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Robust periodic reference tracking by stable uncertain infinite-dimensional linear systems

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***Abstract*— A robust control scheme for tracking of periodic signals, consisting of a finite number of sinusoids, by uncertain exponentially stable infinite dimensional linear systems is presented. The scheme consists in constructing a cascade interconnection of the stable linear system and a partitioning filter and augmenting this cascade system with a simple internal model based filter. The stable system model is presumed to be unknown, but its transfer function gain at the frequencies to be tracked is assumed to be known and non-zero. A theorem guaranteeing the robust stability and performance of this scheme while tracking a sinusoidal reference is proved. The general theorem for tracking periodic signals is stated and can be established analogously. A discussion on quantitatively estimating the robustness of this scheme is presented. The efficacy of the scheme is demonstrated via simulation of an example. The simplicity of the proposed scheme, its quantitatively ascertainable robustness and a virtual lack of modeling requirements make it well suited for industrial applications.**

***Index Terms*—Internal model principle, well-posed linear system, regular linear system, exponential stability.**

I. Introduction

INTERNAL model principle [1] enables tracking of periodic signals with zero steady state error by embedding the generator of the signal into the closed-loop system. This approach has been used in [2]-[6] for linear finite dimensional plants. In these works the plant model is assumed to be known. This paper addresses the tracking of periodic signals consisting of a finite number of sinusoids assuming no knowledge of the plant, other than that it is an exponentially stable regular linear system and its transfer function (TF) gain at the frequencies to be tracked, readily found by experiments, is known and non-zero. Davison [7-9] provides a solution methodology for a similar problem in case of stable finite dimensional plants. This methodology has been extended to exponentially stable regular linear systems (a large subset of well-posed linear system) in [10] (only for step reference), to the class of stable plants with transfer function in Callier-Desoer algebra in [11] and to exponentially stable well-posed linear systems in [12].

The solution technique proposed in the present paper consists in introducing a novel topology obtained by cascading the given stable, infinite-dimensional in general, system $\boldsymbol{P}$ with a stable finite dimensional partitioning filter $\boldsymbol{Q}$, adding positive and negative feedback paths that cancel one another (Fig. 1), and forming an extended loop through a path containing an internal model based filter $\boldsymbol{F}$. The resulting system (Fig. 3) is further referred to as the augmented system, while the system in Fig. 1 is referred to as the unaugmented system. The stability of the augmented system and asymptotic tracking of a periodic reference are guaranteed by choosing $\boldsymbol{Q}$ and $\boldsymbol{F}$ appropriately.

The solution technique in this work differs fundamentally from those in [7-12] in the topology of the internal model introduction. In the present work, a loop augmentation by an internal model based filter is structured so that the difference between the steady state responses of the augmented and unaugmented systems near frequencies to be tracked becomes quantifiable, and away from these frequencies becomes minimal. This enables guaranteeing stability and performance robustness of the augmented system to quantifiable variations in plant TF gains near the frequencies to be tracked, and to large variations away from these frequencies, assuming stability of $\boldsymbol{P}$. Estimates for the quantifiable variations are obtained in Section V. Such estimates of practical value are not provided in [7-12]. Furthermore, the gain based proof technique in the present work renders the effect of the small controller parameter (to be selected in all approaches) transparent, making it intuitive to tune.

This work is motivated by a longstanding motion distortion problem in steel casting mold oscillation systems. The problem is recreated on an industrial scale physical testbed - a servo consisting of a beam with an electro-hydraulic actuator attached at one end and a mass at the other. This servo can be modeled as a system of coupled nonlinear ordinary and linear partial differential equations whose input-output behavior resembles that of a nonlinearly perturbed stable linear system. Simulations and experiments indicate that the control technique in this paper, although developed for linear systems, attains rejection of sinusoidal disturbance in the servo model. Details of this motivating example and the results of controller implementation on the testbed can be found in [13] wherein the plant is assumed to

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be modeled by a nonlinearly perturbed stable linear finite dimensional system.

The organization of this paper is as follows. Section II contains a list of notations. Section III presents the problem setting and contains the problem statement. The control schemes addressing the problem statement are presented in Section IV, while Section V presents the estimates for the robustness of these schemes. Section VI demonstrates the performance of the control schemes on an example. This is followed by a brief conclusion.

