_U(\gamma) := \frac{1}{\mu(A)} \log\left(1 + \gamma^{-1}\mu(A)\right).$$

where $\mu(A)$ denote the logarithmic norm of A:

$$\mu(A) := \lambda_{\max}\left(\frac{A + A^*}{2}\right).$$

Then X_1, X_2, and any $h_s \in (h_k,\, h_{k+1})$ satisfy (1) for all i, $j \in \{1,\, 2\}$.

Remark 2: The symbol σ denotes a trivial multiplier (See Fig. 2) which will be used later.

Remark 3: The condition (6) represents $2k + 2(N - k) = 2N$ inequalities.

In principle Lemma 2 solves the both difficulties of Problem 1. Once we fix the candidate of h_s, the infinitely many inequalities in Condition 1 reduce to a finite number of LMIs, by invoking Lemma 2 with a given σ. Indeed, an LMI based condition is suggested in [9] for a given candidate of h_s and $\sigma = 1$. Moreover it is enough to consider $\{(h_i + h_{i+1})/2\}$ as a set of candidates of h_s, and one could try all the elements in turn. For generating $\mathcal{G}$, it is not hard to extend the algorithm in [6] based on Lemma 2. One might also expect that, once T_d is modeled as a set of finite number of uncertain systems, it is not hard to apply the method developed in [3] if we consider a multi-modal switched Lyapunov function.

However a direct application or a straightforward extension of existing methods should be avoided from the viewpoint of computational complexity: The grid $\mathcal{G}$ may have a large number of elements. In our numerical experiences [6], $\mathcal{G}$ can contains more than 20 elements. If we apply the method in [3] to construct a multi-modal switched Lyapunov function, we have to solve several hundreds (roughly speaking, the square of the number of the elements in $\mathcal{G}$) of LMIs simultaneously, that are not tractable. If we extend the method in [6] to construct a bimodal switched Lyapunov function in a straightforward fashion, the total size of resultant LMIs to be solved would be around four times as large as that in [6], which should be avoided. Hence we need to develop a computationally cheap and tractable method for stability analysis by using a switched Lyapunov function. This will be the subject of the next subsection.

B. Recovering Robustness Considering Total Size of LMIs

In earlier studies based on the associate discrete-time system T_d [1], [21], [28], the range of sampling periods is approximated by a grid in the range. In other words, T_d is approximated by a set of time-invariant discrete-time systems related to the grid. If there exists a common quadratic Lyapunov function for all the time-invariant discrete-time systems in the set, one could expect that it can play as a quadratic Lyapunov function of the associate discrete-time system. There is, however, no theoretical guarantee. In order to get a rigorous guarantee, the algorithm in [6] constructs a quadratic Lyapunov function for T_d by searching a common solution to the bounded real LMIs (instead of the Lyapunov inequalities) for all the discrete-time systems in the set.

In the sequel we will, however, use the approximated set of systems defined by the gridding to construct a candidate of a switched Lyapunov function. The purpose is to use the Lyapunov inequality instead of the bounded real LMI so that the total size of LMIs to be solved does not grow much. On the other hand, as opposed to the earlier studies, we will not ignore the approximation error introduced by the gridding. Indeed we will see that the candidate of the switched Lyapunov function satisfies a robustness property derived from Lemma 2.

As a preliminary we pose two trivial properties related to the following condition:

Condition 2: There exist $\beta > 0$, $0 < X_1 = X_1^T \le I$, $0 < X_2 = X_2^T \le I$, and $h_s \in (h_\ell,\, h_u)$ satisfying

$$\Phi_h X_i \Phi_h^T - X_j \le -\beta I, \quad \forall h \in \mathcal{H}_i \tag{8}$$

for all i, $j \in \{1,\, 2\}$.

The following two properties are trivial:

Property 2: Conditions 1 and 2 are equivalent.

Property 3: Suppose that Condition 2 holds. Given $\mathcal{G}$ in (3), there exist $0 < \hat{X}_1 = \hat{X}_1^T \le I$, $0 < \hat{X}_2 = \hat{X}_2^T \le I$, $k \in \{1,\, \ldots,\, N-1\}$, and $\hat{\beta} \ge \beta$ such that

$$\Phi_h \hat{X}_i \Phi_h^T - \hat{X}_j \le -\hat{\beta} I, \quad \forall h \in \mathcal{G}_i \tag{9}$$

for all i, $j \in \{1,\, 2\}$, where β is given in Condition 2.

