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# Consensus for multi-agent systems — synchronization and regulation for complex networks

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*Abstract*— **In this paper, we consider three problems, namely, the consensus (synchronization) problem, the model-reference consensus problem, and the regulation of consensus problem, for a network of identical linear time-invariant (LTI) multi-input and multi-output (MIMO) agents. For each problem, we propose a distributed LTI protocol to solve such a problem for a broad class of time-invariant network topologies including not only Laplacian topologies, but a wide family of asymmetric topologies.**

## I. Introduction

A multitude of networks in nature automatically *synchronize*, that is, states of individual network components or agents dynamically evolve toward a common value or trajectory. In complement, control-theorists have recently sought to develop a decentralized protocol that brings a network's components into *consensus*, that is, to deliberately drive the states of network components to a common value or trajectory. Consensus problems have a long history in the computer science community [5]. The control-theoretic approach to consensus - that is, the use of a distributed protocol to synchronize agents' local states in a network to a common value or trajectory - is relatively new, but has been extensively studied in the control community during the past decade and has yielded some advances in e.g., sensor networking [7], [8], [6], [17] and autonomous vehicle control applications [16], [13], [12], [14]. Although this literature is extensive, much of it fundamentally derives from the classical and pioneering work of Wu and Chua [23], [24] in the circuit community, which gives conditions on a network topology for synchronization of coupled nonlinear oscillators. Pogromsky [9], [10] has given a control-theoretic interpretation of the classical synchronization result, that captures the essence of the consensus problem.

The consensus literature can be categorized according to various types of network observation models and internal models for each agent. Regarding the network observation model, the efforts on consensus have focused on Laplacian topologies. Along this line, most work assume that the relative state of the agent and its neighbors is available for each agent, see [7], [8], [6], [13], [11], [12]. A more realistic scenario, that is, the relative output rather than the relative state is available, has been considered in [4]. We refer the reader to [16], [17], [25] for a more general network model. Also, consensus for networks with time-varying topologies has been studied extensively; we refer the reader to Blondel's summary [1], which shows that general results in the time-varying case can be extracted from an early result of Tsitsiklis [19].

Regarding the internal model of each agent, the ongoing research on consensus is progressing toward increasing complexity. For a network of identical LTI agents, the consensus problem has been solved for first-order dynamics, [7], [8], [6], [17], [13], second-order dynamics [12], [16], integrator-chain dynamics [14] and general dynamics [20], [21], [22], [4], [25].

The consensus by itself does not impose any requirements on the consensus trajectory. In many applications, the goal is to design a protocol such that the states of each agent asymptotically approach an, a priori given, reference trajectory, generated by a reference model (virtual leader). This is called the model-reference consensus problem in the literature and has been considered in [14] for identical LTI agents with purely integrator dynamics and in [4] for identical LTI agents with general dynamics.

The main contributions of this paper lie in the following two aspects. First, we extend the results given in [4] for the consensus problem and the model-reference consensus problem to general time-invariant network topologies. Second, we consider the regulation of consensus problem, where the objective is to design a distributed protocol such that the controlled output of each agent tracks the same trajectory, generated by an arbitrary autonomous exosystem.

The remainder of the article is organized as follows. In Section II, we give some basic notations and introduce some useful results on stabilizing on a matrix by scaling. In Section III, we formally formulate the three problems, namely, the consensus problem, the model-reference consensus problem, and the regulation of consensus problem. In Section IV, Section V, and Section VI, we propose a distributed observer-type protocol to solve each problem for general network topologies including Laplacian topologies respectively.

## II. Preliminaries and Notations

Let us first give some notations which we use throughout the paper. For a set of vectors $x_1,\dots,x_n$, we denote by $\mathrm{col}(x_1,\dots,x_n)$ the column vector obtained by stacking the elements of $x_1,\dots,x_n$. $\mathbb{R}^{n\times n}$ and $\mathbb{C}^{n\times n}$ represent the set of $n\times n$ real matrices and complex matrices, respectively. $A^*$

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denotes the conjugate transpose of the matrix $A \in \mathbb{C}^{n\times n}$. $I_N$ denotes the identity matrix of dimension $N \times N$; we sometimes drop the subscript if the dimension is clear in the context. Similarly, $0_N$ represents the square matrix of dimension $N \times N$ with all entries equal to zero. $\mathbf{1}$ denotes the column vector with all entries equal to one. A matrix $A \in \mathbb{C}^{n\times n}$ is Hurwitz stable if all its eigenvalues have negative real parts. $\lambda(A)$ is an eigenvalue of the matrix $A \in \mathbb{C}^{n\times n}$. $\mathrm{Diag}(a_1,\dots,a_n)$ denotes the diagonal matrix with diagonal entries $a_1,\dots,a_n$. In this paper, we use some known results on stabilizing a matrix by scaling. Let us therefore recall a useful result from Fisher and Fuller's paper [2].