II. Notation

$\mathcal{L}(X,Y)$ - Space of bounded linear operators from X to Y. Let $\mathcal{L}(X,X)=\mathcal{L}(X)$.

$L^2([0,\infty),X)$ - Space of square integrable functions from $[0,\infty)$ to X.

$L^2_{loc}([0,\infty),X)$ - Space of locally square integrable functions from $[0,\infty)$ to X.

$D(A)$ / $\rho(A)$ - Domain / resolvent set of an operator A.

$\|\cdot\|_X$ - Norm in space X.

$\mathbb{R}/\mathbb{C}$ - Space of real/complex numbers.

$\mathbb{C}_\alpha$ - $\{s\in\mathbb{C}: \mathrm{real}(s)>\alpha\}$ and $\mathbb{C}^+_\alpha$ - $\{s\in\mathbb{C}: \mathrm{real}(s)\geq\alpha\}$.

H^∞_α - Space of analytic functions from $\mathbb{C}_\alpha$ to $\mathbb{C}$ bounded in the supremum norm. Let $H^\infty_0=H^\infty$.

$\bar{x}$ - Complex conjugate of x.

$L^2_\alpha([0,\infty),X)$ - $\left\{f\in L^2_{loc}([0,\infty),X): \int_0^\infty \|f(t)\|^2_X e^{-2\alpha t}dt<\infty\right\}$.

III. Problem formulation

This work proposes a control scheme to ensure tracking of periodic signals, containing a finite number of sinusoids, by uncertain exponentially stable infinite-dimensional systems belonging to the class of regular linear systems (RLS). The following definitions and background on RLS can be found in [14, 15] and references therein, but are reproduced here to enhance the clarity of presentation.

A well-posed linear system is a linear time-invariant system such that on any finite time interval the operator from the initial state and the input function to the final state and the output function is bounded. A detailed definition can be found in [15]. The input, state, and output spaces considered are Hilbert spaces and the input and output functions are of class L^2_{loc}.

Consider a well-posed linear system Σ with input space U, state space X, and output space Y. Associated with every such Σ is a C_0-semigroup $T(\tau)$ on X which describes the evolution of the state of Σ with zero input function, a control operator $B\in\mathcal{L}(U,X_{-1})$ and a observation operator $C\in\mathcal{L}(X_1,Y)$. Let A be the generator of $T(\tau)$. The Hilbert spaces X_1 and X_{-1} are defined as follows: X_1 is $D(A)$ with the norm $\|x\|_{X_1}=\|(\beta I-A)x\|_X$, where $\beta\in\rho(A)$ is fixed and X_{-1} is the completion of X with respect to the norm $\|x\|_{X_{-1}}=\|(\beta I-A)^{-1}x\|_X$. The semigroup T can be extended to a semigroup on X_{-1} which is isomorphic to T and will be denoted by the same symbol. Using the above notation, the state of Σ at time $\tau\geq 0$ is expressed as

$$x(\tau)=T(\tau)x(0)+\int_0^\tau T(\tau-s)Bu(s)ds,$$

where $x(0)$ is the initial state, $u\in L^2_{loc}([0,\infty),U)$ is the input function and $x(\tau)\in X$, $\forall\tau\geq 0$. Also with $u=0$ and $x(0)\in X_1$, the output of Σ for all $t\geq 0$ is given by

$$y(t)=CT(t)x(0).$$

The Λ-extension of C is defined by

$$C_\Lambda x_0=\lim_{\lambda\to\infty} C\lambda(\lambda I-A)^{-1}x_0$$

with λ real and for all x_0 in the domain

$$D(C_\Lambda)=\{x_0\in X: \text{the limit above exists}\}.$$

The input-to-output operator of any well-posed linear system can be described by a TF which is an operator valued analytic function defined on some right half complex plane and bounded on some right half complex plane. Let $\boldsymbol{G}$ denote the TF of Σ. $\boldsymbol{G}$ is called regular if the following limit exists $\forall v\in U$,

