2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Stability analysis of networked control systems with asynchronous sampling and input delay

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***Abstract*— This article proposes a novel approach to assess stability of linear systems with delayed and sampled-data inputs. The paper considers both asynchronous sampling and input delay. the proposed results are based on an extension of a recent research on stability of sampled-data systems to the case where a delay is introduced in the control loop. The proposed method provides easy tractable sufficient conditions for asymptotic stability of sampled-data systems under asynchronous sampling and transmission delays. The period and delay-dependent conditions are expressed using computable linear matrix inequalities. Several examples show the efficiency of the stability criteria.**

I. Introduction

In the last decades, a large attention has been taken to Networked Control Systems (NCS) (see [7], or [22]). Such systems are controlled systems containing several distributed plants which are connected through a communication network. In such applications, one has to check the robustness of a control law with respect to the additional dynamics introduced by the communication networks. Among these dynamics, this article focuses on the influence of transmission delay and asynchronous samplings. The transmission of a data packet through a network can not be achieved instantaneously. Transmission delays are unavoidably introduced. Those delays may lead to instability [18]. It is thus an important issue to develop robust stability criteria with respect to transmission delays and asynchronous samplings.

Sampled-data systems have extensively been studied in the literature [1], [3], [5], [23], [24] and the references therein. It is now reasonable to design controllers which guarantee the robustness of the solutions of the closed-loop system under periodic samplings. However the case of asynchronous samplings still leads to several open problems such that the guarantee of stability whatever the sampling period lying in an interval. Recently, several articles drive the problem of time-varying periods based on a discrete-time approach, [8], [16], [21]. Note that the discrete-time approaches do not fit with the case of uncertain systems or systems with time-varying parameters. Recent papers considered the modelling of continuous-time systems with sampled-data control in the form of continuous-time systems with delayed control input. In [3], a Lyapunov-Krasovskii approach is introduced. Improvements are provided in [5], [13], using the small gain theorem and in [14] based on the analysis of impulsive systems. These approaches are very relevant because they deal with time-varying sampling periods and with uncertain systems (see [3] and [14]). Nevertheless, these sufficient conditions are still conservative. This means that the conditions obtained by continuous-time approaches are not able to guarantee asymptotic stability whereas the system is stable. Recently several authors [2], [19] refine those approaches and obtain tighter conditions.

When transmission delays are introduced in the control loop, the problem becomes more complex. It is indeed well known that delays require a more accurate analysis since the time-delay systems are of infinite dimension [6], [18]. Several articles have been provided to cope with stability of NCS under sampling and transmission delays. In [3], [11], [15], stability conditions of systems under asynchronous sampling and transmission delays are presented. However those conditions are still conservative and require improvements. In the present article, we provide a novel method to assess asymptotic stability of such systems. The conditions are presented as an extension of [20] to the case of time-varying transmission delays. This problem of NCS is hybrid since we consider a continuous time model of the plant and a discrete-time communication. Thus an important improvement addressed inthis article consists in employing the discrete-time Lyapunov theorem to continuous-time modelling of sampled-data systems with delays. The main contribution of this paper is to consider separately the two types of delays.

This article is organized as follows. The next section formulates the problem. Section III exposes the novel stability criteria based on the discrete-time Lyapunov theorem. Then,in Sections IV and V, asymptotic stability criteria sampled-data systems are exposed to cope respectively with the cases of constant and time-varying input delays. Some examples are provided in Section VI which shows the efficiency of the method.

Notations. *The sets $\mathbb{R}^+$, $\mathbb{R}^{n\times n}$ and $\mathbb{S}^n$ denote respectively the set of positive scalar, the set of $n\times n$ matrices and the set of symmetric matrices of $\mathbb{R}^{n\times n}$. The notation $|\cdot|$ refers to the Euclidian norm. For any function f defined over an interval of the form $[a,\ b]$ where $a<b$ are saclars, the notation $\|f\|$ refers to $\sup_{s\in[a,b]}|f(s)|$. The superscript 'T' stands for the*

This research is supported by FeedNetBack project, FP7- ICT-2007-2: http://www.feednetback.eu/.

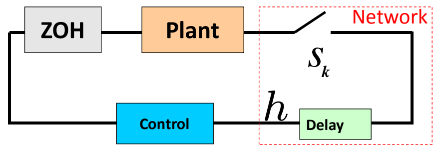

Fig. 1. Control loop of Networked control systems under transmission and sampling delays

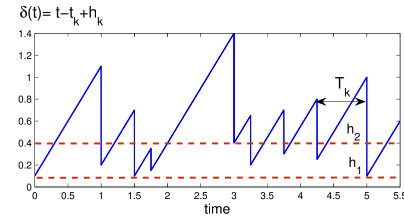

Fig. 2. Examples of a delay generated by a transmission delay h_k bounded by h_1 and h_2 and an asynchronous sampling of periods T_k

matrix transposition. The notation $P > 0$ for $P \in \mathbb{S}^n$ means that P is positive definite. The symbols I and 0 represent the identity and the zero matrices of appropriate dimensions.