*Lemma 1:* There exists a diagonal matrix $D$ such that the eigenvalues of $DG$ are all in the open left half complex plane (or, alternatively, in the open right half complex plane) if there exists a permutation matrix $P_1$ such that all the leading principal minors of $P_1GP_1^{-1}$ are nonzero.

Recently, generalizations of the above Lemma were given in [15]. Also, note that the proofs of Lemma 1 in [2] and the generalizations of Lemma 1 in [15] are constructive.

Due to the space limitation, we have omitted some proofs.

## III. Problem statement

Consider a network of $N$ identical LTI agents of the following form:

$$\begin{cases} \dot{x}_i = Ax_i + Bu_i, \\ y_i = Cx_i, \qquad \forall i \in \{1,\dots,N\}, \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}^n$ is agent $i$'s *local state*, $u_i \in \mathbb{R}^m$ is agent $i$'s *local input*, and $y_i \in \mathbb{R}^q$ is agent $i$'s output.

For the state consensus problem, we want achieve:

$$x_i(t) - x_j(t) \to 0 \qquad \forall i,j \in \{1,2,\dots,N\}, \quad \text{as} \quad t \to \infty.$$

State consensus by itself does not impose any requirements on the consensus trajectory. In other words, we do not impose any conditions on the asymptotic behavior of the state of an individual agent as long as the asymptotic behavior is the same for all agents.

In this paper we are, however, also interested in the model-reference consensus problem. In this problem, the states of each agent asymptotically approach the reference state of the reference model (virtual leader) given by

$$\begin{cases} \dot{x}_r = Ax_r + Bu_r, \\ y_r = Cx_r, \end{cases} \tag{2}$$

where $x_r \in \mathbb{R}^n$ is a reference trajectory, which all the $x_i$ need to approach asymptotically, $u_r \in \mathbb{R}^m$ is the input variable and $y_r \in \mathbb{R}^q$ is the output variable of the reference model, respectively. That is, we want to achieve:

$$x_i(t) - x_r(t) \to 0 \qquad \forall i \in \{1,2,\dots,N\}, \quad \text{as} \quad t \to \infty.$$

Notice that the reference model has the same dynamics as each individual agent.

Finally, we consider the regulation of consensus problem. Consider a network of $N$ identical LTI agents of the following form:

$$\begin{cases} \dot{x}_i = Ax_i + Bu_i, \\ y_i = Cx_i, \\ z_i = C_z x_i, \qquad \forall i \in \{1,\dots,N\}, \end{cases} \tag{3}$$

where the new term $z_i \in \mathbb{R}^p$ is agent $i$'s *local controlled output*.

We want the controlled output $z_i$ of each agent tracks the same trajectory, generated by an arbitrary autonomous exosystem given by

$$\begin{cases} \dot{\omega} = S\omega, \quad \omega(0) = \omega_0 \\ z_r = C_r\omega, \end{cases} \tag{4}$$

where $\omega \in \mathbb{R}^r$ is the state of the exosystem, and $z_r \in \mathbb{R}^p$ is the output of the exosystem, which is the consensus trajectory. That is, we want to achieve:

$$z_i(t) - z_r(t) \to 0 \qquad \forall i \in \{1,2,\dots,N\}, \quad \text{as} \quad t \to \infty$$

for all the initial conditions of the agents and of the exosystem.