$$Dv=\lim_{\lambda\to\infty} G(\lambda)v,\ \lambda \text{ real}.$$

Then $D\in\mathcal{L}(U,Y)$ is called the feedthrough operator of $\boldsymbol{G}$. If $\boldsymbol{G}$ is regular, Σ is called RLS. If Σ is regular, then

$$G(s)=C_\Lambda(sI-A)^{-1}B+D \qquad (1)$$

for every $s\in\mathbb{C}$ with real part of s sufficiently large. (A,B,C,D) as discussed above are referred to as the generating operators (GOs) of Σ. A RLS Σ is called exponentially stable if the associated C_0-semigroup $T(\tau)$ satisfies $\|T(\tau)\|_{\mathcal{L}(X)}\leq Me^{\beta\tau}$ for all $\tau\geq 0$, for some $M\geq 1$ and $\beta<0$. In this case A is called exponentially stable and $\forall\delta>\beta$, the TF for Σ belongs to H^∞_δ, and is given by (1).

Consider the RLS $\boldsymbol{P}$, with GOs (A_P,B_P,C_P,D_P) with A_P being exponentially stable and let $U=Y=\mathbb{R}$. Let the TF for $\boldsymbol{P}$, $G_P(s)$, satisfy $G_P(-j\tau)=\bar{G}_P(j\tau)$ $\forall\tau\in\mathbb{R}$. Henceforth, in this paper, the linear system $\boldsymbol{P}$ refers to the RLS described above satisfying all the assumptions.

<u>Problem statement</u>: *given the linear system* $\boldsymbol{P}$*, design a controller to ensure that the output* y_p *of* $\boldsymbol{P}$ *tracks a*

reference signal r consisting of a finite number of sinusoids, such that $\left|y_P(t)-r(t)\right| \in L^2_\alpha([0,\infty),\mathbb{R})$ *for some* $\alpha<0$.

A solution to this tracking problem is equally applicable to rejecting similar periodic disturbances. This paper, considers single-input/single-output (SISO) plants. Extension to multi-input/multi-output (MIMO) plants, where for each output an input to be tracked is identified, can be accomplished by choosing an appropriate set of internal model based filters on the basis of the principles expounded in the SISO case. As in [10, 12], due to the generality of the class of linear systems considered, obtaining $\lim_{t\to\infty}\left|y_P(t)-r(t)\right|=0$ is unrealistic.

IV. CONTROLLER DESIGN AND ANALYSIS

This section presents controllers that address the problem statement in Section III. Lemma 1 considers an interconnection, referred to as the unaugmented system, of the given linear system ***P*** with a stable partitioning filter. The stability of the augmented system, obtained via augmenting the unaugmented system with a simple internal model based filter, and the implied tracking of a sinusoidal reference under appropriate choice of the filters are established in Theorem 1. Theorem 2, which extends this result to the tracking of arbitrary periodic signal with a finite number of sinusoids, is stated without proof.

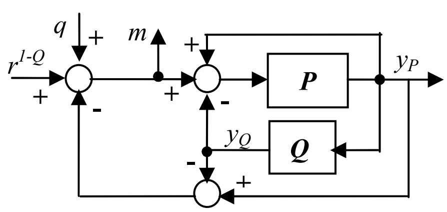

Fig. 1 Unaugmented system obtained via interconnection of *P*, *Q*

Lemma 1: Consider an interconnection of the given RLS ***P*** *and* ***Q*** *shown in Fig. 1, referred to as the unaugmented system, where* ***Q*** *is the stable finite dimensional SISO linear system*

$$\begin{aligned}\dot{x}_Q &= A_Q x_Q + B_Q u\\ y_Q &= C_Q x_Q + D_Q u\end{aligned}.$$

Let $G_P(s)$ *and* $G_Q(s)$ *be the TFs of* ***P*** *and* ***Q****, respectively. Let* $r^{1-Q}+q$ *be the input and* y_P *and m be the outputs of interest where the signal* r^{1-Q} *is defined as the output of the stable system 1-****Q*** *with input r and zero initial conditions. Then the unaugmented system is an exponentially stable RLS and the Laplace transforms for* y_P *and m are given by*

$$\begin{aligned}\hat{y}_P(s) &= G_P(s)\left(\hat{q}(s)+\hat{r}^{1-Q}(s)\right),\\ \hat{m}(s) &= \left(1-G_P(s)+G_Q(s)G_P(s)\right)\left(\hat{q}(s)+\hat{r}^{1-Q}(s)\right)\end{aligned} \tag{2}$$

which are valid $\forall s\in\mathbb{C}_\alpha$ *(for some* $\alpha\in\mathbb{R}$*) on which the r.h.s are well defined. Hence, the input-output representation of the unaugmented system is as shown in Fig. 2.*