The following theorem shows that if there exists a bimodal switched Lyapunov function for T_d, it can be obtained as that for the discrete-time switched system defined by the set of time-invariant discrete-time systems related to a sufficiently fine grid:

Theorem 1: Suppose that Condition 1 holds. Given $\mathcal{G}$ in (3), there exist $\hat{\beta} \ge \beta$, $0 < \hat{X}_1 = \hat{X}_1^T \le I$, $0 < \hat{X}_2 = \hat{X}_2^T \le I$ and $k \in \{1,\, \ldots,\, N-1\}$ satisfying (9) for all i, $j \in \{1,\, 2\}$, where β is given in Condition 2. Moreover $X_1 = \hat{X}_1$, $X_2 = \hat{X}_2$, and $h_s \in (h_k,\, h_{k+1})$ satisfy Condition 1 if $\mathcal{G}$ satisfies (4) and (5) for $\gamma > \gamma_0$, where γ_0 is defined by

$$\gamma_0 := \min_{r \in (0,\,1)} \max_{h \in [h_\ell,\, h_u]} \|\Psi_h\| \sqrt{\frac{1}{r\beta}\left(1 + \frac{\|\Phi_h\|^2}{(1-r)\beta}\right)}. \tag{10}$$

The proof is found in the appendix.

Remark 4: Although Theorem 1 is based on the LFT uncertainty modeling approach [6], [8], [24], it is expected that alternative results can be derived based on the polytopic uncertainty modeling approach [1], [2], [11], [12] or the robust LMI approach [19], [20], by appropriately changing the condition "if $\mathcal{G}$ satisfies (4) and (5) for $\gamma > \gamma_0$" so that corresponding robustness property is guaranteed.

Theorem 1 guarantees that in principle one can find a set of (X_1, X_2, h_s) satisfying Condition 1 in a finite step if one exists. A concrete algorithm will be provided in the next subsection, and the outline is as follows: Once $\mathcal{G}$ is fixed, it is enough to consider $\{(h_i + h_{i+1})/2\}$ as a set of candidates of h_s, and one could try all the elements in turn. Hence one should clarify how to find a candidate of $(\hat{X}_1, \hat{X}_2)$ for a given candidate of h_s. For the purpose let us consider the following optimization problem:

Problem 2 (OPβ): Given finite sets $\mathcal{G}_1, \mathcal{G}_2 \subset [h_\ell, h_u]$. Maximize β over (X_1, X_2, β) subject to

$$\begin{cases} 0 < X_1 = X_1^T \le I, \quad 0 < X_2 = X_2^T \le I, \\ \Phi_h X_1 \Phi_h^T - X_j \le -\beta I, \quad \forall j \in \{1, 2\}, h \in \mathcal{G}_1, \\ \Phi_h X_2 \Phi_h^T - X_j \le -\beta I, \quad \forall j \in \{1, 2\}, h \in \mathcal{G}_2. \end{cases}$$

The problem OPβ is obviously an LMI optimization problem. For each $k \in \{1, \ldots, N-1\}$, $\mathcal{G}_1$ and $\mathcal{G}_2$ are determined as in (7). By solving OPβ for the fixed $\mathcal{G}_1$ and $\mathcal{G}_2$, the optimizer (X_1, X_2, β) is obtained. Since β is maximized in OPβ, it can play the role of $\hat{\beta}$ in Theorem 1 as long as it is strictly positive, and hence the optimizer (X_1, X_2) can be used as a candidate of $(\hat{X}_1, \hat{X}_2)$.

If the optimized β is not strictly positive for some k, $h_s \in (h_k, h_{k+1})$ does not satisfy Condition 1 by invoking Property 3:

Property 4: Given $\mathcal{G}$ in (3) and $k \in \{1, \ldots, N-1\}$. Suppose that there do not exist $0 < \hat{X}_1 = \hat{X}_1^T \le I$, $0 < \hat{X}_2 = \hat{X}_2^T \le I$, and $\hat{\beta} > 0$ satisfying (9) for all $i, j \in \{1, 2\}$, where $\mathcal{G}_1$ and $\mathcal{G}_2$ are defined in (7). Then there do not exist $X_1 = X_1^T > 0$, $X_2 = X_2 > 0$, and $h_s \in (h_k, h_{k+1})$ satisfying (1) for all $i, j \in \{1, 2\}$.