### A. Available Information

Clearly in order to resolve the above problems, the information available to each agent plays a crucial role. We assume availability of the following information:

1) The agents *share information*, that is, each agent observes a linear combination of the outputs of the different agents

$$\zeta_i = \sum_{j=1}^{N} g_{ij} y_j, \tag{5}$$

where $g_{ij} \in \mathbb{R}$ are scalars, referred as *observation weights*. The observation weight $g_{ij}$ represents the influence (through sensing or networked communication) of agent $j$'s output on agent $i$'s observation. $g_{ij} \neq 0$ if and only if agent $i$ can obtain information from agent $j$, and $g_{ij} = 0$ if and only if agent $i$ cannot obtain information from agent $j$. The set of *neighbors* of agent $i$ is $N_i \triangleq \{j \,|\, g_{ij} \neq 0\}$. Therefore, agent $i$ can only obtain information from its neighbors. Let us assemble the weights into an $N \times N$ *communication/network topology* $G = [g_{ij}]$, which describes the observation model of the agents. From the above, the communication topology matrix $G$ is represented by a directed graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$, where $\mathscr{V} = \{1,\dots,N\}$ is the set of nodes/agents and $\mathscr{E} \subset \mathscr{V} \times \mathscr{V}$ is the set of weighted edges with weight $g_{ij}$.
2) The protocols *share information*, that is, each agent observes a linear combination of some of the states of the different control protocols, that is

$$\zeta_i^c = \sum_{j=1}^{N} g_{ij}^c H^c x_j^c, \tag{6}$$

where $g_{ij}^c \in \mathbb{R}$ are scalars, $x_j^c$ is the state of the protocol of agent $j$, and $H^c$ is a matrix of appropriate dimension.
3) In the model-reference consensus and the regulation of consensus problems, some agents observe the *output or input of the reference model*. Obviously, at least one agent needs to observe the output of the reference

model to ensure that the consensus trajectory tracks the reference trajectory.

In many examples, the only information sharing among agents that is occurring, is constructed based on relative information of the form $y_i - y_j$. This would result in a Laplacian structure for the topology matrix $G$. However, in other cases one agent might for instance transmit its state or part of its state to its neighbors in which case the topology matrix $G$ will not be Laplacian because at least part of the topology does not consist of relative information.

It actually makes sense that in the communication about the states or part of states of the different control protocols we use the same network topology as in the communication about the outputs of the different agents, that is, $g_{ij}^c = g_{ij}$. Then (6) becomes:

$$\zeta_i^c = \sum_{j=1}^N g_{ij} H^c x_j^c. \tag{7}$$

## IV. The consensus problem

In this section, we design a slight different distributed observer-type protocol based on the proposed protocol in [4] for solving the consensus problem for general network topologies.

Consider a distributed observer-type protocol given by

$$\begin{cases} \dot{\hat{x}}_i = (A+BF)\hat{x}_i + K_i(\sum_{j=1}^N g_{ij} C\hat{x}_j - \zeta_i), \\ u_i = F\hat{x}_i, \end{cases} \tag{8}$$

where $\hat{x}_i \in \mathbb{R}^n$ is the state of the protocol of agent $i$, which is an estimate of the deviation of the state $x_i$ from the consensus trajectory.

Note that we have chosen $H^c = C$ and the state of the protocol $x_j^c \stackrel{\triangle}{=} \hat{x}_j$ in the protocol shared information (7) to make $\sum_{j=1}^N g_{ij} C\hat{x}_j$ available to agent $i$.

Notice that we have chosen an identical state feedback gain matrix $F \in \mathbb{R}^{m\times n}$ for different agents. However, for each agent $i$ we choose

$$K_i = K d_i, \tag{9}$$

where $K \in \mathbb{R}^{n\times q}$ and $d_i$ is a scalar. Therefore, the observation feedback gain matrix $K_i \in \mathbb{R}^{n\times q}$ for different agents differ but only by a scalar factor $d_i$.

We first present the sufficient conditions under which the consensus problem is solvable with the distributed observer-type protocol of the form (8).

*Theorem 1:* Consider a network of $N$ identical LTI agents of the form (1). Assume that

1) the pair $(A,B)$ is stabilizable,
2) the matrix $G$ has only one eigenvalue at the origin, with right eigenvector $\mathbf{1}$,
3) there exist matrices $K$ and $D$, where $D = \mathrm{Diag}(d_1,\dots,d_N)$, such that the matrices $A+\lambda_i KC$, for $i=2,\dots,N$, are Hurwitz stable,[1] where $\lambda_i$ are the nonzero eigenvalues of the matrix $DG$.

