2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Collision Avoidance Control with Sensing Uncertainties

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Abstract— **Sensing and localization mechanisms, employed by mobile robots for the detection of obstacles and other nearby agents, may inaccurately estimate the position of obstacles due to noise, delays, and interferences incurred during the detection process. Therefore, it is critical to design collision avoidance strategies that are robust to the presence of measurement errors. In this paper, we present a decentralized, cooperative collision avoidance strategy for a pair of agents considering bounded sensing uncertainties and acceleration constraints. The avoidance control can be appended to any other stable control law (i.e., main control objective) and is active only when the vehicle is close to the other agent. A numerical example is presented that validates the proposed avoidance strategy.**

I. Introduction

Recent advances in intelligent systems and wireless communication have allowed the use of multiple mobile robots in a wide range of commercial, military, and scientific applications [1], [2]. Nowadays, mobile robotic systems are employed in diverse complex tasks such as surveillance [3], space exploration [4], and warehouse management [5]. Despite many applications and progress in distributed mobile robotics, the field still faces plenty of control challenges [1], [2]. One of these challenges is to guarantee collision avoidance between neighboring agents and obstacles at all times independently of sensing uncertainties. Mobile agents, such as unmanned vehicles, typically rely on navigation and localization sensors to estimate the location of nearby agents and obstacles or on wireless communication networks for the broadcast of position coordinates among agents. These sensing mechanisms may inaccurately estimate the position of obstacles and agents as a result of process delays, interferences, and quantization, as well as other sources of measurement errors [6]–[8]. Therefore, it is critical to design avoidance control policies that are robust to such uncertainties.

The design of collision avoidance control laws for the case of accurate position estimation has arguably been well documented in the literature (see [9]–[15] and references therein for examples). This sharply contrast the case of collision-free navigation under sensing uncertainties, which has not been comprehensively studied from a control perspective. Instead, the orthodox solution in the presence of uncertainties has been to improve sensory perception [16]–[18]. Some of the few examples that effectively deal with inaccurate obstacle position estimation include the certainty grid [19] and the occupancy grid [20], both based on probabilistic methods, and a noncooperative strategy for unicycle models [21] based on the concept of reachable sets [12]. However, the aforementioned strategies do not address collision avoidance with moving obstacles. Recently in [22], a noncooperative collision avoidance algorithm based on the velocity occupancy space, a variation of the occupancy grid, was proposed to guarantee the safe navigation of vehicles interacting with dynamic obstacles. Yet, the performance of the avoidance algorithm in the case of time-varying speed obstacles, such as other agents, was not explicitly investigated.

In this paper, we now present a decentralized, cooperative collision avoidance strategy for a team of two agents with bounded sensing uncertainties and limited sensing range. The control formulation relates to the concept of avoidance control presented in [9], [23], yet the control inputs proposed herein are bounded. Advantages of the proposed avoidance strategy include the use of relative position information rather than absolute position and the robustness to time-varying delays, quantization, and other measurement errors. The strategy is also reactive (or real-time), meaning that collision avoidance control inputs are computed on-line as obstacles are detected, rather than computed according to a predetermined (i.e., planned) collision-free trajectory. Furthermore, the avoidance control can be appended to any other stable control law and is exclusively active when the vehicle is close to another agent. This implies that the agent's main objective, such as flocking, trajectory tracking, or set-point regulation, is unaffected when both agents (including obstacles) are safely apart. Finally, a numerical example is presented to illustrate the performance of the proposed avoidance control.

II. Problem Formulation

A. Dynamics of the Two-Agent System

We aim to control a pair of n-degree-of-freedom (DOF) agents with double-integrator dynamics given by

$$\ddot{\mathbf{q}}_i(t) = \mathbf{u}_i(t), \qquad \mathbf{q}_i \in \Re^n, \ \ \mathbf{u}_i \in \mathcal{U}_i \subset \Re^n \qquad (1)$$

where $\mathbf{q}_i$ represents the position and $\mathcal{U}_i$ denotes the set of admissible control inputs $\mathbf{u}_i$ for the first ($i = 1$) and second ($i = 2$) agent. We assume that the magnitudes of the control inputs are radially upper bounded by $\mu_i > 0$, i.e., $\|\mathbf{u}_i(t)\| \leq \mu_i$, $\forall t \geq 0$. The case where limits on the control inputs vary according to the Cartesian coordinates can be similarly covered by means of a coordinate transformation.

This work has been supported by The Boeing Company via the Information Trust Institute, University of Illinois at Urbana-Champaign, and by the National Science Foundation Grant ECCS 07-25433.

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We also assume that each agent can locate the other agent within some margin of error. Specifically, we suppose that the ith agent is able to sense the jth agent as being located at $\hat{\mathbf{q}}_j(t) = \mathbf{q}_j(t) + \mathbf{d}_i(t)$, where $\mathbf{d}_i \in \Re^n$ is a time-varying vector representing the uncertainty on the localization process (e.g., errors due to delays, noise, and quantization) and which is considered to be upper bounded by some positive constant Δ_i, i.e., $\|\mathbf{d}_i(t)\| \leq \Delta_i$, $\forall t \geq 0$.

In what follows, we will omit time dependence of signals except when considered necessary.

B. Control Objective

Our main objective is to develop a control policy that enforces the completion of the agents' main tasks, such as flocking and trajectory tracking, while guaranteeing a safe separation between both agents at all times independently of bounded sensing uncertainties. In general, we would like to define a safety region around each agent and design a control law that keeps the vehicles from entering into each other's safety region. According to this idea, and inspired by the concept of avoidance sets [9], we introduce the following definitions.