II. Problem formulation

A. System definition

Consider the linear system with a sampled and delayed input as shown in Figure 1:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ represent the state variable and the input vector. The matrices A and B are constant and of appropriate dimension. It is assumed that the network induces a time-varying transmission delay h and a sampling of the transmitted signal. The control law is a piecewise-constant static state-feedback of the form:

$$u(t) = Kx(s_k), \; s_k + h(s_k) \leq t < s_{k+1} + h(s_{k+1}),$$

where the gain K in $\mathbb{R}^{n\times m}$ is given and where $0 = s_0 < s_1 < ... < s_k < ...$ represent the sampling instants from the sensors. The sequence of $\{s_k\}_{k\geq 0}$ is strictly increasing and goes to infinity as k increases. The transmission delay $h(t)$ is assumed to be constant or time-varying and such that

$$\forall t, \quad h(t) \in [h_1 \; h_2], \quad -1 < \epsilon_1 \leq \dot{h}(t) \leq \epsilon_2 < 1, \tag{2}$$

where $0 \leq h_1 < h_2$ and $\epsilon_1 < \epsilon_2$. In order to simplify the notation, $h_k = h(s_k)$ is introduced. Denote $t_k = s_k + h_k$. These instants t_k represent the instants where the control input is updated. The condition $\dot{h} > -1$ ensures that the sequence of t_k's is strictly increasing. The closed loop system is thus rewritten as

$$\dot{x}(t) = Ax(t) + A_d x(t_k - h_k), \quad t_k \leq t < t_{k+1}, \tag{3}$$

where $A_d = BK$. Assume that there exists two positive scalars $\mathcal{T}_1 < \mathcal{T}_2$ such that the difference between two successive sampling instants $T_k = s_{k+1} - s_k$ satisfies

$$\forall k \geq 0, \quad 0 \leq \mathcal{T}_1 \leq T_k \leq \mathcal{T}_2. \tag{4}$$

Then the length of the sampling interval in the actuator, $\bar{T}_k = t_{k+1} - t_k$, satisfies

$$\bar{T}_k = t_{k+1} - t_k = s_{k+1} - s_k + h_{k+1} - h_k = T_k + h_{k+1} - h_k.$$

This show the influence of a time-varying delay in the synchrony of the sampling at the actuator. This means that even if the sampling at the sensor is periodic, the sampling at the actuator becomes asynchronous. The chronological order of the control values is ensured by the positivity of $\bar{T}_k$. Finally the following bounds of the sampling period of the actuator is bounded by:

$$\forall k \geq 0 \leq \bar{\mathcal{T}}_1 \leq \bar{T}_k \leq \bar{\mathcal{T}}_2, \tag{5}$$

where $\bar{\mathcal{T}}_1 = \mathcal{T}_1 - (h_2 - h_1)$ and $\bar{\mathcal{T}}_2 = \mathcal{T}_2 + (h_2 - h_1)$. Note that the sampling period at the sensor and at the actuators are equal if the transmission delay is constant. In other words, $\bar{\mathcal{T}}_1 = \mathcal{T}_1$ and $\bar{\mathcal{T}}_2 = \mathcal{T}_2$.

Several authors investigated in guaranteeing stability of such systems. In [3], a continuous-time approach to model sampled-data systems was developed. It allows assimilating sampling effects as the ones of a particular delay. We will further consider (3) as a linear system with uncertain and bounded delay $\delta(t) = t - t_k + h_k$. An example of such delays is presented in Figure 2. In [3], [12] or [11], the authors propose an aggregated delay formulation. They develop stability criteria which take into account the delay δ. However they did not consider the different natures of the transmission and the sampling delay. More especially the additional characteristic of sampled delay which is $\dot{\delta} = 1$ has not been included and thus leads to conservative conditions.

The discrete-time modelling of such systems is obtained by integrating the differential equation (3) over the interval $[t_k, \; t_k + \tau]$, for any τ in $[0, \; \bar{T}_k]$,

$$\begin{array}{l} x(t_k + \tau) = \tilde{A}(\tau)x(t_k) + \tilde{A}_d(\tau)x(t_k - h_k), \\ \tilde{A}(\tau) = e^{A\tau}, \quad \tilde{A}_d(\tau) = \int_0^\tau e^{A(\tau-\theta)}d\theta BK. \end{array} \tag{6}$$

This equality leads naturally to the introduction of a novel notation. Define, for all integer k, the function $\chi_k : [0, \; T_k] \times [-h_k, \; 0] \to \mathbb{R}^n$ such that for all τ in $[0, \; T_k]$ and all θ in $[-h_k, \; 0]$, $\chi_k(\tau, \theta) = x(t_k + \tau + \theta)$. The set $\mathbb{K}$ represents the set of functions defined by χ_k as the set of continuous functions from $\mathcal{I} \times \mathcal{J}$ to $\mathbb{R}^n$, where $\mathcal{I} \subset [0, \; \bar{T}]$ and $\mathcal{J} \subset [-h_2, \; 0]$.

Taking $\tau = t_{k+1} - t_k$ in (6), a recurrence equation is obtained of the form $x(t_{k+1}) = \tilde{A}(T_k)x(t_k) + \tilde{A}_d(T_k)x(t_k - h_k)$. In the particular situation of a delay h_k equal to $t_k - t_{k-i}$ where i is an integer, the model can be rewritten as $x(t_{k+1}) = \tilde{A}(T_k)x(t_k) + \tilde{A}_d(T_k)x(t_{k-i})$. In this situation, several stability conditions can be seen in the literature (see for instance [4]) based on the increment of a Lyapunov function. However, in practice there is no guarantee that the instant $t_k - h_k$ corresponds exactly to a previous sampling instant. Thus there is a need to introduce novel stability conditions to cope with this type of discrete-time systems.