Then the distributed observer-type protocol (8) with any matrix $F$ such that the matrix $A+BF$ is Hurwitz stable, and matrices $K$ and $D$ as given in condition 3) solves the consensus problem. Moreover,

$$x_i(t) \to (\omega_0^{\mathrm{T}} \otimes e^{At})\,\mathrm{col}(x_1(0),\dots,x_N(0)), \tag{10}$$
$$\hat{x}_i(t) \to 0, \quad \forall i \in \{1,\dots,N\}, \quad \text{as} \quad t\to\infty, \tag{11}$$

where $\omega_0^{\mathrm{T}}$ is the normalized left eigenvector of the matrix $DG$ associated with the zero eigenvalue.

*Proof:* Define

$$\bar{x}_i = x_i - x_N, \quad \forall i = 1,\dots,N-1.$$

To prove the first part of the theorem, we need to prove the asymptotic stability of the manifold $\bar{x}_1 = \dots = \bar{x}_{N-1} = 0$.

Since condition 2) is satisfied, we get $G\mathbf{1} = 0$, thus it is clear that $DG\mathbf{1} = 0$. With some algebra, we find that the dynamics of the relative state vector equals:

$$\dot{q} = \begin{pmatrix} I_{N-1}\otimes A & I_{N-1}\otimes (BF) \\ -(\overline{DG})\otimes(KC) & I_{N-1}\otimes(A+BF)+(\overline{DG})\otimes(KC) \end{pmatrix} q, \tag{12}$$

where $q = \mathrm{col}(\bar{x},\bar{\hat{x}})$, $\bar{x} = \mathrm{col}(\bar{x},\dots,\bar{x}_{N-1})$, $\bar{\hat{x}} = \mathrm{col}(\bar{\hat{x}}_1,\dots,\bar{\hat{x}}_{N-1})$, while $\bar{\hat{x}}_i = \hat{x}_i - \hat{x}_N$ for $i=1,\dots,N-1$, and $\overline{DG}$ is formed by removing the last row and column from $DG - d_N \mathbf{1} g_N^{\mathrm{T}}$ where $g_N^{\mathrm{T}}$ is the last row of the matrix $G$. Note that the eigenvalues of $\overline{DG}$ are nonzero eigenvalues of $DG$.

Consider a state transformation, $\bar{q} = Tq$, where

$$T = \begin{pmatrix} I_{(N-1)n} & -I_{(N-1)n} \\ 0_{(N-1)n} & I_{(N-1)n} \end{pmatrix}.$$

With some algebra, we obtain

$$\dot{\bar{q}} = \begin{pmatrix} I_{N-1}\otimes A + (\overline{DG})\otimes(KC) & 0 \\ -(\overline{DG})\otimes(KC) & I_{N-1}\otimes(A+BF) \end{pmatrix} \bar{q}. \tag{13}$$

Since the system matrix of the closed-loop dynamics (15) is a block lower triangular matrix, its eigenvalues are the union of the eigenvalues of the matrices $I_{N-1}\otimes A + (\overline{DG})\otimes(KC)$ and $I_{N-1}\otimes(A+BF)$. It is clear that the eigenvalues of the matrix $I_{N-1}\otimes(A+BF)$ are the eigenvalues of the matrix $A+BF$ repeated $N-1$ times. With some algebra, we can show that the eigenvalues of the matrix $I_{N-1}\otimes A + (\overline{DG})\otimes(KC)$ are the union of the eigenvalues of $A+\lambda_i KC$ for all the eigenvalues $\lambda_i$ of the matrix $\overline{DG}$ (that is, all the nonzero eigenvalues of $DG$). Since condition 3) is satisfied and $(A,B)$ is stabilizable, all the poles of the closed-loop system (12) can be placed to the open left-half complex plane, thus, asymptotic stabilization of the closed-loop system (12) is achieved. Hence, consensus is achieved.

Next let us try to figure out the consensus trajectory. Let $x = \mathrm{col}(x_1,\dots,x_N)$, $\hat{x} = \mathrm{col}(\hat{x}_1,\dots,\hat{x}_N)$, and $\xi = \mathrm{col}(x,\hat{x})$. With some algebra, we obtain the closed-loop system

$$\dot{\xi} = \begin{pmatrix} I_N\otimes A & I_N\otimes(BF) \\ -(DG)\otimes(KC) & I_N\otimes(A+BF)+(DG)\otimes(KC) \end{pmatrix} \xi, \tag{14}$$