Proof: Although the unaugmented system consists of positive and negative feedback loops, these cancel one another. Therefore the unaugmented system is simply a cascade interconnection of two exponentially stable RLS, ***P*** and ***Q***, and hence is an exponentially stable RLS with TFs from $r^{1-Q}+q$ to y_P and y_Q being $G_P(s)$ and $G_Q(s)G_P(s)$, respectively [14]. Since $m(t)=\left(q(t)+r^{1-Q}(t)\right)-y_P(t)+y_Q(t)$ (2) follows. □

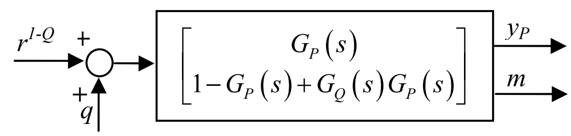

Fig. 2 Input-output representation of the unaugmented system

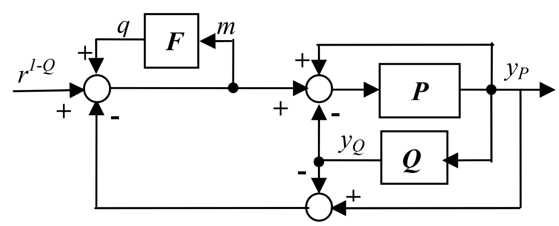

Fig. 3 Augmented system

Let $(A_{UA},B_{UA},C_{UA},D_{UA})$ be the GOs of the unaugmented system with input $r^{1-Q}+q$ and outputs y_P and m, and let $G_{UAm}(s)=1-G_P(s)+G_Q(s)G_P(s)$. The following theorem presents the controller design to ensure the tracking of a single sinusoid.

Theorem 1: Consider the unaugmented system shown in Fig. 1 where ***P****,* ***Q*** *and* r^{1-Q} *are as in Lemma 1. Let* ***Q*** *be chosen such that*

$$\left|\left(1-G_P(j\omega)+G_Q(j\omega)G_P(j\omega)\right)\right|<1 \tag{3}$$

where ω *is the frequency of r, the reference sinusoid to be tracked. Next consider the system shown in Fig. 3, referred to as the augmented system. Let* ***F*** *be the linear stable SISO system with TF*

$$G_F(s)=\frac{2\zeta\omega^2}{s^2+2\zeta\omega s+\omega^2}\frac{s-\omega}{s+\omega} \tag{4}$$

where $0<\zeta\ll 1$ *is a parameter to be chosen. Then,* $\forall\zeta\le\zeta^*$*, with* ζ^* *sufficiently small, the augmented system is an exponentially stable RLS and* y_P *tracks r asymptotically, i.e.* $\left(y_p(t)-r(t)\right)\in L^2_\gamma([0,\infty),\mathbb{R})$ *with* $\gamma<0$.

Remark 1: $G_F(s)$ *is a stable TF whose frequency response at* ω *has gain one and phase lag zero. In Fig. 3 it forms a positive feedback loop to generate poles at* $\pm j\omega$.

Proof: From Lemma 1, the augmented system in Fig. 3 is equivalent to the feedback interconnection of two exponentially stable RLS, the unaugmented system and $[0,\, \boldsymbol{F}]$, as shown in Fig. 4. For this feedback system to be well posed, $1-\left[0\ G_F\right]\times\left[G_P\ G_{UAm}\right]^T=1-G_{UAm}G_F\left(s\right)$ must have a bounded inverse on $\mathbb{C}_\alpha$ for some α. Since the feedthrough operator D_F for $[0,\, \boldsymbol{F}]$ is [0, 0], 1-$D_F D_{UA}$ =1, which is invertible. Moreover, since G_P, G_{UAm}, $G_F \in H^\infty$, if $\left(1-G_{UAm}G_F\right)^{-1}\in H^\infty$, then the augmented system is an exponentially stable RLS [14].