We define an *Antitarget Region* (see Fig. 1), $\mathcal{T}$, as the collision zone for both agents, i.e.,

$$\mathcal{T} = \left\{\mathbf{q} : \mathbf{q} \in \Re^{2n}, \|\mathbf{q}_i - \mathbf{q}_j\| \leq r^*\right\}$$

where r^* denotes the minimum safe, separation distance between both vehicles and $\mathbf{q} = [\mathbf{q}_i^T, \mathbf{q}_j^T]^T$. Similarly, we define an *Avoidance Region*, Ω, as the zone for which the two agents are not allowed to enter at any given time. Mathematically,

$$\Omega = \left\{\mathbf{q} : \mathbf{q} \in \Re^{2n}, \|\mathbf{q}_i - \mathbf{q}_j\| \leq r\right\}$$

where $r \geq r^*$ is the desired minimum separation distance between both agents. Note that if we design a control policy such that $\mathbf{q}_1$ and $\mathbf{q}_2$ avoid Ω, then we have that they must also avoid $\mathcal{T}$.

Now, consider the dynamic limitations of the ith agent. Since its control inputs and acceleration components are bounded, a control policy aimed to avoid Ω needs to be implemented with enough anticipation, such that the ith vehicle has sufficient time to decelerate and prevent a collision. Consequently, we define a *Conflict Region*, $\mathcal{W}_i$, as

$$\mathcal{W}_i = \left\{\mathbf{q} : \mathbf{q} \in \Re^{2n}, r < \|\mathbf{q}_i - \mathbf{q}_j\| \leq \bar{r}_i\right\}$$

where $\bar{r}_i > r$ is a lower bound on the distance that the ith agent can come from the other agent and still be able to decelerate and avoid Ω. Thus any avoidance strategy for the ith agent must take effect as soon as $\mathbf{q}_i$ and $\mathbf{q}_j$ enter $\mathcal{W}_i$.

Finally, in order for the problem to be well-defined, it is assumed that $\mathcal{W}_i$ lies within the *Detection Region*, $\mathcal{D}_i$, of the ith agent, defined as

$$\mathcal{D}_i = \left\{\mathbf{q} : \mathbf{q} \in \Re^{2n}, \|\mathbf{q}_i - \mathbf{q}_j\| \leq R_i\right\}$$

where $R_i > \bar{r}_i$ is the detection radius. That is, the ith agent can detect any obstacle or agent inside the Detection Region.

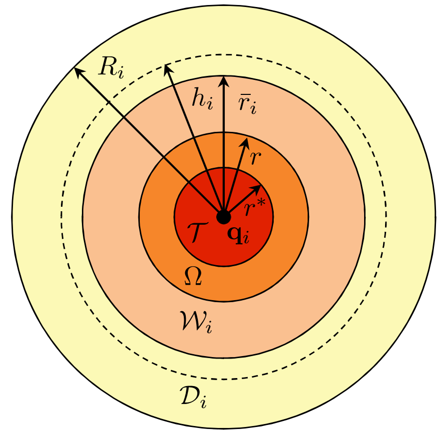

Fig. 1. Antitarget ($\mathcal{T}$), Avoidance (Ω), Conflict ($\mathcal{W}_i$), and Detection ($\mathcal{D}_i$) Regions for the ith agent.

In addition, note that whereas $\mathcal{T}$ and Ω are equal for both agents, $\mathcal{W}_i$ and $\mathcal{D}_i$ can differ.

According to the above definitions, we can state the control objective as follows. Given Δ_1, Δ_2 and $\mathcal{T}$, design control inputs $\mathbf{u}_1(t)$ and $\mathbf{u}_2(t)$ such that $[\mathbf{q}_1^T(t), \mathbf{q}_2^T(t)]^T \notin \Omega$ for all $t \geq 0$, where $\Omega \supseteq \mathcal{T}$.

III. Collision Avoidance Control

In order to achieve our control objective, we propose the use of the following control input:

$$\mathbf{u}_i = \left(1 - \frac{\|\mathbf{u}_i^a\|}{\bar{\mu}_i}\right)\mathbf{u}_i^o + \mathbf{u}_i^a - k_i\dot{\mathbf{q}}_i \tag{2}$$

where $\mathbf{u}_i^o \in \Re^n$ represents a known objective control law satisfying the constraint $\|\mathbf{u}_i^o(t)\| \leq \bar{\mu}_i$ for all $t \geq 0$ and $\bar{\mu}_i = \frac{1}{2}\mu_i$, and where k_i is a positive constant given by

$$k_i = \frac{\bar{\mu}_i}{\eta_i}, \qquad \text{for some } \eta_i > 0.$$

The objective control law $\mathbf{u}_i^o$ is designed such that the ith agent can accomplish its primary task, whereas the term $k_i\dot{\mathbf{q}}_i$ is injected into the system to regulate the maximum velocity of the agent, as will be shown at the end of this section. The control term $\mathbf{u}_i^a \in \Re^n$ is the avoidance control input designed to guarantee collision-free trajectories. It is computed according to

$$\mathbf{u}_i^a = -\frac{\partial V_{ij}^a(\mathbf{q}_i, \hat{\mathbf{q}}_j)}{\partial \mathbf{q}_i}^T \tag{3}$$

where V_{ij}^a, called the avoidance function [23], is given by

$$V_{ij}^a = \begin{cases} \Gamma_i\left(\min\left\{0, \dfrac{\|\hat{\mathbf{q}}_{ij}\|^2 - R_i^2}{\|\hat{\mathbf{q}}_{ij}\|^2 - r^2}\right\}\right)^2, & \text{if } \|\hat{\mathbf{q}}_{ij}\| \geq h_i \\ -\bar{\mu}_i\|\hat{\mathbf{q}}_{ij}\| + c_i, & \text{otherwise} \end{cases} \tag{4}$$

for $\hat{\mathbf{q}}_{ij} = \mathbf{q}_i - \hat{\mathbf{q}}_j$, $h_i = \bar{r}_i + \Delta_i$, and

$$\Gamma_i = \frac{\bar{\mu}_i\left(h_i^2 - r^2\right)^3}{4h_i(R_i^2 - h_i^2)(R_i^2 - r^2)}, \quad c_i = \Gamma_i\frac{(h_i^2 - R_i^2)^2}{(h_i^2 - r^2)^2} + \bar{\mu}_i h_i.$$