Consider a state transformation, $\bar{\xi} = T_1\xi$, where

$$T_1 = \begin{pmatrix} I_{Nn} & -I_{Nn} \\ 0_{Nn} & I_{Nn} \end{pmatrix}.$$

[1] We assume without loss of generality that $\lambda_1 = 0$

Clearly, we have, $\bar{\xi} = \text{col}(x-\hat{x},\hat{x})$. With some algebra, we obtain

$$\dot{\bar{\xi}} = \begin{bmatrix} I_N \otimes A + (DG)\otimes(KC) & 0 \\ -(DG)\otimes(KC) & I_N \otimes (A+BF) \end{bmatrix} \bar{\xi}. \quad (15)$$

With some algebra, we obtain

$$\xi(t) = T_1^{-1} \begin{bmatrix} (\mathbf{1}\omega_0^T)\otimes e^{At} & 0 \\ 0 & (\mathbf{1}\omega_0^T)\otimes e^{(A+BF)t} \end{bmatrix} T_1 \xi(0).$$

Since $A+BF$ is Hurwitz stable, we get (10) and (11). ■

Next, we show that condition 3) of Theorem 1 are indeed satisfied for a broad class of internal agent's dynamics and network topologies. In order to present our result, let us first recall the following lemma, which is Proposition 1 of [4] and then we give an alternative proof.

*Lemma 2:* Given the agent dynamics (1), there exists a matrix $K$ such that $A+(x+iy)KC$ is Hurwitz stable for all $x \in [1,\infty)$, and $y \in (-\infty,\infty)$, if and only if the pair $(C,A)$ is detectable.

*Proof:* The necessity is trivial by setting $x = 1$ and $y = 0$.

Now, let us show the sufficiency. Since the pair $(C,A)$ is detectable, we know that the $H_2$ continuous-time algebraic Riccati equation (CARE) defined as:

$$AP + PA^{\mathrm{T}} - PC^{\mathrm{T}}CP + I = 0 \quad (16)$$

has a unique solution $P > 0$.

Now, choose $K = -PC^{\mathrm{T}}$, we get

$$\begin{aligned} &[A+(x+yi)KC]P + P[A+(x+yi)KC]^* \\ &= AP + PA^{\mathrm{T}} - 2xPC^{\mathrm{T}}CP \\ &= AP + PA^{\mathrm{T}} - PC^{\mathrm{T}}CP + (1-2x)PC^{\mathrm{T}}CP < 0, \end{aligned}$$

where the last inequality follows from (16), $PC^{\mathrm{T}}CP \geq 0$, and $x \geq 1$.

Thus, $A+(x+yi)KC$, with $K = -PC^{\mathrm{T}}$, where the matrix $P > 0$ is the solution of (16), is Hurwitz stable. ■

Now, we are ready to present our Theorem for existence of $K$ and $G$ such that condition 3) of Theorem 1 is satisfied.

*Theorem 2:* Assume that

1) the pair $(C,A)$ is detectable,
2) the network topology $G$ has only one eigenvalue at the origin, with right eigenvector $\mathbf{1}$,
3) there exists a permutation matrix $P_1$ such that all the leading principal minors of $P_1GP_1^{-1}$ of size less than $N$ are nonzero.

Then there exist $K = -PC^{\mathrm{T}}$, where the matrix $P > 0$ is the solution of (16), and $D$ such that $A+\lambda_i KC$, for $i = 2,\ldots,N$ are Hurwitz stable for all nonzero eigenvalues $\lambda_i$ of the matrix $DG$.

*Proof:* Since there exists a permutation matrix $P_1$ such that all the leading principal minors of $P_1GP_1^{-1}$ of size less than $N$ are nonzero, from the constructive proof of Lemma 1 given in [2], we can design a diagonal matrix $D$ such that $DG$ has all its eigenvalue in the closed right-half complex plane, except only one eigenvalue at the origin. We can further place all the nonzero eigenvalues of $DG$ with real parts greater than or equal to 1 while the single zero eigenvalue is unchanged by positively scaling the matrix $D$. Since the pair $(C,A)$ is detectable, we can choose $K = -PC^{\mathrm{T}}$, where the matrix $P > 0$ is the solution of (16) as in Lemma 2, the rest of the proof follows from Lemma 2. ■

Let us make several comments regarding to Theorem 2:

- As a special case, it is easy to check conditions 2) and 3) given in Theorem 2 are satisfied when $G$ is a Laplacian matrix. Hence, the above theorem recovers the result in [4].
- The network topology conditions, that is conditions 2) and 3) given in Theorem 2 are satisfied for a broad class of matrices, including a Laplacian topology for the connected network, and a class of matrices known as *D-semistable matrices*, which have a single eigenvalue at the origin with the corresponding right eigenvector $\mathbf{1}$. For the definition of $D$-semistability, please see [16], [3]. It is clear that $D$-semistable matrices includes a wide family of matrices with more general entry sign pattern than the Laplacian matrix, and hence admits consensus control for a wider set of observation capabilities.
- Furthermore, the conditions can be weakened if we use the generalizations of Fisher and Fuller's Theorem given in [15].