From Lemma 1, A_{UA} is exponentially stable and hence $G_{UAm} \in H^\infty_{\beta_1}$ for some $\beta_1<0$ with $\sup\limits_{s\in\mathbb{C}_{\beta_1}}\left|G_{UAm}\left(s\right)\right|=M<\infty$. Note that all the TFs considered are continuous on the imaginary axis. By (3), $\exists\delta,\,\varepsilon>0$ and the set $I=\left[\omega-\delta,\omega+\delta\right]\cup\left[-\omega-\delta,-\omega+\delta\right]$ such that $\forall\tau\in I$, $\left|G_{UAm}\left(j\tau\right)\right|<1-\varepsilon$. There exists ζ^* sufficiently small such that $\forall\zeta\le\zeta^*$, $\left|G_F\left(j\tau\right)\right|\le 1\big/\sqrt{1-\zeta^2}<1+\varepsilon$, $\forall\tau\in\mathbb{R}$ and $\left|G_F\left(j\tau\right)\right|<\left(1-\varepsilon^2\right)\big/M$, $\forall\tau\in\mathbb{R}-I$. For any such ζ, it follows that $\left|G_{UAm}G_F\left(j\tau\right)\right|<1-\varepsilon^2$, $\forall\tau\in\mathbb{R}$. As $\left|s\right|\to\infty$ with $s\in\mathbb{C}_0^+$, $\left|G_{UAm}G_F\left(s\right)\right|\to 0$ uniformly. Since $G_{UAm}G_F\left(s\right)$ is analytic on $\mathbb{C}_0^+$, by the maximum modulus principle [16], $\left|G_{UAm}G_F\left(s\right)\right|<1-\varepsilon^2$ and consequently $\left|1-G_{UAm}G_F\left(s\right)\right|>\varepsilon^2$ $\forall s\in\mathbb{C}_0^+$. This implies that $\left(1-G_{UAm}G_F\right)^{-1}\in H^\infty$. Therefore the augmented system is an exponentially stable RLS. Hence the TFs from r^{1-Q} to y_P and m, $H_p=G_P\left(1-G_{UAm}G_F\right)^{-1}$ and $H_m=G_{UAm}\left(1-G_{UAm}G_F\right)^{-1}$, respectively, belong to $H^\infty_{\beta_2}$ for some $\beta_2<0$.

Assume all initial conditions are zero. Let $r=A_0\sin\left(\omega t\right)$. Then $\hat{r}\left(s\right)=A_0\omega\big/\left(s^2+\omega^2\right)$ and $\hat{r}^{1-Q}\left(s\right)=\left(1-G_Q\left(s\right)\right)\hat{r}\left(s\right)$. For the error $e_0\left(t\right)=y_p\left(t\right)-r\left(t\right)$,

$$\begin{aligned}\hat{e}_0\left(s\right)&=\hat{y}_P\left(s\right)-\hat{r}\left(s\right)=\left(G_P\left(1-G_Q\right)\left(1-G_{UAm}G_F\right)^{-1}-1\right)\hat{r}\\&=A_0\omega\left(G_F-1\right)G_{UAm}\left(1-G_{UAm}G_F\right)^{-1}\left(s^2+\omega^2\right)^{-1}.\end{aligned}$$

Now, $G_F\left(s\right)-1$ can be written as $G_R\left(s\right)\left(s^2+\omega^2\right)\big/\left(s+1\right)$ where $G_R\left(s\right)$ is a rational, stable, proper TF corresponding to an exponentially stable RLS $\boldsymbol{R}$. Hence

$$\begin{aligned}\hat{e}_0\left(s\right)&=A_0\omega G_R G_{UAm}\left(1-G_{UAm}G_F\right)^{-1}\left(s+1\right)^{-1}\\&=\left[0\ A_0\omega G_R\right]\left[H_P\ \mathrm{H}_m\right]^T\left(s+1\right)^{-1}.\end{aligned}$$