The reader can verify that V_{ij}^a is non-negative, continuously differentiable (except at $\|\hat{\mathbf{q}}_{ij}\| = 0$), and that $\mathbf{u}_i^a$ reduces to

$$\mathbf{u}_i^a = \begin{cases} \mathbf{0}, & \text{if } \|\hat{\mathbf{q}}_{ij}\| \geq R_i \\ \dfrac{K_i^a(R_i^2 - \|\hat{\mathbf{q}}_{ij}\|^2)}{(\|\hat{\mathbf{q}}_{ij}\|^2 - r^2)^3}\hat{\mathbf{q}}_{ij}, & \text{if } h_i \leq \|\hat{\mathbf{q}}_{ij}\| < R_i \\ \bar{\mu}_i \dfrac{\hat{\mathbf{q}}_{ij}}{\|\hat{\mathbf{q}}_{ij}\|}, & \text{if } 0 < \|\hat{\mathbf{q}}_{ij}\| < h_i \\ \text{not defined}, & \text{if } \|\hat{\mathbf{q}}_{ij}\| = 0 \end{cases} \tag{5}$$

where $K_i^a = 4\Gamma_i(R_i^2 - r^2)$. Note that in contrast to the unboundedness of the avoidance functions and control inputs in [23], (4) and (5) are bounded by c_i and $\bar{\mu}_i$, respectively. Similarly, the overall control input $\mathbf{u}_i$ can be shown to be bounded by μ_i.

We now prove that the control law in (2) guarantees boundedness of the velocities if $k_i > 0$.

Lemma 3.1: Consider the system in (1) with control law (2) and (5). Let $k_i = \bar{\mu}_i/\eta_i$ for $\bar{\mu}_i = \frac{\mu_i}{2}, \eta_i > 0$. Then, for all initial conditions satisfying $\|\dot{\mathbf{q}}_i(0)\| \leq \eta_i$, we have that $\|\dot{\mathbf{q}}_i(t)\| \leq \eta_i \ \forall t \geq 0$.

Proof: Consider the following Lyapunov function

$$V_\eta = \frac{1}{2}\|\dot{\mathbf{q}}_i\|^2.$$

Taking its time derivative we obtain that

$$\dot{V}_\eta = \dot{\mathbf{q}}_i^T \ddot{\mathbf{q}}_i \leq \|\dot{\mathbf{q}}_i\| \bar{\mu}_i - k_i \|\dot{\mathbf{q}}_i\|^2 = \|\dot{\mathbf{q}}_i\| (\bar{\mu}_i - k_i \|\dot{\mathbf{q}}_i\|).$$

Since $\dot{V}_\eta$ is negative for all $\|\dot{\mathbf{q}}_i\| > \eta_i$, we can conclude that the velocity solutions of (1) are bounded by η_i. ■

IV. Cooperative Collision Avoidance under Bounded Sensing Uncertainties

In this section we present the main results of this paper. In general, we show that the proposed control law enforces collision-free trajectories for the system in (1) given that the design parameters r and $\bar{r}_i$ satisfy a set of inequality constraints. We start the analysis with the following lemma, which will aid us to show that if the ith vehicle has control input given by (2) and (5), then it will try to evade the other agent.

Lemma 4.1: Consider the two-agent system in (1). Assume that the ith agent has control input given by (2) and (5) for $k_i = \bar{\mu}_i/\eta_i$ and $\eta_i > 0$. Define $\beta_{ij}(t) = (\mathbf{q}_i(t) - \mathbf{q}_j(t))^T\dot{\mathbf{q}}_i(t)$, $\theta_i \in \left(0, \sin^{-1}\left(\frac{\sqrt{r_\epsilon^2 - \Delta_i^2}}{r_\epsilon}\right)\right)$, and $\delta_i = \frac{\theta_i r_\epsilon}{\eta_i + \eta_j}$, where $r_\epsilon \in (r, \bar{r}_i]$ and $r > \Delta_i$. Let $t_0 \leq t_f - \delta_i$ and suppose that $\|\dot{\mathbf{q}}_i(t_0)\| \leq \eta_i$, $\|\dot{\mathbf{q}}_j(t)\| \leq \eta_j$ for some $\eta_j \geq 0$, $\|\mathbf{d}_i(t)\| \leq \Delta_i$, and $\|\mathbf{q}_i(t) - \mathbf{q}_j(t)\| \in [r_\epsilon, \bar{r}_i]$ $\forall\, t \in [t_0, t_f]$. Then, $\beta_{ij}(t_f)$ is bounded from below by (6), where $\omega_{ij} = -\frac{\eta_i + \eta_j}{r_\epsilon}$.

Proof: To simplify the notation, let $t_\delta = t_f - \delta_i$ and $\mathbf{q}_{ij}(t) = \mathbf{q}_i(t) - \mathbf{q}_j(t)$. From the assumption that $[\mathbf{q}_i^T(t), \mathbf{q}_j^T(t)]^T \in \mathcal{W}_i \ \forall t \in [t_0, t_f]$ we have that the velocity solution for the ith agent can be computed as

$$\dot{\mathbf{q}}_i(t_f) = e^{-k_i\delta_i}\dot{\mathbf{q}}_i(t_\delta) + \int_{t_\delta}^{t_f} e^{-k_i(t_f - \tau)}\mathbf{u}_i^a(\tau)d\tau.$$