## V. The Model-reference consensus problem

In this section, we design a slight different distributed observer-type protocol based on the proposed protocol in [4] for solving the model-reference consensus problem for general network topologies.

In order to solve such a problem, we make one assumption which is the same as the assumption in [4], that is, we assume that only a non-empty subset of agents has access to the output variable $y_r$ of the reference model (2), while all the agents have access to the input variable $u_r$ of the reference model (2). With this assumption, the information available to agent $i$ is

$$\bar{\zeta}_i = \begin{pmatrix} \sum_{j=1}^N g_{ij} y_j \\ e_i(y_i - y_r) \end{pmatrix}, \quad (17)$$

where $e_i = 1$ for the agents which have access to the reference output while $e_i = 0$ otherwise.

Consider a distributed observer-type protocol given by

$$\begin{cases} \dot{\hat{x}}_{e,i} = (A+BF)\hat{x}_{e,i} \\ \qquad\quad + \begin{pmatrix} K_1 d_{i,1} & K_2 d_{i,2} \end{pmatrix} \left( \begin{pmatrix} \sum_{j=1}^N g_{ij} C\hat{x}_{e,j} \\ e_i C\hat{x}_{e,i} \end{pmatrix} - \bar{\zeta}_i \right) \\ u_i \ = F\hat{x}_{e,i} + u_r, \end{cases} \quad (18)$$

where $\hat{x}_{e,i} \in \mathbb{R}^n$ is the state of the protocol of the agent $i$, which is an estimate for $x_{e,i} = x_i - x_r$, $F \in \mathbb{R}^{m\times n}$ is a feedback gain matrix, which is the same for all agents, $K_1, K_2 \in \mathbb{R}^{n\times q}$ and $d_{i,1}, d_{i,2}$ are scalars.

Note that we have chosen $H^c = C$ and the state of the protocol $\hat{x}_j^c \triangleq \hat{x}_{e,j}$ in the protocol shared information (7) to make $\sum_{j=1}^N g_{ij} C\hat{x}_{e,j}$ available to agent $i$.

Note that matrices $F, K_1, K_2$ and $d_{i,1}, d_{i,2}$ for all $i = 1,\ldots,N$ are design parameters of the protocols. In the following theorem, we will give explicit conditions on the network topology under which the model-reference consensus problem can be solved by the distributed observer-type protocol.

*Theorem 3:* Consider a network of $N$ identical LTI agents of the form (1) and the reference model/virtual leader (2). Assume that

1) the pair $(A,B)$ is stabilizable,
2) the pair $(C,A)$ is detectable,
3) network topology $G$ has only one zero eigenvalue, with right eigenvector $\mathbf{1}$,
4) there exists a permutation matrix $P_1$ and a constant $\alpha$ such that all the leading principal minors of $P_1(G+\alpha E)P_1^{-1}$ are nonzero, where $E = \text{Diag}(e_1,\dots,e_N)$.

Then the distributed observer-type protocol (18) with any matrix $F$ such that the matrix $A+BF$ is Hurwitz stable, $K_1 = K_2 = -PC^{\text{T}}$, where the matrix $P>0$ is the solution of (16), $D_1 = D$ and $D_2 = \alpha D$, where $D_1 = \text{Diag}(d_{1,1},\dots,d_{N,1})$ and $D_2 = \text{Diag}(d_{1,2},\dots d_{N,2})$ and $\alpha > 0$, such that all the eigenvalues of $D(G+\alpha E)$ have real parts greater than or equal to 1 solves the model-reference consensus problem.

*Proof:* Omitted. ■

*Remark 1:* Note that if the network topology $G$ satisfies the properties of Theorem 2 then the conditions of Theorem 3 are always satisfied with $\alpha = 1$ provided $E \neq 0$. Hence, Theorem 3 recovers the result in [4] where the network topology $G$ is Laplacian.