Considering $e_0\left(t\right)$ to be the response of the cascade interconnection of the two exponentially stable RLS, the augmented system and $\left[0\ A_0\omega\mathbf{R}\right]$, to the input e^{-t}, $e_0\left(t\right)\in L^\infty_{\alpha_1}\left(\left[0,\infty\right),\mathbb{R}\right)$ for some $\alpha_1<0$ [15]. Now let $z\left(t\right)$ be the contribution of all the initial conditions to y_P. Then the total error $e\left(t\right)=e_0\left(t\right)+z\left(t\right)$. Since the augmented system is an exponentially stable RLS $z\left(t\right)\in L^\infty_{\alpha_2}\left(\left[0,\infty\right),\mathbb{R}\right)$ for some $\alpha_2<0$ [15]. Hence the error $e\left(t\right)\in L^\infty_\gamma\left(\left[0,\infty\right),\mathbb{R}\right)$ for some $\gamma<0$. □

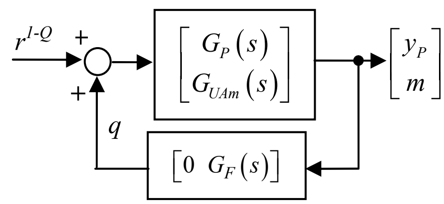

Fig. 4 Input-output representation of the augmented system

Theorem 2 addresses the tracking of periodic signals with finite number of sinusoids.

Theorem 2: Let $\boldsymbol{P}$*, unaugmented system, augmented system and* r^{1-Q} *be as in Theorem 1 and let* $\boldsymbol{Q}$ *be a stable finite dimensional SISO system satisfying*

$$\left|1-G_P\left(j\omega_i\right)+G_Q\left(j\omega_i\right)G_P\left(j\omega_i\right)\right|<1\quad\forall i=1\ldots n,$$

where ω_i *are the frequencies of the sinusoids in r, the reference signal to be tracked. Let* $\boldsymbol{F}$ *be a linear stable system with TF* $G_F\left(s\right)$ *satisfying the following:*

i) 1- $G_F\left(s\right)$ *has zeros at* $\pm j\omega_i$, $\forall i=1\ldots n$,

ii) $\left|G_F\left(j\tau\right)\right|\le 1+\varepsilon$, $\forall\tau\in\mathbb{R}$ *with* $\varepsilon>0$ *and* $\left|G_F\left(j\tau\right)\right|$ *decays in the intervals* $\left(\pm\omega_i-\delta_1,\pm\omega_i+\delta_1\right)$ *such that for* $\tau\notin\left(\pm\omega_i-\delta_1,\pm\omega_i+\delta_1\right)$, $\left|G_F\left(j\tau\right)\right|<\delta_2$ $\forall i=1\ldots n$.

Then, for all sufficiently small values of ε *and* $\delta_{1,2}$*, the augmented system is an exponentially stable RLS and* y_P *tracks r asymptotically, i.e.* $e\left(t\right)\in L^2_\gamma\left(\left[0,\infty\right),\mathbb{R}\right)$ *for some* $\gamma<0$*. Here* $\left(\pm\omega-x,\pm\omega+y\right)$ *stands for* $\left(-\omega-x,-\omega+y\right)\cup\left(+\omega-x,+\omega+y\right)$.

Proof: The proof of Theorem 2 is analogous to that of Theorem 1 and is hence omitted. A filter $G_F\left(s\right)$ satisfying the conditions in Theorem 2 can be obtained by considering sums of filters of the type (4) with some modifications. The example in Section VI shows one such filter when n=2. □

In Theorem 1, the parameter ζ localizes the effect of the filter $\boldsymbol{F}$ near the frequency to be tracked and can be chosen by tuning near zero, since for all sufficiently small values of ζ stability is guaranteed. While applying Theorem 2, $\boldsymbol{F}$ can be similarly parameterized and the parameter can be tuned to reduce ε and $\delta_{1,2}$, thereby localizing $\boldsymbol{F}$ and guaranteeing stability.

V. Robustness Estimates

The controller design of Section IV addresses tracking of periodic references by stable plants with minimal plant information. For controller implementation on a physical system, it is of interest to understand the controller robustness properties. Robustness estimate of the controller designed in Theorem 1 is presented below. Estimates for the design in Theorem 2 can be obtained similarly.