In this paper, the aggregated delay δ representing the effect of the transmission and the sampling delays is split into two parts. In this paper, a novel method to assess stability of systems subject to this type of delay is proposed. The main idea is to consider separately the two types delays. To do so, the stability conditions are based on the discrete-time Lyapunov Theorem and leads to less conservative necessary conditions.

III. MAIN RESULT

This section is motivated by the difference between the discrete and continuous-time Lyapunov Theorems. As the problem of sampled-data systems is at the boundary of the discrete and the continuous-time theories, it is important to put in clear the difference between them. More especially, the main idea of this section consists in developing a novel stability criterion for systems, taken in a continuous-time model, using the discrete-time Lyapunov Theorem.

Theorem 1: Let $V : \mathbb{K} \to \mathbb{R}^+$ be a functional for which there exist real numbers $0 < \mu_1 < \mu_2$ and $p > 0$ such that

$$\forall \chi_k \in \mathbb{K}, \quad \mu_1 |\chi_k(0,\cdot)|^p \leq V(\chi_k(0,\cdot)) \leq \mu_2 |\chi_k(0,\cdot)|^p. \tag{7}$$

The two following statements are equivalent.

(i) The increment of the functional V is strictly negative for all $k \in \mathbb{N}$ and all $\bar{T}_k \in [\bar{\mathcal{T}}_1, \bar{\mathcal{T}}_2]$

$$\Delta V(k) = V(\chi_k(\bar{T}_k,\cdot)) - V(\chi_k(0,\cdot)) < 0;$$

(ii) There exists a continuous functional $\mathcal{V} : \mathbb{R} \times \mathbb{K} \to \mathbb{R}$, which satisfies for all $k \in \mathbb{N}$, and all $\bar{T}_k \in [\bar{\mathcal{T}}_1, \bar{\mathcal{T}}_2]$

$$\mathcal{V}(T_k, \chi_k(\cdot,\cdot)) = \mathcal{V}(0, \chi_k(\cdot,\cdot)), \tag{8}$$

and such that, for all $k > 0$ and for all τ in $[0\ \bar{T}_k]$, the following inequality holds

$$\dot{\mathcal{W}}(\tau, \chi_k) = \frac{d}{d\tau}\left\{[V(\chi_k(\tau,\cdot) + \mathcal{V}(\tau, \chi_k(\cdot,\cdot))]\right\}, \tag{9}$$

Moreover, if one of these two statements is satisfied, the solutions of the system (3) are asymptotically stable.

Proof: Consider a positive integer k, $\bar{T}_k$ satisfying (5) and $\tau \in [0,\ \bar{T}_k]$. Assume (ii) is satisfied. Integrating $\dot{\mathcal{W}}$ over the interval $[0,\ \bar{T}_k[$ and assuming that (8) holds, this directly implies $\Delta V(k) < 0$ and that (i) holds.

Assume now that (i) is satisfied. Inspired by Lemma 2 in [17], consider the functional $\mathcal{V}(\tau, \chi_k) = -V(\chi_k(\tau,\cdot) + \tau/\bar{T}_k \Delta V(k)$. Indeed, $\mathcal{V}$ is a functional since it is expressed with respect to $\Delta V(k)$ which depends on the function χ_k. By simple computations, it is easy to obtain that this functional satisfies (8) and that $\dot{\mathcal{W}}(\tau, \chi_k) = \Delta V(k)/\bar{T}_k$. Thus, $\dot{\mathcal{W}}$ has the same sign as $\Delta V(k)$. This proves the equivalence between (i) and (ii).

From the discrete-time Lyapunov theorem, the equilibrium of the discrete-time system is asymptotically stable.

The end of the proof consists in ensuring that the solutions of the continuous-time system are not diverging within a sampling period. Consider any $\tau \in [0,\ T_k]$. From (7), it follows that $V(\chi_k(\tau,\cdot) < \lambda_2 |\chi_k(\tau,\cdot|$. From (6), the following equality holds

$$x(t_k + \tau) = \tilde{A}(\tau)\chi_k(0,0) + \tilde{A}_d(\tau)\chi_k(0,-h_k),$$

where $\tilde{A}$ and $\tilde{A}_d$ are given in (6). Since those matrix functions are considered on $\tau \in [0,\ \bar{\mathcal{T}}_2]$, is continuous over this interval and are independent of the indice k, it is clear that there exist a constant parameter λ^* such that $|\chi_k(\tau, 0)| \leq \lambda^* \|\chi_k\|$. Since $\|\chi_k\|$ is converging to zero as t_k tends to infinity, it is clear that x also converges to zero as t_k tends to infinity. ■