Therefore,

$$\begin{aligned}\beta_{ij}(t_f) &= \mathbf{q}_{ij}(t_f)^T e^{-k_i\delta_i}\dot{\mathbf{q}}_i(t_\delta) \\ &\quad + \mathbf{q}_{ij}(t_f)^T \int_{t_\delta}^{t_f} e^{-k_i(t_f-\tau)}\bar{\mu}_i \frac{\hat{\mathbf{q}}_{ij}(\tau)}{\|\hat{\mathbf{q}}_{ij}(\tau)\|}d\tau \\ &\geq -\eta_i \|\mathbf{q}_{ij}(t_f)\| e^{-k_i\delta_i} \\ &\quad + \bar{\mu}_i \|\mathbf{q}_{ij}(t_f)\| \int_{t_\delta}^{t_f} e^{-k_i(t_f-\tau)} \frac{\mathbf{q}_{ij}(t_f)^T\hat{\mathbf{q}}_{ij}(\tau)}{\|\mathbf{q}_{ij}(t_f)\|\,\|\hat{\mathbf{q}}_{ij}(\tau)\|}d\tau \\ &= -\eta_i \|\mathbf{q}_{ij}(t_f)\| e^{-k_i\delta_i} \\ &\quad + \bar{\mu}_i \|\mathbf{q}_{ij}(t_f)\| \int_{t_\delta}^{t_f} e^{-k_i(t_f-\tau)} \cos\phi_{ij}(\tau)d\tau \end{aligned} \tag{7}$$

where $\phi_{ij}(\tau)$ defines the angle between $\mathbf{q}_{ij}(t_f)$ and $\hat{\mathbf{q}}_{ij}(\tau)$ for $\tau \in [t_\delta, t_f]$ and where we used the fact that $\|\dot{\mathbf{q}}_i(t)\| \leq \eta_i \ \forall t$ (due to Lemma 3.1). Now, our first objective in developing the proof is to compute a lower bound on $\int_{t_\delta}^{t_f} e^{-k_i(t_f-\tau)}\cos\phi_{ij}(\tau)d\tau$. In order to do so, we would like to first find an upper bound on $\phi_{ij}(\tau)$ at every instance of time τ. That is, we would like to define a function $\overline{\phi}_{ij}(\tau)$ such that $\phi_{ij}(\tau) \leq \overline{\phi}_{ij}(\tau)$ for all $\tau \in [t_\delta, t_f]$. Therefore, let us consider the illustration in Fig. 2. Observe that $\phi_{ij}(\tau)$ is always upper bounded by the summation of the angle between $\mathbf{q}_{ij}(t_f)$ and $\mathbf{q}_{ij}(\tau)$ and the angle between $\mathbf{q}_{ij}(\tau)$ and $\hat{\mathbf{q}}_{ij}(\tau)$, denoted as $\vartheta_{ij}(\tau)$ and $\varphi_{ij}(\tau)$, respectively, i.e., $\|\phi_{ij}(\tau)\| \leq \|\vartheta_{ij}(\tau)\| + \|\varphi_{ij}(\tau)\|$. Hence, a suitable function would be

$$\overline{\phi}_{ij}(\tau) = \underbrace{\sup_{t\in[t_\delta,\tau]} \|\vartheta_{ij}(t)\|}_{\overline{\vartheta}_{ij}(\tau)} + \underbrace{\sup_{t\in[t_\delta,\tau]} \|\varphi_{ij}(t)\|}_{\overline{\varphi}_{ij}(\tau)}$$

where $\overline{\vartheta}_{ij}(\tau)$ and $\overline{\varphi}_{ij}(\tau)$ are yet to be determined.

Now, consider $\vartheta_{ij}(\tau)$. Since $\|\mathbf{q}_{ij}(\tau)\| \geq r_\epsilon \ \forall\, \tau \in [t_\delta, t_f]$ and the velocities of the agents are bounded, we have that $\vartheta_{ij}(\tau)$ attains its maximum when the agents approach each other at maximum speed along the boundary of Ω_ϵ, where $\Omega_\epsilon = \left\{\mathbf{q} : \mathbf{q} \in \Re^{2n}, \|\mathbf{q}_{ij}\| < r_\epsilon\right\}$ (see Fig. 3 for an illustration). Then, using the arc-length formula to compute

$$\begin{aligned}\beta_{ij}(t_f) \geq \|\mathbf{q}_i(t_f) - \mathbf{q}_j(t_f)\| \Bigg[&-e^{-k_i\delta_i}\eta_i + \frac{\bar{\mu}_i}{r_\epsilon(k_i^2 + \omega_{ij}^2)}\Bigg(k_i\sqrt{r_\epsilon^2 - \Delta_i^2} + \omega_{ij}\Delta_i \\ &-e^{-k_i\delta_i}\left(k_i\sqrt{r_\epsilon^2 - \Delta_i^2}\cos\theta_i - k_i\Delta_i\sin\theta_i + \omega_{ij}\sqrt{r_\epsilon^2 - \Delta_i^2}\sin\theta_i + \omega_{ij}\Delta_i\cos\theta_i\right)\Bigg)\Bigg]\end{aligned} \tag{6}$$

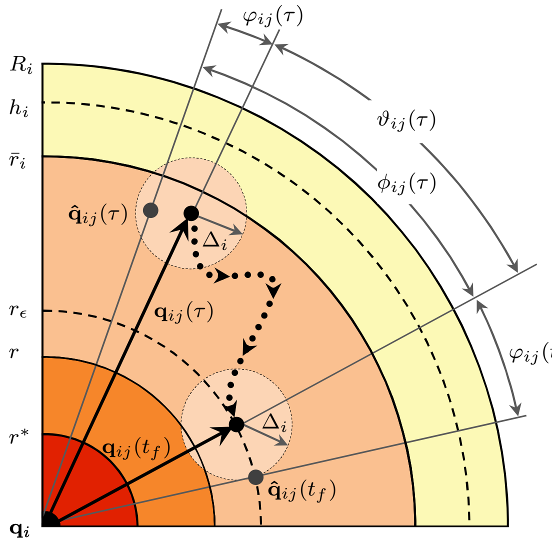

Fig. 2. Hypothetical motion of $\mathbf{q}_j(\tau)$ with respect to $\mathbf{q}_i(\tau)$ for $t_\delta \leq \tau \leq t_f$. The larger black dots represent the vectors (i.e., distances) $\mathbf{q}_{ij}(\tau)$ and $\mathbf{q}_{ij}(t_f)$, whereas the gray dots denote $\hat{\mathbf{q}}_{ij}(\tau)$ and $\hat{\mathbf{q}}_{ij}(t_f)$.

