2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Multi-agent Consensus Algorithm with Obstacle Avoidance via Optimal Control Approach

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Abstract—**Multi-agent consensus problem in an obstacle-laden environment is addressed in this paper. A novel optimal control approach is proposed for the multi-agent system to reach consensus as well as avoid obstacles with a reasonable control effort. An innovative nonquadratic penalty function is constructed to achieve obstacle avoidance capability from an inverse optimal control perspective. The asymptotic stability and optimality of the consensus algorithm are proven. In addition, the optimal control law of each agent only requires local information from the neighbors to guarantee the proposed behaviors, rather than all agents' information. The consensus and obstacle avoidance are validated through various simulations.**

I. Introduction

Multi-agent cooperative missions are becoming increasingly important and feasible owing to the rapid advances in computing, communication, sensing, and actuation. Cooperative control has been recognized to be of critically importance to the successful accomplishment of these cooperative missions.

As a core of multi-agent cooperative control, consensus problem has been extensively studied in recent years [1-4]. From the optimization perspective, consensus algorithms have been developed along two lines: 1) fastest convergence time: the algorithms were designed to achieve the fastest convergence time by finding an optimal weighting matrix [5], constructing a proper configuration that maximizes the second smallest eigenvalue of the Laplacian [6], and exploring an optimal interaction graph for the average consensus problem [7]; 2) Optimal control design: the consensus problem was formulated as an optimal control problem and solved using a linear matrix inequality (LMI) approach [8], a LQR-based optimal linear consensus algorithm [9], a distributed subgradient method for multi-agent optimization [10], and a locally optimal nonlinear consensus strategy by imposing individual objectives [11].

In the realistic environment, if obstacles emerge right on the trajectory, the multiple agents may not be able to safely achieve desired cooperative behaviors. Therefore, intensive attention has been paid to the cooperative control problem with obstacle/collision avoidance. In [12], three flocking algorithms were proposed to achieve both flocking and obstacle avoidance by adding obstacle avoidance terms to the group objective. In [13], a constraint force, directly converted from the structural distance constraints for a desired formation, was introduced to achieve the formation as well as the collision avoidance between multiple agents. In [14], a new distributed robust model predictive control algorithm was developed for multi-agent trajectory optimization utilizing constraint tightening to ensure safety in the presence of the environmental changes and generate an intelligent trajectory around known obstacles. A cooperative control law for the individual agent to guarantee collision avoidance in multi-agent systems was proposed in [15]. However, it is assumed that every agent knows its desired state and a LQR based optimal control is designed to track the desired state.

Most of the obstacle avoidance strategies are designed either for path planning of the single agent or for multiple agents without considering their interaction topologies and the information consensus problem. In this paper, we address both consensus problem and obstacle avoidance in a unified optimal control framework. A novel avoidance penalty function is constructed based on an inverse optimal control strategy [16, 17] such that an analytical optimal control law can be obtained. In addition, it can be shown that the resultant consensus algorithm is a linear function of the Laplacian, and thus only local information from the communication topology is required to implement the optimal cooperative control law.

The remainder of this paper is organized as follows. The consensus problem is described in Section II and Section III presents the main result of this paper. Simulation results and analysis are shown in Section IV. Some conclusion remarks are given in Section V.

II. Problem Statement

Consider n agents with double-integrator dynamics:

$$\begin{cases} \dot{\boldsymbol{p}}_i = \boldsymbol{v}_i \\ \dot{\boldsymbol{v}}_i = \boldsymbol{a}_i \end{cases}, \quad i = 1, \dots, n \tag{1a}$$

or in a matrix form

$$\dot{\boldsymbol{X}} = A\boldsymbol{X} + B\boldsymbol{U} \tag{1b}$$

$$A = \begin{bmatrix} 0_{n\times n} & I_n \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} \otimes I_m,\ B = \begin{bmatrix} 0_{n\times n} \\ I_n \end{bmatrix} \otimes I_m$$