Let us consider the case of the maximal β of OPβ is strictly positive. One can also check whether the maximizer (X_1, X_2) and $h_s \in (h_k, h_{k+1})$ satisfy Condition 1 by invoking Lemma 2. In particular there exist $\gamma > 0$ and $\sigma > 0$ satisfying $M_h(X_i, X_j, \gamma, \sigma) < 0$ if and only if there exist $\alpha > 0$ and $\delta > 0$ satisfying

$$\begin{bmatrix} \Phi_h & I \\ \Psi_h & 0 \end{bmatrix} \begin{bmatrix} X_i & 0 \\ 0 & \alpha I \end{bmatrix} \begin{bmatrix} \Phi_h & I \\ \Psi_h & 0 \end{bmatrix}^T - \begin{bmatrix} X_j & 0 \\ 0 & \delta I \end{bmatrix} < 0 \quad (11)$$

which is convex in (α, δ). Indeed σ and γ are respectively given by $\sigma = \sqrt{\alpha}$ and $\gamma = \sqrt{\delta/\alpha}$. Thus one can find a candidate of $(\hat{X}_1, \hat{X}_2)$.

Note that $\hat{X}_1$ and $\hat{X}_2$ can be obtained by solving OPβ which contains simultaneous Lyapunov inequality-like LMIs. This is contrast to the algorithm in [6], where LMIs related to (11) are directly solved. Namely one can expect computational complexity reduction in the present algorithm in compare to that in [6]. In addition, the condition (11) will be checked without solving LMIs in the algorithm below.

It is not trivial in general to conclude that no (X_1, X_2, h_s) satisfying Condition 1 exists. Note that β in Condition 2 cannot be obtained a priori and hence one cannot numerically verify if (4) and (5) are satisfied for the β. The following Corollary shows that one can conclude either (a) there is no (X_1, X_2, h_s) satisfying Condition 1 or (b) the decay rate is slower than the specified level, if one cannot find (X_1, X_2, h_s) satisfying Condition 1 by using a fine grid:

Corollary 1: Let $\beta > 0$ and $\mathcal{G}$ in (3) be given. Suppose that $\mathcal{G}$ satisfy (4) and (5) for γ_0 in (10) determined by the given β. If there are no $X_1 = X_1^T > 0$, $X_2 = X_2 > 0$, $k \in \{1, 2, \ldots, N-1\}$, $\gamma > 0$, and $\sigma > 0$ satisfying (6) for all $i, j \in \{1, 2\}$, there do not exist X_1, X_2, and h_s satisfying (8) for the given β and all $i, j \in \{1, 2\}$.

C. Algorithm for Stability Analysis

This subsection provides an algorithm for solving Problem 1 based on the discussions above. The algorithm obtains either one of the following three outcomes in a finite steps:

(a) (X_1, X_2, h_s) satisfying Condition 1
(b) a conclusion that Condition 1 does not hold.
(c) a conclusion that either Condition 1 does not hold or the decay rate is slow in the sense of Corollary 1.

Now we are ready to state the algorithm for stability analysis. We will use the following definition:

$$W_h(X, Y, \sigma) := \frac{1}{\sigma^2}\Psi_h \Big(X - X\Phi_h^T \times \left(\Phi_h X \Phi_h^T - Y + \sigma^2 I\right)^{-1} \Phi_h X \Big) \Psi_h^T.$$

Algorithm 1: Given: T, $0 < \varepsilon_\gamma \ll 1$, $0 < \varepsilon_\theta \ll 1$, $0 < \varepsilon_h \ll 1$, $r \in (0, 1)$;

Step 0 (initialization)

$\mathcal{G}_1 \leftarrow \{h_\ell\}$, $\mathcal{G}_2 \leftarrow \{h_u\}$;

Step 1 (find a candidate of the pair X_1 and X_2)

Solve OPβ;
if $\beta \le 0$ **then**
  $\mathcal{G} \leftarrow \mathcal{G}_1 \cup \mathcal{G}_2$, $k = 1$;
  do
    $\mathcal{G}_1 \leftarrow$ the set of the smallest k elements of $\mathcal{G}$,
    $\mathcal{G}_2 \leftarrow \mathcal{G} \setminus \mathcal{G}_1$,
    solve OPβ;
    $k \leftarrow k + 1$;
  while ($\beta \le 0$ and $k <$ (# of elements in $\mathcal{G}$) $- 1$)
  if $k =$ (# of elements of $\mathcal{G}$) $- 1$ **then**
    Condition 1 does not hold;
    return
  end
end

Step 2 (verification)