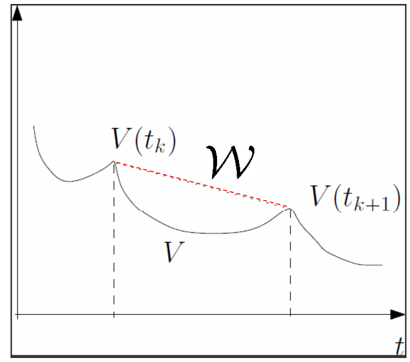

Fig. 3. Illustration of the proof of Theorem 1

A graphical illustration of Theorem 1 is shown in Figure 3. The main idea remains in showing the equivalence between the conditions on the decreasing increment $\Delta V(k) = V(\chi_k(T_k,\cdot)) - V(\chi_k(0,\cdot)) < 0$ and the existence of a continuous functional $\mathcal{W}$ which coincides with the Lyapunov function V at the sampling instants and which is strictly decreasing within all sampling intervals. The main contribution of Theorem 1 is that the introduction of the functional $\mathcal{V}$ allows the Lyapunov-Krasovskii functional V to be locally increasing.

IV. CASE OF CONSTANT INPUT DELAY

In this section, a study on asymptotic stability of the solutions of sampled-data systems with constant input delay is provided. The objective is to design a class of functionals which satisfy the conditions proposed in Theorem 1.

Theorem 2: For given delay $h > 0$ and two positive scalar $\mathcal{T}_1 < \mathcal{T}_2$, assume that there exist $Q > 0$, $R_1 > 0$ and $R_2 > 0 \in \mathbb{S}^n$, $P > 0$, $U > 0$ and $S_1 \in \mathbb{S}^{2n}$ and three matrices S_2 and $X \in \mathbb{R}^{2n \times 2n}$, $Y \in \mathbb{R}^{5n \times 2n}$ that satisfy for $i = 1, 2$

$$\Pi_1(h) + T_i(N_2^T X N_2 + \Pi_2) < 0, \tag{10}$$

$$\begin{bmatrix} \Pi_1(h) - T_i N_2^T X N_2 & T_i Y \\ T_i Y^T & -T_i U \end{bmatrix} < 0, \tag{11}$$

where

$$\begin{aligned} \Pi_1(h) = {} & 2N_1^T P N_0 - N_{12}^T S_1 N_{12} - 2N_2^T S_2 N_{12} \\ & + M_1^T Q M_1 - M_2^T Q M_2 - M_5^T R_1 M_5 \\ & + M_0^T (R_1 + hR_2) M_0 - M_{12}^T R_2 / h M_{12} - 2Y N_{12} \\ \Pi_2 = {} & N_0^T (U N_0 + 2S_1 N_{12} + 2S_2^T N_2), \end{aligned}$$

and

$$M_0 = \left[\begin{array}{ccccc} A & 0 & 0 & A_d & 0 \end{array}\right], M_1 = \left[\begin{array}{ccccc} I & 0 & 0 & 0 & 0 \end{array}\right], \\ M_2 = \left[\begin{array}{ccccc} 0 & I & 0 & 0 & 0 \end{array}\right], \quad M_3 = \left[\begin{array}{ccccc} 0 & 0 & I & 0 & 0 \end{array}\right], \\ M_4 = \left[\begin{array}{ccccc} 0 & 0 & 0 & I & 0 \end{array}\right], \quad M_5 = \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & I \end{array}\right] \\ N_0 = [M_0^T M_5^T]^T, \ N_1 = [M_1^T M_2^T]^T, \ N_2 = [M_3^T M_4^T]^T, \\ M_{12} = M_1 - M_2, \ N_{12} = N_1 - N_2$$

Then the system (3) is asymptotically stable for any asynchronous sampling lying in $[\mathcal{T}_1, \ \mathcal{T}_2]$ and the delay h.

Proof: Consider the classical functional for time-delay systems:

$$V(x) = y^T(t)Py(t) + \int_{t-h}^{t} x^T(s)Qx(s)ds \\ + \int_{t-h}^{t} \dot{x}^T(s)(R_1 + (h-\tau+s)R_2)\dot{x}(s)ds \tag{12}$$

where $y(t) = [x^T(t) \ x^T(t-h)]^T$. Rewriting the previous functional using the notation

$$V(\chi_k) = \bar{\chi}_k^T(\tau)P\bar{\chi}_k(\tau) + \int_{-h}^{0} \chi_k^T(\tau,s)Q\chi_k(\tau,s)ds \\ + \int_{-h}^{0} \dot{\chi}_k^T(\tau,\theta)(R_1 + (h+\theta)R_2)\dot{\chi}_k(\tau,\theta)d\theta \tag{13}$$

where $\bar{\chi}_k(\tau) = [\chi_k^T(\tau,0) \ \chi_k^T(\tau,-h)]^T$. The objective is here to ensure that the variation of the of V between two successive sampling instant is negative. This means that $\Delta V = V(\chi_k(T_k,\cdot)) - V(\chi_k(0,\cdot))$ is definite negative for all positive integer k. For any integer k consider the additional functional

$$\mathcal{V}(\tau,\chi_k) = (T_k - \tau)\zeta_k^T(\tau)[S_1\zeta_k(\tau) + 2S_2\bar{\chi}_k(0)] \\ +(T_k-\tau)\int_0^\tau \dot{\bar{\chi}}_k^T(s)U\dot{\bar{\chi}}_k(s)ds \\ +(T_k-\tau)\tau\bar{\chi}_k^T(0)X\bar{\chi}_k(0), \tag{14}$$

where $\bar{T}_k$ is in the interval $[\bar{\mathcal{T}}_1, \ \bar{\mathcal{T}}_2]$ and $\zeta_k(\tau) = \bar{\chi}_k(\tau) - \bar{\chi}_k(0)$. It is clear from the definition of $\mathcal{V}$ that the condition (8) is satisfied. As suggested in the theorem, no additional constraint is introduced on S_1, S_2, U and X and $\mathcal{V}$ is not necessary positive definite within two sampling instants. This corresponds to the improvement with respect to the previous approaches exposed in [11], [15]. Note that the positivity of U is not required but we will be introduced in the sequel.