$$\boldsymbol{X} = [\underbrace{\boldsymbol{p}_1^T, \dots, \boldsymbol{p}_n^T}_{\boldsymbol{p}^T}, \underbrace{\boldsymbol{v}_1^T, \dots, \boldsymbol{v}_n^T}_{\boldsymbol{v}^T}]^T, \boldsymbol{U} = [\boldsymbol{a}_1^T, \dots, \boldsymbol{a}_n^T]^T$$

where $\boldsymbol{p}_i(t) \in R^m$, $\boldsymbol{v}_i(t) \in R^m$ and $\boldsymbol{a}_i(t) \in R^m$ are, respectively, the position, velocity and control input of agent i. $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{p}^T & \boldsymbol{v}^T \end{bmatrix}^T$ and $\boldsymbol{U}$ are the aggregate state and control input of all agents. $\otimes$ denotes the Kronecker product.

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The consensus problem in this paper is to design a distributed control law $\boldsymbol{a}_i(t)$ based on the information exchange topology such that $\|\boldsymbol{p}_i(t)-\boldsymbol{p}_j(t)\|\to 0$ and $\|\boldsymbol{v}_i(t)-\boldsymbol{v}_j(t)\|\to 0$. In addition, each agent is guaranteed to avoid the obstacle along its trajectory.

Fig. 1 shows an example scenario of four agents' consensus problem. R denotes the radius of the obstacle detection region and r denotes the radius of the obstacle. The dashed line denotes the original consensus trajectory without obstacle. The proposed consensus law will be able to not only drive all the agents along the solid lines to reach consensus but also avoid the obstacle with an optimal control effort.

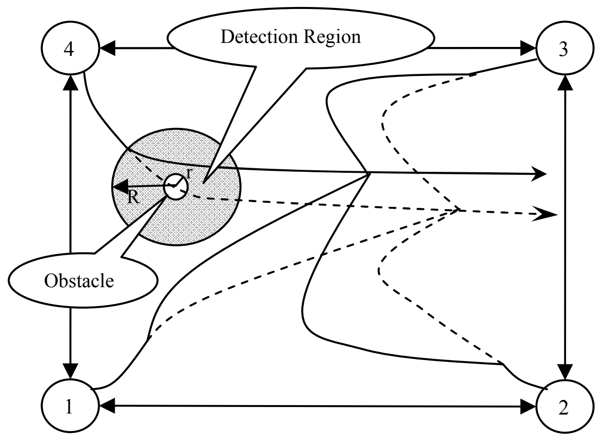

Fig. 1: Multi-agent consensus scenario with an obstacle

III. OPTIMAL CONSENSUS WITH OBSTACLE AVOIDANCE

In this section, we propose a unified inverse optimal control approach to address the consensus problem with obstacle avoidance capability. For the convenience of formulation, we define the error state

$$\hat{\boldsymbol{X}}=\begin{bmatrix}\hat{\boldsymbol{p}}^T & \hat{\boldsymbol{v}}^T\end{bmatrix}^T \triangleq \boldsymbol{X}-\boldsymbol{X}_{cs} \tag{2}$$

where $$\boldsymbol{X}_{cs}=\begin{bmatrix}\boldsymbol{p}_{cs}^T & \boldsymbol{v}_{cs}^T\end{bmatrix}^T \tag{3}$$

is the final consensus state. For instance, in a planar motion,

$$\boldsymbol{X}_{cs}=\begin{bmatrix}\boldsymbol{p}_{cs}^T & \boldsymbol{v}_{cs}^T\end{bmatrix}^T=\begin{bmatrix}\boldsymbol{1}_{1\times n}\otimes\begin{bmatrix}\alpha_x & \alpha_y\end{bmatrix} & \boldsymbol{1}_{1\times n}\otimes\begin{bmatrix}\beta_x & \beta_y\end{bmatrix}\end{bmatrix}^T \tag{4}$$

where α_x,α_y are the final consensus position along x axis and y axis, respectively; β_x,β_y are the final consensus velocity along x axis and y axis, respectively. Note that the consensus state $\boldsymbol{X}_{cs}$ is not known *a priori*.