$h_{s1} \leftarrow h_\ell$, $h_{s2} \leftarrow h_u$;
repeat

$$\beta_1 := -\max_{j \in \{1,2\}} \lambda_{\max}(\Phi_{h_{s1}} X_1 \Phi_{h_{s1}}^T - X_j);$$

$$\gamma_1 := (1 + \varepsilon_\gamma) \max_{j \in \{1,2\}} \lambda_{\max}^{1/2} \Big\{ W_{h_{s1}}\Big(X_1, X_j, \sqrt{r\beta_1}\Big) \Big\};$$

$$h_{s1} \leftarrow h_{s1} + \theta_U(\gamma_1);$$

until $(\theta_U(\gamma_1) < \varepsilon_\theta(h_u - h_\ell))$

repeat

$$\beta_2 := -\max_{j\in\{1,2\}} \lambda_{\max}(\Phi_{h_{s2}} X_2 \Phi_{h_{s2}}^T - X_j);$$

$$\gamma_2 := (1+\varepsilon_\gamma) \max_{j\in\{1,2\}} \lambda_{\max}^{1/2}\left\{W_{h_{c2}}\left(X_2,\, X_j,\, \sqrt{r\beta_2}\right)\right\};$$

$$h_{c2} \leftarrow h_{c2} - \theta_L(\gamma_2);$$

until $(\theta_L(\gamma_2) < \varepsilon_\theta(h_u - h_\ell))$
if $(h_{s2} \le h_{s1})$ **then**
  $h_s \leftarrow (h_{s1} + h_{s2})/2$;
  $(X_1,\, X_2,\, h_s)$ satisfies Condition 1;
  return
end
if $(|h_i - h_j| \le \varepsilon_h(h_u - h_\ell),\ \forall h_i,\, h_j \in \mathcal{G}_1 \cup \mathcal{G}_2)$ **then**
  No $(X_1,\, X_2,\, h_s)$ found with the specified fineness of $\mathcal{G}$;
  More information from Corollary 1;
  return
end
$\mathcal{G}_1 \leftarrow \mathcal{G}_1 \cup \{h_{s1} + (h_{s2} - h_{s1})/3\}$;
$\mathcal{G}_2 \leftarrow \mathcal{G}_2 \cup \{h_{s2} - (h_{s2} - h_{s1})/3\}$;
goto Step 1

Remark 5: In Step 1, T_d is approximated by a discrete-time switched systems with finite modes and then a switched Lyapunov function is searched for the approximated system. There exists a switched Lyapunov function for T_d only if there exists one for the approximated system. See Property 4.

Remark 6: Step 2 verifies if there exist h_{s1} and h_{s2} such that X_1 and X_2 found in Step 1 satisfy

$$\Phi_h X_i \Phi_h^T - X_j < 0, \quad \forall h \in \mathcal{H}_i,$$

for all $i,\ j \in \{1, 2\}$ and $h_s \in [h_{s2}, h_{s1}]$ by invoking Lemma 2. Note that (6) is verified by checking the maximal eigenvalue of W_h, which is much cheaper than solving LMIs related to (11).

IV. NUMERICAL EXAMPLES

This section demonstrates the effectiveness of the proposed algorithm by numerical examples. Let us consider the following parameters for T:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -3 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$F = -\begin{bmatrix} 1 & 2 & 2 & 1 \end{bmatrix}, \quad h_\ell = 0.100, \quad h_u = 3.293.$$

By checking the quadratic stability of T_d, one can verify that there is no $X = X^* > 0$ satisfying

$$\Phi_{0.32} X \Phi_{0.32}^T - X < 0$$

and

$$\Phi_{3.13} X \Phi_{3.13}^T - X < 0$$

simultaneously. In other words, T_d is not quadratically stable. Hence we emphasize that most of methods in the literature cannot conclude stability of T.

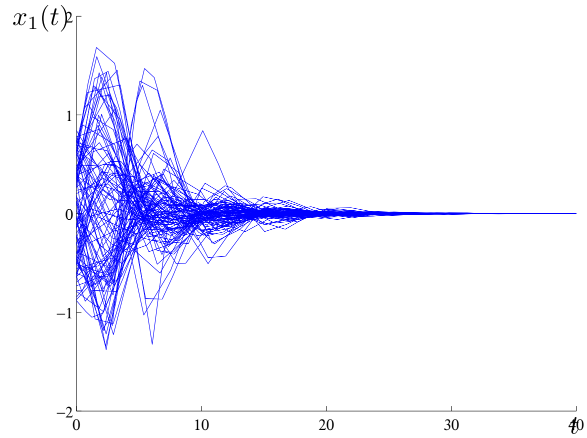

Fig. 3. Initial value responses

We have also computed initial value responses for 100 sets of randomly generated $x(0)$, which is normalized, and $\{\tau_k\}$ with $h_\ell = 0.32$ and $h_u = 3.14$. The results are shown in Fig. 3, where the initial value responses of the first state x_1 are plotted. We observe the convergence of x_1. Similar convergences of all the other states are observed.