the maximum length traveled by the agents and invoking its relation with the central angle ϑ_{ij} we obtain

$$\vartheta_{ij}(\tau) \leq \frac{\int_\tau^{t_f} \|\dot{\mathbf{q}}_{ij}(s)\|\, ds}{r_\epsilon} \leq \frac{(\eta_i+\eta_j)(t_f-\tau)}{r_\epsilon} = \overline{\vartheta}_{ij}(\tau)$$

for $\tau \in [t_\delta, t_f]$. Note that $\overline{\vartheta}_{ij}(t_\delta) = \frac{(\eta_i+\eta_j)\delta_i}{r_\epsilon} = \theta_i$ while $\overline{\vartheta}_{ij}(t_f) = 0$. Now, we are left to find $\overline{\varphi}_{ij}(\tau)$.

Since $\|\mathbf{q}_{ij}(\tau)\| \geq r_\epsilon \;\forall\, \tau \in [t_\delta, t_f]$ and $\|\mathbf{d}_i\| \leq \Delta_i < r$, we have that $\varphi_{ij}(\tau)$ is maximized when $\mathbf{q}_{ij}(\tau)$ is close to the boundary of Ω_ϵ. Thus, let us consider the diagram in Fig. 3, which details this case. First, observe that the maximum angle $\overline{\varphi}_{ij}(\tau)$ is constant whenever $\mathbf{q}_{ij}(\tau)$ lies on the boundary of Ω_ϵ. Consequently, it is sufficient to find $\overline{\varphi}_{ij}(\tau)$ when $\tau = t_\delta$. To this end, let us choose the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ as an orthonormal basis for the plane containing $\mathbf{q}_{ij}(t_\delta)$ and $\hat{\mathbf{q}}_{ij}(t_\delta)$ and let $\mathbf{e}_2$ be oriented along the same direction and origin as $\mathbf{q}_{ij}(t_\delta)$, as shown in Fig. 3. Then, $\mathbf{q}_{ij}(t_\delta)$ can be rewritten as $\mathbf{q}_{ij}(t_\delta) = r_\epsilon \mathbf{e}_2$. Similarly, $\hat{\mathbf{q}}_{ij}(t_\delta)$ can be written as $\hat{\mathbf{q}}_{ij}(t_\delta) = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$, where c_1 and c_2 are constants. Now, from the constraint $\|\hat{\mathbf{q}}_{ij}(t_\delta) - \mathbf{q}_{ij}(t_\delta)\| = \|\mathbf{d}_i(t)\| \leq \Delta_i$, we have that c_1 and c_2 must satisfy the following equation: $c_1^2 + (c_2 - r_\epsilon)^2 \leq \Delta_i$. Likewise, we have that $\overline{\varphi}_{ij}$ is maximized when the ratio $|c_1/c_2|$ attains its maximum. Such conditions are satisfied when

$$c_1 = \pm \frac{\Delta_i}{r_\epsilon}\sqrt{r_\epsilon^2 - \Delta_i^2}, \qquad c_2 = \frac{r_\epsilon^2 - \Delta_i^2}{r_\epsilon}.$$

Therefore, $\overline{\varphi}_{ij}(\tau) = \overline{\varphi}_{ij}$ can be computed as

$$\overline{\varphi}_{ij} = \cos^{-1}\left(\frac{\mathbf{q}_{ij}(t_\delta)^T \hat{\mathbf{q}}_{ij}(t_\delta)}{\|\mathbf{q}_{ij}(t_\delta)\|\,\|\hat{\mathbf{q}}_{ij}(t_\delta)\|}\right) = \cos^{-1}\left(\frac{\sqrt{r_\epsilon^2 - \Delta_i^2}}{r_\epsilon}\right).$$

Now, let us return to (7). Since $\overline{\vartheta}_{ij}(\tau) \leq \theta_i < \sin^{-1}\left(\frac{\sqrt{r_\epsilon^2-\Delta_i^2}}{r_\epsilon}\right)$ and $\overline{\varphi}_{ij}(\tau) = \cos^{-1}\left(\frac{\sqrt{r_\epsilon^2-\Delta_i^2}}{r_\epsilon}\right)$, we have that $\overline{\phi}_{ij}(\tau) = \overline{\vartheta}_{ij}(\tau) + \overline{\varphi}_{ij}(\tau) < \frac{\pi}{2}$. Therefore, $\cos\phi_{ij}(\tau) \geq \cos\overline{\phi}_{ij}(\tau) > 0$ for all τ and

$$\begin{aligned}
\int_{t_\delta}^{t_f} e^{-k_i(t_f-\tau)} \cos\phi_{ij}(\tau) d\tau &\geq \int_{t_\delta}^{t_f} e^{-k_i(t_f-\tau)} \cos\overline{\phi}_{ij}(\tau) d\tau \\
&= \frac{1}{k_i^2+\omega_{ij}^2}\left(k_i\cos\overline{\phi}_{ij}(t_f) + \omega_{ij}\sin\overline{\phi}_{ij}(t_f)\right) \\
&\quad - \frac{e^{k_i\delta_i}}{k_i^2+\omega_{ij}^2}\left(k_i\cos\overline{\phi}_{ij}(t_f) + \omega_{ij}\sin\overline{\phi}_{ij}(t_\delta)\right) \qquad (8)
\end{aligned}$$

where we used the fact that $\dot{\overline{\phi}}_{ij}(t) = \omega_{ij} = -\frac{\eta_i+\eta_j}{r_\epsilon}$ is constant. Also note that $\overline{\phi}_{ij}(t_f) = \overline{\varphi}_{ij}$ and hence

$$\begin{aligned}
\cos\overline{\phi}_{ij}(t_f) &= \cos\overline{\varphi}_{ij} = \frac{\sqrt{r_\epsilon^2-\Delta_i^2}}{r_\epsilon} \\
\sin\overline{\phi}_{ij}(t_f) &= \sin\overline{\varphi}_{ij} = \frac{\|\mathbf{q}_{ij}(t_\delta)\times\hat{\mathbf{q}}_{ij}(t_\delta)\|}{\|\mathbf{q}_{ij}(t_\delta)\|\,\|\hat{\mathbf{q}}_{ij}(t_\delta)\|} = \frac{\Delta_i}{r_\epsilon}.
\end{aligned}$$