The rest of the proof consists in ensuring $\dot{\mathcal{W}} < 0$ over $[0 \ \bar{T}_k[$. The computation of the derivative of $\dot{\mathcal{W}}$ leads to

$$\dot{\mathcal{W}}(\tau,\chi_k) = 2\bar{\chi}_k^T(\tau)P\dot{\bar{\chi}}_k(\tau) + \dot{\chi}_k^T(\tau,0)(R_1+hR_2)\dot{\chi}_k(\tau,0) \\ +\chi_k^T(\tau,0)Q\chi_k(\tau,0) - \chi_k^T(\tau,-h)Q\chi_k(\tau,-h) \\ -\dot{\chi}_k^T(\tau,-h)R_1\dot{\chi}_k(\tau,-h) - \int_{-h}^{0}\dot{\chi}_k^T(\tau,s)R_2\dot{\chi}_k(\tau,s)ds \\ +(T_k-\tau)\dot{\bar{\chi}}_k^T(\tau)[U\dot{\bar{\chi}}_k(\tau) + 2S_1\zeta_k(\tau) + 2S_2\bar{\chi}_k(0)] \\ -\zeta_k^T(\tau)[S_1\zeta_k(\tau) + 2S_2\bar{\chi}_k(0)] \\ -\int_0^\tau \dot{\bar{\chi}}_k^T(s)U\dot{\bar{\chi}}_k(s)ds + (T_k - 2\tau)\bar{\chi}_k^T(0)X\bar{\chi}_k(0). \tag{15}$$

Applying Jensen inequality to the first integral, we have that

$$-\int_{-h}^{0}\dot{\chi}_k^T(\tau,s)R_2\dot{\chi}_k(\tau,s)ds \leq \\ -(\chi_k(\tau,0) - \chi_k(\tau,-h))^T\frac{R_2}{h}(\chi_k(\tau,0) - \chi_k(\tau,-h))$$

Consider the augmented vector $\xi_k(\tau) = [\bar{\chi}_k^T(\tau) \ \bar{\chi}_k^T(0) \ \dot{\chi}_k^T(\tau,-h)]^T$ and a matrix $Y \in \mathbb{R}^{5n\times 2*n}$. The following equality holds

$$2\xi_k^T(\tau)Y\left[\bar{\chi}_k(\tau) - \bar{\chi}_k(0)\right] = \int_0^\tau \left[2\xi_k^T(\tau)Y\dot{\bar{\chi}}_k(s)\right]ds. \tag{16}$$

Since U is assumed to be positive definite and thus non singular, a classical bounding ensures that for all $\tau \in [0, \ T_k]$ and for all $s \in [0, \ \tau]$

$$2\xi_k^T(\tau)Y\dot{\bar{\chi}}_k(s) \leq \xi_k^T(\tau)YU^{-1}Y^T\xi_k(\tau) + \dot{\bar{\chi}}_k^T(s)U\dot{\bar{\chi}}_k(s).$$

Integrating the previous inequality over $[0, \ \tau]$, the following inequality is obtained

$$-\int_0^\tau \dot{\bar{\chi}}_k^T(s)U\dot{\bar{\chi}}_k(s)ds \leq -2\xi_k^T(\tau)Y(\bar{\chi}_k(\tau) - \chi_k(0)) \\ +\tau\xi_k^T(\tau)YU^{-1}Y^T\xi_k(\tau). \tag{17}$$

Noting that

$$\dot{\chi}_k(\tau,0) = A\chi_k(\tau,0) + A_d\chi_k(0,-h) = M_0\xi_k(\tau), \\ \chi_k(\tau,0) = M_1\xi_k(\tau), \quad \chi_k(\tau,-h) = M_2\xi_k(\tau), \\ \chi_k(\tau,0) - \chi_k(\tau,-h) = M_{12}\xi_k(\tau), \quad \bar{\chi}_k(\tau) = N_1\xi_k(\tau), \\ \bar{\chi}_k(0) = N_2\xi_k(\tau), \quad \zeta_k(\tau) = \bar{\chi}_k(\tau) - \chi_k(0) = N_{12}\xi_k(\tau) \\ \dot{\bar{\chi}}_k(\tau) = [(M_0\xi_k(\tau))^T \quad \dot{\chi}_k^T(\tau,-h)]^T = N_0\xi_(\tau)$$

and substituting (17) into (15), the following inequality is obtained for all $\tau \in [0, \ T_k[$

$$\dot{\mathcal{W}}(\tau,\chi_k) \leq \xi_k^T(\tau)[\Pi_1(h) + (T_k-\tau)\Pi_2 \\ +(T_k - 2\tau)N_2^TXN_2 + \tau NU^{-1}N^T]\xi_k(\tau). \tag{18}$$