## VI. The regulation of consensus problem

The main disadvantage of the result in Section V is that the input to the reference model needs to be known by each agent. Moreover, the reference model (virtual leader) has to have the same dynamics as all the agents. Both of these conditions are quite restrictive. In this section, we consider the regulation of consensus, where the objective is to design a distributed protocol such that the controlled output $z_i$ of each agent (3) tracks the same polynomial or sinusoidal signal or the combination of these generated by an arbitrary autonomous exosystem (4).

In order to solve such a problem, at least some of the agents clearly need to know the output $z_r$ of exosystem (4). Then the information available to agent $i$ is

$$\tilde{\zeta}_i = \begin{pmatrix} \sum_{j=1}^{N} g_{ij} y_j \\ e_i(z_i - z_r) \end{pmatrix}, \tag{19}$$

where $e_i = 1$ for the agents which have access to the output $z_r$ of (4) while $e_i = 0$ otherwise.

Let us first check whether it is even possible to track the output of the exosystem by the controlled output of one individual agent of the form (3) when it has access to both its own state and the state of the exosystem. This will obviously generate a necessary condition for consensus of regulation.

*Lemma 3:* Consider one agent of the form (3) and the exosystem (4). There exists a, possibly nonlinear, feedback $u = f(x_i,\omega)$ such that

$$\lim_{t\to\infty} (z_i(t) - z_r(t)) = 0 \tag{20}$$

for all initial conditions $x_i(0) \in \mathbb{R}^n$ and $\omega(0) \in \mathbb{R}^r$ if and only if there exist matrices $\Pi$ and $\Gamma$ satisfying:

$$\begin{aligned} \Pi S &= A\Pi + B\Gamma, \\ C_r &= C_z \Pi. \end{aligned} \tag{21}$$

This result is well-known and can, for instance, be found in the book [18]. Note that $\Pi$ and $\Gamma$ describe the asymptotic behavior of the state and the input respectively when tracking is achieved, that is, if we have:

$$\Pi\omega(t) - x_i(t) \to 0, \qquad \Gamma\omega(t) - u_i(t) \to 0, \quad \text{as} \quad t\to\infty,$$

then (20) is satisfied. Next, we recall the following known lemma which can also be found in [18].

*Lemma 4:* Consider one agent of the form (3) and the exosystem (4). Assume that (21) is solvable, then these equations have a unique solution $\Pi$ and $\Gamma$ if and only if $(C_z,A,B)$ is right-invertible. Moreover, in that case, a controller $u = f(x_i,\omega)$ is such that (20) is satisfied if and only if

$$\Pi\omega(t) - x_i(t) \to 0, \qquad \Gamma\omega(t) - u_i(t) \to 0 \quad \text{as} \quad t\to\infty.$$

Lemma 4 implies that, for right-invertible agents $(C_z,A,B)$, output consensus, where all the outputs $z_i$ must converge to the output $z_r$, is equivalent to state consensus, where all the states $x_i$ must converge to $\Pi\omega$. However, for non-right-invertible systems, output consensus does not require state consensus.

Our design for regulation of consensus will achieve state consensus. However, as noted before, this is not necessary. On the other hand, we will show that solving the regulation of consensus problem including state consensus requires only weak additional conditions in addition to the necessary condition of solvability of the output regulation equations (21).

Consider a distributed observer-type protocol given by

$$\begin{cases} \dot{\hat{x}}_i &= A\hat{x}_i + Bu_i + \begin{pmatrix} K_1 d_{i,1} & K_2 d_{i,2}\end{pmatrix} \left( \begin{pmatrix} \sum_{j=1}^{N} g_{ij} C\hat{x}_j \\ e_i C_r \hat{\omega}_i \end{pmatrix} - \tilde{\zeta}_i \right), \\ \dot{\hat{\omega}}_i &= S\hat{\omega}_i + F_1\hat{\omega}_i + \begin{pmatrix} L_1 d_{i,1} & L_2 d_{i,2}\end{pmatrix} \left( \begin{pmatrix} \sum_{j=1}^{N} g_{ij} C\hat{x}_j \\ e_i C_r \hat{\omega}_i \end{pmatrix} - \tilde{\zeta}_i \right), \\ u_i &= F_2\hat{x}_i + \Gamma\hat{\omega}_i, \end{cases} \tag{22}$$

where $\hat{x}_i \in \mathbb{R}^n$ and $\hat{\omega}_i \in \mathbb{R}^r$ are states of the protocol, $F_1 \in \mathbb{R}^{r\times r}, F_2 \in \mathbb{R}^{m\times n}$ are same for all agents, $K_1, K_2 \in R^{n\times q}$, $L_1, L_2 \in \mathbb{R}^{r\times q}$ and $d_{i,1}, d_{i,2}$ for $i = 1,\dots,N$ are scalars. Note that the fact that in addition to a differential equation for the state $x_i$ we also need the differential equation for $\hat{\omega}_i$ actually follows directly from the internal model principle [18].