In order to evaluate $\cos\overline{\phi}_{ij}(t_\delta)$ and $\sin\overline{\phi}_{ij}(t_\delta)$, let us rewrite $\mathbf{q}_{ij}(t_f)$ using $\mathbf{e}_1$ and $\mathbf{e}_2$ as orthonormal basis, i.e., $\mathbf{q}_{ij}(t_f) = r_\epsilon\sin\theta_i\mathbf{e}_1 + r_\epsilon\cos\theta_i\mathbf{e}_2$. Then, we have that

$$\begin{aligned}
\cos\overline{\phi}_{ij}(t_\delta) &= \cos(\theta_i + \overline{\varphi}_{ij}) = \frac{\mathbf{q}_{ij}(t_f)\hat{\mathbf{q}}_{ij}(t_\delta)}{\|\mathbf{q}_{ij}(t_f)\|\,\|\hat{\mathbf{q}}_{ij}(t_\delta)\|} \\
&= \frac{\sqrt{r_\epsilon^2-\Delta_i^2}\cos\theta_i - \Delta_i\sin\theta_i}{r_\epsilon} \\
\sin\overline{\phi}_{ij}(t_\delta) &= \sin(\theta_i + \overline{\varphi}_{ij}) = \frac{\|\mathbf{q}_{ij}(t_f)\times\hat{\mathbf{q}}_{ij}(t_\delta)\|}{\|\mathbf{q}_{ij}(t_f)\|\,\|\hat{\mathbf{q}}_{ij}(t_\delta)\|} \\
&= \frac{\sqrt{r_\epsilon^2-\Delta_i^2}\sin\theta_i + \Delta_i\cos\theta_i}{r_\epsilon}
\end{aligned}$$

and returning to (8), we obtain

$$\begin{aligned}
&\int_{t_\delta}^{t_f} e^{-k_i(t_f-\tau)}\cos\phi_{ij}(\tau)d\tau \\
&\geq \frac{1}{r_\epsilon(k_i^2+\omega_{ij}^2)}\Big(k_i\sqrt{r_\epsilon^2-\Delta_i^2} + \omega_{ij}\Delta_i \\
&\quad - e^{-k_i\delta_i}k_i\sqrt{r_\epsilon^2-\Delta_i^2}\cos\theta_i + e^{-k_i\delta_i}k_i\Delta_i\sin\theta_i \\
&\quad - e^{-k_i\delta_i}\omega_{ij}\sqrt{r_\epsilon^2-\Delta_i^2}\sin\theta_i - e^{-k_i\delta_i}\omega_{ij}\Delta_i\cos\theta_i\Big).
\end{aligned}$$

Therefore, substituting the above equation into (7) yields (6), and the proof is complete. ■

Remark 4.1: Mathematically, we can interpret $\frac{\beta_{ij}(t)}{\|\mathbf{q}_{ij}(t)\|}$ as the scalar projection of the velocity vector $\dot{\mathbf{q}}_i$ onto the collision threat vector $\mathbf{q}_{ij}$. Accordingly, we can say that the previous lemma provides an indication of the direction of the ith agent's velocity vector with respect to the collision threat. For instance, if for some time t_f, $\beta_{ij}(t_f) > 0$, then we can conclude that at time t_f the ith agent is moving away from the jth agent.

Remark 4.2: Although the analysis in this paper will focus on the design of cooperative avoidance strategies, it is worth mentioning that Lemma 4.1 provides much general results. Note that Lemma 4.1 does not make any assumption on the control used by the jth agent and, therefore, it can be applied to a noncooperative scenario.

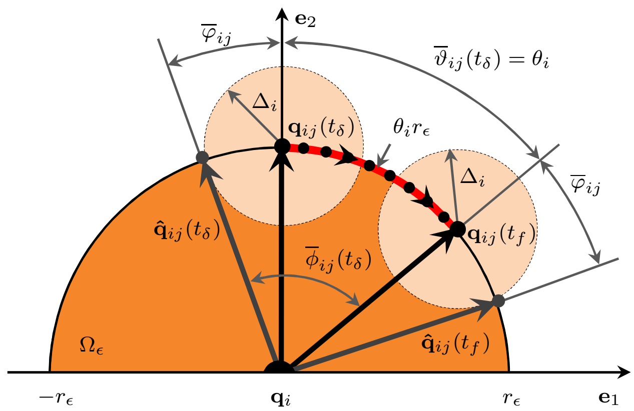

Fig. 3. Extreme case in Lemma 4.1. The agents approach each other at maximum speed along the boundary of Ω_ϵ for $\tau \in [t_\delta, t_f]$.

Having established Lemma 4.1, we now proceed to state the main result of this paper.