Based on a convexity argument on τ, the right hand-side term is negative definite if and only if

$$\Pi_1(h) + T_k(\Pi_2 + N_2^TXN_2) < 0,$$

and

$$\Pi_1(h) + T_k(YU^{-1}Y^T - N_2^TXN_2) < 0.$$

Using the same convexity argument on $T_k \in [\mathcal{T}_1, \ \mathcal{T}_2]$, the LMI's (10) and (11) are retrieved. By virtue of Theorem 1, asymptotically stability of the system (3) is guaranteed. ■

Note that the conditions from Theorem 2 include robust stability properties with respect to the input delay h. This means that (10) and (11) require the system to be stable at least for the transmission delay h and $\mathcal{T}_1 = \mathcal{T}_2 = 0$.

V. Case of time-varying input delay

Consider that input delay h is now time-varying and satisfies (2). In the sequel the notations $\eta_k(\tau)$ and $\dot{\eta}_k(\tau)$ stands for the time-varying delay $h(t)$ $(= h(t_k+\tau))$ and its time-varying derivative $\dot{h}(t)$ $(= \dot{h}(t_k+\tau))$. Since the transmission is time-varying, the samplings at the actuator and at the sensor are not the same. Then it the sequel, the stability criteria focuses on the sampling at the actuator denote by $\bar{T}_k \in [\bar{\mathcal{T}}_1, \ \bar{\mathcal{T}}_2]$. The following theorem is proposed:

Theorem 3: Consider a time-varying delay h which satisfies (2) and two positive scalar $0 \leq \bar{\mathcal{T}}_1 < \bar{\mathcal{T}}_2$, assume that there exist $Q > 0$, $R_1 > 0$ and $R_2 > 0 \in \mathbb{S}^n$, $P > 0$, $U > 0$ and $S_1 \in \mathbb{S}^{2n}$ and three matrices S_2 and $X \in \mathbb{R}^{2n\times 2n}$, Y_1 and $Y_2 \in \mathbb{R}^{5n\times 2n}$ that satisfy for $i = 1,2$ and $j = 1,2$

$$\left[\begin{array}{cc} \Pi_3(h_j) + \bar{T}_i(N_2^TXN_2 + \Pi_2) & (1-\epsilon_2)h_jY_2 \\ (1-\epsilon_2)h_jY_2^T & -(1-\epsilon_2)h_jR_2 \end{array}\right] < 0, \tag{19}$$

$$\begin{bmatrix} \Pi_3(h_i) - \bar{T}_i N_2^T X N_2 & \bar{T}_i Y_1 & (1-\epsilon_2) h_j Y_2 \\ \bar{T}_i Y_1^T & -\bar{T}_i U & 0 \\ (1-\epsilon_2) h_j Y_2^T & 0 & -(1-\epsilon_2) h_j R_2 \end{bmatrix} < 0, \quad (20)$$

where Π_2 and the matrices N_i and M_i are given in Theorem 2 and

$$\begin{aligned} \Pi_3(h_j) = & 2N_1^T P N_0 - N_{12}^T S_1 N_{12} - 2N_2^T S_2 N_{12} \\ & + M_1^T Q M_1 - (1-\epsilon_2) M_2^T Q M_2 \\ & + M_0^T (R_1 + h_j R_2) M_0 - 1/(1-\epsilon_1) M_5^T R_1 M_5 \\ & - 2Y_1 N_{12} - 2(1-\epsilon_2) Y_2 M_{12}, \end{aligned}$$

The system (3) is thus asymptotically stable for the sampling period T and the time-varying input delay h.

Proof: Consider the same functional as defined in Theorem 2 but with a time-varying delay $\eta_k(\tau) = h(t_k+\tau)$:

$$\begin{aligned} V(\chi_k) = & \bar{\chi}_k^T(\tau,0) P \bar{\chi}_k(\tau,0) \\ & + \int_{-\eta_k(\tau)}^0 \chi_k^T(\tau,s) Q \chi_k(\tau,s) ds \\ & + \int_{-\eta_k(\tau)}^0 \dot{\chi}_k^T(\tau,\theta)(R_1 + (\eta_k+\theta)R_2)\dot{\chi}_k(\tau,\theta) d\theta \end{aligned} \quad (21)$$

where $z_k(\tau) = [\chi_k^T(\tau)\ \chi_k^T(\tau, -\eta_k(\tau))]^T$ and the additional functional $\mathcal{V}$

$$\begin{aligned} \mathcal{V}(\tau,\chi_k) = & (\bar{T}_k - \tau) \zeta_k^{*T}(t)[S_1 \zeta_k^*(t) + 2S_2 z_k(0)] \\ & + (\bar{T}_k - \tau) \int_{t_k}^t \dot{z}^T(s) U \dot{z}(s) ds \\ & + (\bar{T}_k - \tau)\tau z_k^T(0) X z_k(0), \end{aligned} \quad (22)$$