Note that we have chosen $H^c = \begin{pmatrix} C & 0 \end{pmatrix}$ and the state of the protocol $x_j^c \stackrel{\triangle}{=} \text{col}(\hat{x}_j, \hat{\omega}_j)$ in the protocol shared information (7) to make $\sum_{j=1}^{N} g_{ij} C\hat{x}_j$ available to agent $i$.

*Theorem 4:* Consider a network of $N$ identical LTI agents of the form (3) and the consensus trajectory generated by an exosystem (4). Assume that

- the matrix $G$ has only one eigenvalue at the origin, with right eigenvector $\mathbf{1}$,
- there exist matrices $\Pi$ and $\Gamma$ satisfying (21),

- the pair $(A,B)$ is stabilizable,
- there exist matrices $K_{f,1} = \begin{pmatrix} L_1 \\ K_1 \end{pmatrix}$ and $K_{f,2} = \begin{pmatrix} L_2 \\ K_2 \end{pmatrix}$ such that the matrix
$$I_N \otimes A_f + (D_1 G) \otimes (K_{f,1} C_f) + (D_2 E) \otimes (K_{f,2} C_{z,f}) \quad (23)$$
is Hurwitz stable, where
$$A_f = \begin{pmatrix} S & 0 \\ B\Gamma & A \end{pmatrix}, \quad B_f = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix}, \quad C_f = \begin{pmatrix} 0 & C \end{pmatrix},$$
$$C_{z,f} = \begin{pmatrix} 0 & C_z \end{pmatrix}, \quad D_1 = \mathrm{Diag}(d_{1,1}, \ldots, d_{N,1}),$$
$$D_2 = \mathrm{Diag}(d_{1,2}, \ldots, d_{N,2}), \quad \text{and} \quad E = \mathrm{Diag}(e_1, \ldots, e_N).$$

Then the regulation of consensus problem is solved via a distributed observer-type of the form (22) with matrices $\Pi$ and $\Gamma$ satisfying (21), $D_1$, $D_2$, $K_{f,1}$ and $K_{f,2}$ such that (23) is Hurwitz stable and $F_1$ and $F_2$ such that $S+F_1$ and $A+BF_2$ are both Hurwitz stable.

*Proof:* Omitted. ■

In general it is quite hard to verify whether $K_{f,1}$ and $K_{f,2}$ exist such that (23) is Hurwitz stable. However, there is one case where this can be verified quite easily and we have a constructive proof. Moreover in this case, the detectability of the pair $(C\Pi, S)$ is equivalent to the detectability of the pair $(C_r, S)$.

*Theorem 5:* Consider a network of $N$ identical LTI agents of the form (3) and the consensus trajectory generated by an exosystem (4) with $C_z = C$. Let the following conditions be satisfied:

- there exist matrices $\Pi$ and $\Gamma$ satisfying (21),
- matrices $A$ and $S$ have no eigenvalues in common,
- the pairs $(C_r, S)$ and $(C, A)$ are detectable.

Then there exist diagonal matrices $D_1, D_2$ and matrices $K_{f,1}$ and $K_{f,2}$ such that (23) is asymptotically stable if there exists a $\alpha > 0$ and a permutation matrix $P_1$ such that all the leading principal minors of $P_1(G+\alpha E)P_1^{-1}$ are nonzero.

*Proof:* Note that the first conditions imply that $(C_f, A_f)$ is detectable.

Now, we choose $K_{f,2} = K_{f,1}$ and $D_2 = \alpha D_1$. Since $C_{z,f} = C_f$ we find that (23) is equal to:
$$I_N \otimes A_f + [D_1(G+\alpha E)] \otimes (K_{f,1} C_f). \quad (24)$$

Explicit construction of matrices $K_{f,1}$ and $D_1$ to make the matrix (24) Hurwitz stable follows from Theorem 3 and Lemma 2. ■

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