In the following analysis, it is assumed that the perturbed plant $\boldsymbol{P}^D = \boldsymbol{P}+\boldsymbol{D}$, where the perturbation, $\boldsymbol{D}$, is an exponentially stable RLS. Let the TF of $\boldsymbol{P}^D$ be $G_{P^D} = G_P + \Delta_P$. Assume that $\Delta_P(-j\tau) = \bar{\Delta}_P(j\tau) \quad \forall \tau \in \mathbb{R}$. The robustness properties of the controller depend on the choice of the filters $\boldsymbol{Q}$. For this analysis let $\boldsymbol{Q}$ be chosen so as to obtain

$$\left|\left(1 - G_P(j\omega) + G_Q(j\omega)G_P(j\omega)\right)\right| = 0 \qquad (5)$$

where ω is the frequency of the sinusoid to be tracked. From the proof of Theorem 1, stability of the augmented system is guaranteed if $\left|G_{UAm}(j\tau)G_F(j\tau)\right| < 1, \quad \forall \tau \in \mathbb{R}$, which in turn guarantees asymptotic tracking. This is achieved using (3) and appropriately choosing $0 < \zeta \ll 1$ in (4). For the perturbed plant, stability of augmented system and asymptotic tracking are guaranteed if $\forall \tau \in \mathbb{R}$,

$$\left|\begin{pmatrix} 1 - G_P(j\tau) - \Delta_P(j\tau) \\ +G_Q(j\tau)\left(G_P(j\tau) + \Delta_P(j\tau)\right) \end{pmatrix}\right| < 1 / \left|G_F(j\tau)\right|. \qquad (6)$$

From (4), $\forall \tau$ away from $\pm\omega$, i.e. $\tau \notin (\pm\omega - \rho, \pm\omega + \rho)$ for some small ρ (proportional to ζ), $\left|G_F(j\tau)\right|$ is small (again proportional to ζ) and therefore large perturbation $\left|\Delta_P(j\tau)\right|$ at these values of τ will not invalidate the inequality (6) and hence will not cause instability.

For $\tau = \omega$ (or $-\omega$), $G_F(j\omega) = 1$ and hence for stability

$$\left|\left(1 - G_P(j\omega) - \Delta_P(j\omega) + G_Q(j\omega)\left(G_P(j\omega) + \Delta_P(j\omega)\right)\right)\right| < 1.$$

From (5), this is equivalent to $\left|\Delta_P(j\omega)/G_P(j\omega)\right| < 1$. Note that this inequality is optimal in the sense that when $\Delta_P(j\omega) = -G_P(j\omega)$, $\left|\Delta_P(j\omega)/G_P(j\omega)\right| = 1$ and $G_{P^D}(j\omega) = 0$. In this case asymptotic tracking of sinusoid at frequency ω can not be guaranteed by any internal model based feedback technique since G_{P^D} may have multiple zeros at $\pm j\omega$. Let J be the interval $(\omega - \rho, \omega + \rho)$. Note that, since ρ is proportional to $\zeta \ll 1$, it is reasonable to assume that on J, $G_Q(j\tau)$ and $G_P(j\tau)$ do not vary significantly while $\left|G_F(j\tau)\right|$ reduces rapidly, by design. Hence from (5), for η_1 small and $\eta_2 \approx 1$, $\left|\left(1 - G_P(j\tau) + G_Q(j\tau)G_P(j\tau)\right)\right| < \eta_1$ and $\left(\left|1 - G_Q(j\tau)\right| / \left|1 - G_Q(j\omega)\right|\right) < \eta_2$, $\forall \tau \in J$. In this case it can be shown that if $\left|\Delta_P(j\tau)/G_P(j\omega)\right| < (1-\eta_1)/\eta_2 \approx 1$, the inequality (6) is not violated and hence the perturbation does not cause instability of the augmented system. The above argument can be repeated on the interval $(-\omega - \rho, -\omega + \rho)$.

Hence estimates for the admissible magnitude of $\left|\Delta_P(j\tau)\right|$ $\forall \tau \in \mathbb{R}$ can be computed. For τ away from $\pm\omega$, $\left|\Delta_P(j\tau)\right|$ can be large and for τ near ω, $\left|\Delta_P(j\tau)\right|$ must be less than $\left|G_P(j\omega)\right|$. In practice, $G_{P^D}(j\omega)$ can be monitored to recognize scenarios in which stability is at risk.