where $\zeta_k^*(\tau) = z_k(\tau) - z_k(0)$, $\phi_k(\tau) = [z_k^T(\tau)\ z_k^T(0)\ (1-\dot{\eta_k}(\tau))\dot{\chi}_k^T(\tau, -\eta_k(\tau))]^T$. Following the proof of Theorem 2, we prove that the functional $\mathcal{V}$ satisfies (8), is continuous with respect to the time variable and $\mathcal{W}$ and V coincide at all sampling instants. The computation of the derivative of $\dot{\mathcal{W}}$ leads to

$$\begin{aligned} \dot{\mathcal{W}}(\tau,\chi_k) = & 2z_k^T(\tau) P \dot{z}_k(\tau) + \chi_k^T(\tau,0) Q \chi_k(\tau,0) \\ & + \dot{\chi}_k^T(\tau,0) R_1 \dot{\chi}_k(\tau,0) + \eta_k(\tau) \dot{\chi}_k^T(\tau,0) R_2 \dot{\chi}_k(\tau,0) \\ & - (1-\dot{\eta_k}(\tau)) \chi_k^T(\tau,-\eta_k(\tau)) Q \chi_k(\tau,-\eta_k(\tau)) \\ & - (1-\dot{\eta_k}(\tau))[\dot{\chi}_k^T(\tau,-\eta_k(\tau)) R_1 \dot{\chi}_k(\tau,-\eta_k(\tau)) \\ & + \int_{-\eta_k(\tau)}^0 \dot{\chi}_k^T(\tau,s) R_2 \dot{\chi}_k(\tau,s) ds] \\ & + (\bar{T}_k - \tau)\dot{z}_k(\tau)^T [U \dot{z}_k(\tau) + 2S_1 \zeta_k^*(\tau) + 2S_2 z_k(0)] \\ & - \zeta_k^{*T}(\tau)[S_1 \zeta_k^*(\tau) + 2S_2 z_k(0)] - \int_0^\tau \dot{z}_k^T(s) U \dot{z}_k(s) ds \\ & + (\bar{T}_k - 2\tau) z_k^T(0) X z_k(0). \end{aligned} \quad (23)$$

From the definition of the delay $\eta_k(\tau)$ given in (2), and knowing that the matrices Q, R_1 and R_2 are positive definite, we have $-(1-\dot{\eta_k}(\tau)) \leq -(1-\epsilon_2)$ and

$$\begin{aligned} -(1-\dot{\eta}_k(\tau)) & \dot{\chi}_k^T(\tau,-\eta_k(\tau)) R_1 \dot{\chi}_k(\tau,-\eta_k(\tau)) \\ & \leq -1/(1-\dot{\eta}_k(\tau)) \phi_k^T(\tau) M_5^T R_1 M_5 \phi_k(\tau) \\ & \leq -1/(1-\epsilon_1) \phi_k^T(\tau) M_5^T R_1 M_5 \phi_k(\tau) \end{aligned}$$

For any matrix $Y_2 \in \mathbb{R}^{5n\times 2n}$, the first integral of (23) is bounded as follows

$$\begin{aligned} -\int_{-\eta_k(\tau)}^0 & \dot{\chi}_k^T(\tau,s) R_2 \dot{\chi}_k(\tau,s) ds \leq \\ & -2Y_2(\chi_k(\tau,0) - \chi_k(\tau,-\eta_k(\tau))) \\ & + \eta_k(\tau) \phi_k^T(\tau) Y_2 R_2^{-1} Y_2^T \phi_k(\tau), \end{aligned}$$

and for any matrix $Y_1 \in \mathbb{R}^{5n\times 2n}$ and for all $\tau \in [0,\ \bar{T}_k[$, the following inequality is obtained

$$\begin{aligned} -\int_0^\tau \dot{z}_k^T(s) U \dot{z}_k(s) ds \leq & -2\phi_k^T(\tau) Y_1 (z_k(\tau) - z_k(0)) \\ & + \tau \phi_k^T(\tau) Y_1 U^{-1} Y_1^T \phi_k(\tau), \end{aligned} \quad (24)$$

Using the definition of the matrices M_i's and N_i's and using the previous inequality into (23), the following inequality is obtained for all $\tau \in [0,\quad T_k[$

$$\begin{aligned} \dot{\mathcal{W}}(\tau,\chi_k) \leq & \phi_k^T(\tau)[\Pi_1(\eta_k(\tau)) + (\bar{T}_k - \tau)\Pi_2 \\ & + (\bar{T}_k - 2\tau) N_2^T X N_2 + \tau Y_1 U^{-1} Y_1^T \\ & + \eta_k(\tau) Y_2 R_2^{-1} Y_2^T] \phi_k(\tau). \end{aligned}$$

Consider here that η_k and τ are independent variables. Applying a convexity argument on τ and on η_k, the right hand-side term is negative definite and applying the Schur complement, the conditions (19) and (20) are obatined. By virtue of Theorem 1, asymptotically stability of the system (3) is guaranteed. ■

If the minimal bound of the transmission delay h_1 is zero and $\mathcal{T}_1 \leq h_2$, the lower bound of the sampling period at the actuator is $\bar{\mathcal{T}}_1 = 0$ from equation (5). Then the previous theorem can be simplified into the following corollary.