Theorem 4.1: (Cooperative Collision Avoidance with Sensing Uncertainties): Consider the two dynamical systems in (1) with control inputs (2) and (5). Let $k_i = \bar{\mu}_i/\eta_i$ and suppose $\|\dot{\mathbf{q}}_i(0)\| \le \eta_i$, $\|\mathbf{d}_i(t)\| \le \Delta_i$, and $[\mathbf{q}_i^T(0), \mathbf{q}_j^T(0)]^T \notin \mathcal{W}_i \cup \Omega$ for all $i, j \in \{1, 2\}$, $i \neq j$. Furthermore, assume that there exist $\eta_i > 0$, $r \ge r^*$, $\theta_i \in \left(0, \sin^{-1}\left(\frac{\sqrt{r_\epsilon^2 - \Delta_i^2}}{r_\epsilon}\right)\right)$ and an arbitrarily small constant $\epsilon > 0$ such that

$$\bar{r}_i = (\theta_i + 1)(r + \epsilon) < R_i - \Delta_i \tag{9}$$

and

$$\left(k_i + \frac{\omega_{ij}\Delta_i}{\sqrt{r_\epsilon^2 - \Delta_i^2}}\right)\cos\theta_i + \left(\omega_{ij} - \frac{k_i\Delta_i}{\sqrt{r_\epsilon^2 - \Delta_i^2}}\right)\sin\theta_i \le \left(k_i + \frac{\omega_{ij}\Delta_i}{\sqrt{r_\epsilon^2 - \Delta_i^2}}\right)e^{k_i\delta_i} - \frac{r_\epsilon(k_i^2 + \omega_{ij}^2)}{k_i\sqrt{r_\epsilon^2 - \Delta_i^2}} \tag{10}$$

$\forall\ i, j \in \{1, 2\}, i \neq j$. Then, $[\mathbf{q}_1^T(t), \mathbf{q}_2^T(t)]^T \notin \Omega\ \forall t \ge 0$.

Proof: Consider (1) with control inputs (2) and (5) for $i \in \{1, 2\}$. Assume that (9) and (10) hold. Let $k_i = \bar{\mu}_i/\eta_i$ and $\|\dot{\mathbf{q}}_i(0)\| \le \eta_i$. Then, applying Lemma 3.1 we can conclude that $\|\dot{\mathbf{q}}_i(t)\| \le \eta_i$ for all i, $t \ge 0$. Now, let us consider the following Lyapunov candidate function

$$V(t) = \frac{1}{4(\|\mathbf{q}_{12}(t)\|^2 - r^2)^2}.$$

Taking its time derivative along the trajectories of (1) yields

$$\dot{V}(t) = -\frac{\beta_{12}(t) + \beta_{21}(t)}{(\|\mathbf{q}_{12}(t)\|^2 - r^2)^3}.$$

Our approach to prove collision avoidance will be to show that $\beta_{ij}(t)$ is positive semi-definite for $i, j \in \{1, 2\}$, $i \neq j$, which will be used to demonstrate that $\|\mathbf{q}_{ij}(t)\|$ is bounded from below. For simplicity, let us consider first the case of the ith agent. Let $[\mathbf{q}_i^T(0), \mathbf{q}_j^T(0)]^T \notin \mathcal{W}_i \cup \mathcal{W}_j \cup \Omega$ and suppose that for some time $t \ge \delta_i > 0$, $\|\mathbf{q}_{ij}(t)\| \to r + \epsilon = r_\epsilon$ from above. Since $\|\mathbf{q}_{ij}(0)\| > \bar{r}_i$ and the velocities of the agents are bounded, it will take the agents some time Δt to reduce their distance from $\bar{r}_i$ to r_ϵ. Therefore, we have that $[\mathbf{q}_i^T(\tau), \mathbf{q}_j^T(\tau)]^T \in \mathcal{W}_i\ \forall \tau \in [t - \Delta t, t]$, where it is easy to demonstrate that $\Delta t \ge \delta_i = \frac{\bar{r}_i - r_\epsilon}{\eta_i + \eta_j} = \frac{\theta_i r_\epsilon}{\eta_i + \eta_j}$. Consequently, we can apply Lemma 4.1 and, after some manipulation and use of (9) and (10), we can easily show that $\beta_{ij}(t) \ge 0$. Since the above result holds $\forall i, j, i \neq j$, we have that $\dot{V}(t) \le 0$, for $\|\mathbf{q}_{12}(t)\| \le r_\epsilon$. The fact that $\mathbf{q}_1(t)$ and $\mathbf{q}_2(t)$ are continuous and $\dot{V}(t)$ is non-positive for $\|\mathbf{q}_{12}(t)\| \le r_\epsilon$ implies that $V(t) < \infty$. (i.e., $V(t)$ is finite for any $t \ge 0$). Hence, the solutions of $\mathbf{q}_{12}(t)$ are uniformly ultimately bounded by r_ϵ, which further implies that $[\mathbf{q}_1^T(t), \mathbf{q}_2^T(t)]^T \notin \Omega\ \forall\ t \ge 0$. ■

Note that Theorem 4.1 enforces collision-free trajectories for (1) under the presence of bounded sensing uncertainties given the existence of constants θ_i and r satisfying (9) and (10) for all $i \in \{1, 2\}$. The statement, however, does not suggest how to optimally choose θ_i and r. In general, we would like to design θ_i and r such that the extent of the agents' Conflict Regions is minimized. That is, we want to minimize the distance at which the agents start applying maximum avoidance control such that the attenuation of the objective control inside the Detection Region is reduced.

V. Example

In order to illustrate the performance of the proposed avoidance strategy, we now present the following example.