On the contrary to the quadratic stability based analysis, the proposed algorithm can find a switched Lyapunov function determined by

$$h_{s2} = 2.3311, \quad h_{s1} = 3.0532,$$

$$X_1 = \begin{bmatrix} 0.3423 & -0.1151 & -0.0741 & -0.0034 \\ -0.1151 & 0.1978 & -0.0688 & -0.0294 \\ -0.0741 & -0.0688 & 0.1738 & -0.0894 \\ -0.0034 & -0.0294 & -0.0894 & 0.2181 \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 0.7566 & -0.3331 & -0.2036 & 0.0623 \\ -0.3331 & 0.4277 & -0.1971 & 0.0654 \\ -0.2036 & -0.1971 & 0.5439 & -0.3131 \\ 0.0623 & 0.0654 & -0.3131 & 0.3031 \end{bmatrix}.$$

This demonstrates the usefulness of the proposed algorithm.

V. CONCLUDING REMARKS

We have developed an algorithm for stability analysis of non-uniformly sampled-data feedback system which constructs a bimodal switched Lyapunov function [3] for the associate discrete-time system. The proposed algorithm constructs a bimodal switched Lyapunov function in a finite step if one exists, and composed of two stages: In the first stage a candidate of $(X_1,\, X_2,\, h_s)$ is obtained by solving LMIs of reasonable dimensions, and it is verified if the candidate satisfies Condition 1 without solving LMIs. Hence the algorithm is tractable.

Open issues include to extend the proposed algorithm to multi-modal case. It is not difficult to generalize Theorem 1 to multi-modal case, but it is not trivial how to extend the whole algorithm.

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APPENDIX

Here we provide a proof of Theorem 1.

The first statement is a direct consequence of Properties 2 and 3. Hence we prove the second statement.

What we should prove is that there exists $\sigma > 0$ satisfying

$$M_h(\hat{X}_i,\, \hat{X}_j,\, \gamma,\, \sigma) < 0 \tag{12}$$

for all $\gamma > \gamma_0$, $h \in \mathcal{G}_i$, and $i,\, j \in \{1,\, 2\}$.

One can verify that (12) is equivalent to

$$\Phi_h \hat{X}_i \Phi_h^T - \hat{X}_j + \sigma^2 I < 0, \tag{13}$$

$$W_h(\hat{X}_i,\, \hat{X}_j,\, \sigma) - \gamma^2 I < 0 \tag{14}$$

where W_h is defined in Section III-C. Since (9) holds, (13) is satisfied if $\sigma^2 \le \hat{\beta}$. Let us take $\sigma = \sqrt{r\beta}$ with $r \in (0,\, 1)$ where β is determined in Condition 2.

Since $r \in (0,\, 1)$, (9) also implies

$$\Phi_h \hat{X}_i \Phi_h^T - \hat{X}_j + r\beta I \le -(1-r)\beta$$

which is equivalent to

$$-(\Phi_h \hat{X}_i \Phi_h^T - \hat{X}_j + r\beta I)^{-1} \le \frac{1}{(1-r)\beta} I.$$

Hence one has

$$\begin{aligned} & W_h(\hat{X}_i,\, \hat{X}_j,\, \sqrt{r\beta}) \\ \le\;& \frac{1}{r\beta}\Psi_h \left(\hat{X}_i + \frac{1}{(1-r)\beta}\hat{X}_i\Phi_h^T\Phi_h\hat{X}_i\right)\Psi_h^T \\ \le\;& \frac{1}{r\beta}\left\|\Psi_h\right\|^2 \left(\left\|\hat{X}_i\right\| + \frac{\left\|\hat{X}_i\right\|^2 \left\|\Phi_h\right\|^2}{(1-r)\beta}\right) I \\ \le\;& \frac{1}{r\beta}\left\|\Psi_h\right\|^2 \left(1 + \frac{\left\|\Phi_h\right\|^2}{(1-r)\beta}\right) I \\ \le\;& \gamma_0^2 I < \gamma^2 I. \end{aligned}$$

Thus (14) is obtained. This completes the proof.

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