Corollary 1: If $h_1 = 0$, the conditions (19) and (20) become

$$\Pi_1(0) < 0, \quad \Pi_1(0) + \bar{\mathcal{T}}_2(N_2^T X N_2 + \Pi_2) < 0,$$

$$\begin{bmatrix} \Pi_1(h_2) & (1-\epsilon_2) h_2 Y_2 \\ (1-\epsilon_2) h_2 Y_2^T & -(1-\epsilon_2) h_2 R_2 \end{bmatrix} < 0,$$

$$\begin{bmatrix} \Pi_1(h_2) - \bar{\mathcal{T}}_2 N_2^T X N_2 & \bar{\mathcal{T}}_2 Y_1 & (1-\epsilon_2) h_2 Y_2 \\ \bar{\mathcal{T}}_2 Y_1^T & -\bar{\mathcal{T}}_2 U & 0 \\ (1-\epsilon_2) h_2 Y_2^T & 0 & -(1-\epsilon_2) h_2 R_2 \end{bmatrix} < 0,$$

Remark 1: It is clear that discrete-time approaches proposed for instance in [5] or in [10] leads to less conservative stability conditions. However it is possible to extend those stability criteria to the case of polytopic uncertainties. Since all the stability conditions provided in this article are linear with respect to the system matrices A and A_d, it is possible to extend Theorems 2 and 3 to cop with parameter uncertainties. This makes the proposed method relevant with respect compared to discrete-time approaches.

VI. EXAMPLES

- **Example 1** [3], [14]

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix} x(s_k).$$

- **Example 2** [2]

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} x(s_k)$$

The results are summarized in Table I for time-varying sampling and constant delay and in Table II for time-varying sampling and delay.

Consider the case of constant transmission delays indicated obtained by solving Theorem 2 using $\bar{\mathcal{T}}_1 = \mathcal{T}_1 = 0$. It can be seen in Table I that it delivers less conservative results than the existing ones based on a continuous time-approach.

Ex.1 h_2	10^{-3}	0.2	0.4	0.6	0.8	1	1.075
[15]	1.111	0.714	0.469	0.269	0.069	-	-
[12]	1.043	0.846	0.650	0.456	0.262	0.071	-
[11]	1.638	1.063	0.786	0.541	0.301	0.054	-
Th. 2	1.717	1.435	1.149	0.858	0.540	0.167	10^{-3}
Ex.2 h_2	10^{-3}	0.5	1	2	3	4	4.472
[15]	1.278	0.878	-	-	-	-	-
[12]	1.867	1.380	1.042	0.608	0.314	0.090	10^{-3}
[11]	1.970	1.368	0.868	0.212	0.038	-	-
Th. 2	2.624	2.126	1.929	1.530	1.118	0.605	10^{-3}

TABLE I

MAXIMAL SAMPLING PERIOD AT THE ACTUATOR T FOR SEVERAL CONSTANT DELAY USING THEOREM 2 FOR EXAMPLES 1 AND 2.

Ex.1 h_2	10^{-3}	0.2	0.4	0.6	0.8	1	1.075
[15]	1.111	0.804	0.544	0.296	0.070	-	-
[12]	1.042	0.843	0.643	0.443	0.243	0.043	-
[11]	1.638	1.063	0.786	0.541	0.301	0.054	-
Th. 3^1	1.670	1.209	0.994	0.760	0.479	0.095	-
Th. 3^2	1.658	1.084	0.718	0.203	-	-	-
Ex.2 h_2	10^{-3}	0.5	1	2	3	4	4.472
[15]	1.278	0.575	-	-	-	-	-
[12]	1.867	1.368	0.868	-	-	-	-
[11]	1.970	1.368	0.868	0.443	0.212	0.038	-
Th. 3^1	2.407	1.770	1.488	1.010	0.179	-	-
Th. 3^2	2.387	1.590	1.228	-	-	-	-

TABLE II

MAXIMAL SAMPLING PERIOD AT THE ACTUATOR $\bar{T}$ FOR SEVERAL TIME-VARYING TRANSMISSION DELAYS USING THEOREM3 FOR EXAMPLES 1 AND 2.

Concerning the time-varying transmission delay case, we consider $h_1 = 10^{-4}$, $\bar{\mathcal{T}}_1 = 0$, $\epsilon_1 = -0.2$ and the two case $\epsilon_2 = 0.5$ and 0.8, respectively denoted by the superscripts ^{1} and ^{2} in Table II. They show the influence of the delay variation ϵ_2. The influence of ϵ_1 is not presented here because it is not as relevant as the one of ϵ_2. First, we can see that, for small value of ϵ_2, here $\epsilon_2 = 0.5$, the result are less conservative than existing once. However for larger values of ϵ_2, i.e. $\epsilon_2 = 0.8$, the maximum allowable sampling period obtained by solving the Theorem 3 still delivers less conservative results than the existing ones for small values of h_2. However Theorem 3 becomes more conservative when h_2 is larger. Note that in [9], stability conditions based on Theorem 1 have been improved to avoid the dependence on the derivative of the transmission delays, i.e. on the ϵ_i's.

VII. CONCLUSION

An novel analysis of NCS under asynchronous sampling and input delay is provided in this article. This approach is based on the discrete-time Lyapunov Theorem applied to the continuous-time model of the NCS. Tractable conditions are derived to ensure asymptotic stability. The examples show the efficiency of the method and the reduction of the conservatism compared to other results from the literature.

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