Consider a pair of 2-DOF agents with double-integrator dynamics described by (1) and control inputs bounded by $\mu_1 = 50\text{m/s}^2$ and $\mu_2 = 60\text{m/s}^2$. Let the overall control input for both agents be given as in (2) with $k_1 = 6.25\text{s}^{-1}$ and $k_2 = 6.00\text{s}^{-1}$. Then, using Lemma 3.1, we can show that the velocities of the first and second agents are bounded by $\eta_1 = 4\text{m/s}$ and $\eta_2 = 5\text{m/s}$, respectively. Now, assume the minimum separation and detection radii for both agents to be $r^* = 1\text{m}$ and $R_1 = R_2 = 5.5\text{m}$. Furthermore, suppose that the sensing uncertainties are characterized by

$$\mathbf{d}_1(t) = \int_t^{t-T_1} \dot{\mathbf{q}}_2(\tau)d\tau, \quad \mathbf{d}_2(t) = \int_t^{t-T_2} \dot{\mathbf{q}}_1(\tau)d\tau + \boldsymbol{\zeta}(t) - \boldsymbol{\xi}$$

where $T_1 = 0.3\text{s}$ and $T_2 = 0.2\text{s}$ denote the detection delay for the first and second agent, respectively, $\boldsymbol{\zeta}(t)$ represents a random noise with uniform distribution on the set $\mathcal{Z} = \{\boldsymbol{\zeta} : \boldsymbol{\zeta} \in \Re^2, \|\boldsymbol{\zeta}\| < 0.5\text{m}\}$, and $\boldsymbol{\xi} = [0.2\text{m}, 0.0\text{m}]^T$ is a constant error. It is easy to show that $\|\mathbf{d}_i(t)\| \le \Delta_i = 1.5\text{m}\ \forall i$.

The avoidance control inputs (5) are computed according to Theorem 4.1, which yields that the minimum set of conflict radii $\bar{r}_i$ is attained when $r = 2.33\text{m}$, $\theta_1 = 0.63\text{rad}$, and $\theta_2 = 0.66\text{rad}$. Choosing $\epsilon = 0.01$ we obtain that $\bar{r}_1 = 3.81\text{m}$ and $\bar{r}_2 = 3.88\text{m}$. Finally, we take the objective control input to be computed as

$$\mathbf{u}_i^o = \begin{cases} \tilde{\mathbf{u}}_i^o, & \text{if } \|\tilde{\mathbf{u}}_i^o\| \le \bar{\mu}_i \\ \dfrac{\tilde{\mathbf{u}}_i^o}{\|\tilde{\mathbf{u}}_i^o\|}, & \text{otherwise} \end{cases}, \quad \tilde{\mathbf{u}}_i^o = K_p(\mathbf{q}_i^d - \mathbf{q}_i)$$

where $\mathbf{q}_1^d = -\mathbf{q}_2^d = [8\text{m}, 8\text{m}]^T$ and $K_p = 8\text{s}^{-2}$ are the agents' desired final configurations and the proportional control gain, respectively. The initial conditions for the system are set to $\mathbf{q}_1(0\text{s}) = -\mathbf{q}_2(0\text{s}) = [-8\text{m}, -8\text{m}]^T$ and

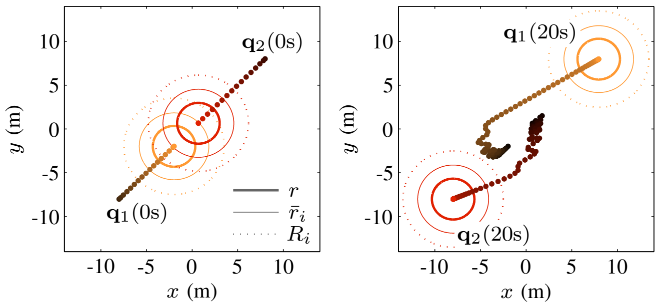

Fig. 4. Cooperative collision avoidance example. The left and right plot illustrate the motion of both agents in the intervals $t \in [0.0\mathrm{s}, 2.4\mathrm{s}]$ and $t \in [2.4\mathrm{s}, 20.0\mathrm{s}]$, respectively, where each mark is time-spaced by 0.15s. The initial position of the two agents at the start of each simulation interval is indicated by the dark-colored dots. The Avoidance, Conflict, and Detection Regions for both agents at the end of the simulation intervals are traced by the bold, thin, and dashed lines, respectively.

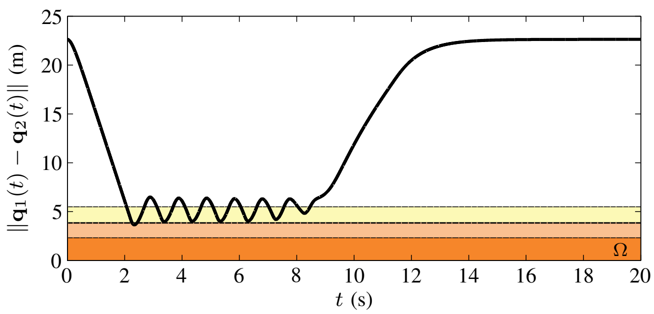

Fig. 5. Distance between both agents.

$\dot{\mathbf{q}}_1(0\mathrm{s}) = \dot{\mathbf{q}}_2(0\mathrm{s}) = [0\mathrm{m/s}, 0\mathrm{m/s}]^T$. Therefore, the main objective is to safely drive both vehicles to the other agent's initial location.

The response of the two-agent system is illustrated Fig. 4. The two agents start traveling toward each other at maximum speed, according to their control objective. Once they enter into each other's Detection, and eventually, Conflict Region (see left-side plot), the two agents decelerate and start an oscillatory motion toward opposite sides while avoiding a collision (corresponding to the right-side plot). After the conflict has been solved and the agents are safely apart, the two systems continue toward their final destinations.

As depicted in Fig. 5, the agents successfully evaded the Avoidance Region (indicated by the dark orange rectangle). The observed oscillatory behavior of both agents can be attributed to the sensing delay and to the symmetry of their desired trajectories (i.e., their desired trajectories are the same but with opposite direction, which maximizes the risk of a collision). In fact, the measurement errors $\boldsymbol{\zeta}$ and $\boldsymbol{\xi}$ introduced during the sensing process of the second agent are what finally break the symmetry in the vehicles' trajectories.

VI. Conclusion

A cooperative collision avoidance strategy for the case of two agents with acceleration constraints, bounded sensing uncertainties, and limited sensing range has been reported. It is shown that for agents with double-integrator dynamics and radially bounded sensing uncertainties, collision-free trajectories can be guaranteed if the initial distance between both agents is equal or greater than a computed conflict radius. Simulations results validated the proposed strategy.

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