VI. Example

In this section the controller of Theorem 2 is applied to an exponentially stable RLS to track two different sinusoids. Robustness of this controller is estimated based on the discussion in Section V. Let the TF representation of the RLS be

$$G_P(s) = \frac{e^{-0.5s}}{s+1}. \qquad (7)$$

Let $$r(t) = \begin{cases} 0.2\sin(t) + 2\sin(10t), & 0 < t < 10 \\ 0.4\sin(t) + 4\sin(10t), & t \geq 10 \end{cases}$$

be the reference to be tracked. The jump in r is induced to observe the transient behavior of the augmented loop. The TFs

$$G_Q = \frac{s-0.6108}{s+2} + \frac{s+41.037}{s+6} + \frac{s+279.4685}{s+10} + \frac{s-676.0468}{s+20},$$
$$G_F = \frac{0.1}{s^2+0.1s+1}\frac{1.12s-1.0779}{s+1} + \frac{10}{s^2+s+100}\frac{1.0009s-10.0134}{s+10}$$

satisfy the assumptions in Theorem 2. For $\omega_1 = 1$, $\omega_2 = 10$

$$\left|\left(1 - G_P(j\omega_i) + G_Q(j\omega_i)G_P(j\omega_i)\right)\right| = 0, \qquad i = 1,2 \qquad (8)$$

and G_F satisfies $G_F(j) = G_F(10j) = 1$ and decays to a sufficiently lower value fast enough to guarantee stability of augmented system (Fig. 3). The output and the tracking error obtained by applying the controller of Theorem 2 to plant (7) are shown in Fig. 5. As seen, the controller ensures asymptotic tracking of r by the plant output and guarantees good transient response.

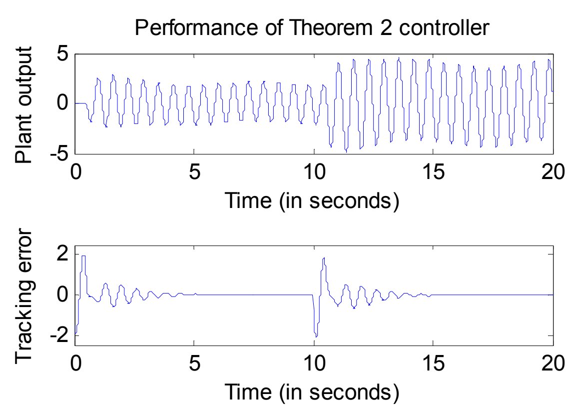

Fig. 5 Output of the plant (8) and tracking error using Theorem 2

Since (8) holds, estimates for robustness of the controller can be obtained on the basis of the discussion in Section V.

It can be shown that, if $\Delta(s)$ is an exponentially stable RLS satisfying $|\Delta(j)| < |G_P(j)| = 0.7071$, $|\Delta(10j)| < |G_P(10j)| = 0.0995$ then the augmented loop in Fig. 3 for the perturbed plant $G_{P^D}(s) = G_P(s) + \Delta(s)$ is stable and asymptotic tracking of r is guaranteed under some reasonable assumptions on the behavior of the TFs (details in Section V). This implies that if delay in the system (7) changes by less than 0.1047, stability and asymptotic tracking will be preserved. This has been observed in simulations.

VII. Conclusion

A control scheme is developed for tracking periodic signals containing a finite number of sinusoids by exponentially stable regular linear systems (RLS). The scheme involves augmenting a cascade RLS and a partitioning filter interconnection by an internal model based filter and requires knowledge of the RLS TF gains only at the frequencies of interest. This scheme permits obtaining quantitative robustness estimates of practical value. Future work involves optimizing the partitioning choices and the internal model based filter.

An equivalent model for the motivating example from steel casting would consist of a linear infinite dimensional beam system coupled to linearized actuator equations, constituting the linear part, with some nonlinear perturbation applied to the linearized actuator equations. To be rigorously applicable to this case, Theorem 1 must be extended to encompass the nonlinearly perturbed RLS and the linear part of the servo must be shown to be an RLS. This extension will be addressed in the future.

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