We follow the standard definitions and concepts from the graph theory to describe the interconnection of multi-agent systems, which can be referred to [18]. In particular, the Laplacian matrix L is commonly used to define the communication topology among agents. In this paper, the information exchange topology is assumed to be undirected and connected. Under this assumption, L is positive semi-definite and the following property holds when the agents reach consensus: [18]

$$\begin{aligned}(L\otimes I_m)\boldsymbol{p}_{cs}&=\boldsymbol{0}_{nm\times 1}\\(L\otimes I_m)\boldsymbol{v}_{cs}&=\boldsymbol{0}_{nm\times 1}\end{aligned} \tag{5}$$

The final consensus state satisfies the dynamic equation

$$\dot{\boldsymbol{X}}_{cs}=A\boldsymbol{X}_{cs}+B\boldsymbol{U}_{cs}=A\boldsymbol{X}_{cs} \tag{6}$$

since $\boldsymbol{U}_{cs}=\boldsymbol{0}_{nm\times 1}$ when the agents reach consensus.

Then, from Eq. (1b) and (6) the error dynamics becomes

$$\dot{\hat{\boldsymbol{X}}}=A\hat{\boldsymbol{X}}+B\boldsymbol{U} \tag{7}$$

The consensus is achieved when the system (7) is asymptotically stable.

In this paper, the consensus problem is formulated as an optimal control problem with three cost function components:

$$\begin{aligned}&\text{Min}: J=J_1+J_2+J_3\\&S.t.\quad \dot{\hat{\boldsymbol{X}}}=A\hat{\boldsymbol{X}}+B\boldsymbol{U}\end{aligned} \tag{8}$$

where J_1,J_2,J_3 represent the consensus cost, obstacle avoidance cost, and control effort, respectively.

The consensus cost has the form of:

$$J_1=\int_0^\infty \hat{\boldsymbol{X}}^T R_1\hat{\boldsymbol{X}}dt=\int_0^\infty\left\{\hat{\boldsymbol{X}}^T\left(\begin{bmatrix}w_p^{\,2}L^2 & 0_{n\times n}\\ 0_{n\times n} & w_v^{\,2}L^2-2w_pw_cL\end{bmatrix}\otimes I_m\right)\hat{\boldsymbol{X}}\right\}dt \tag{9}$$

where w_p, w_v, and w_c represent the weights on the position consensus, velocity consensus, and control effort, respectively. It is necessary that R_1 is positive semi-definite, which can be shown in the following proposition.

Proposition 4.1: R_1 is positive semi-definite if the graph is undirected and connected and

$$w_v^{\,2}e_i^2-2w_pw_ce_i\ge 0 \tag{10}$$

where e_i is the eigenvalue of L.

Proof: Since L is positive semi-definite and it is straightforward to show that L^2 is also positive semi-definite,

$$\begin{aligned}&w_v^{\,2}L^2-2w_pw_cL=w_v^{\,2}Q\Lambda^2Q^{-1}-2w_pw_cQ\Lambda Q^{-1}\\&=Q\begin{bmatrix}w_v^{\,2}e_1^2-2w_pw_ce_1 & 0 & \cdots & 0\\ 0 & w_v^{\,2}e_2^2-2w_pw_ce_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & w_v^{\,2}e_n^2-2w_pw_ce_n\end{bmatrix}Q^{-1}\end{aligned} \tag{11}$$

where Q is composed of the eigenvectors of L and Λ is the diagonal matrix with the diagonal elements being the eigenvalues of L. Since L is positive semi-definite, $e_i\ge 0$. Therefore, $w_v^{\,2}L^2-2w_pw_cL$ is positive semi-definite if $w_v^{\,2}e_i^2-2w_pw_ce_i\ge 0$ and it follows that R_1 is positive semi-definite. ■

Remark 4.1: The condition (10) is required in *Proposition 4.1*. One can always find proper weights to satisfy (10). For instance, a large w_v and small enough w_p and w_c are applicable due to $e_i\ge 0$.

The obstacle avoidance cost has the form of

$$J_2 = \int_0^\infty h(\hat{\boldsymbol{X}})dt \quad (12)$$

where $h(\hat{\boldsymbol{X}})$ will be constructed from an inverse optimal control approach in *Theorem 4.1*.

The control effort cost has the regular quadratic form of

$$J_3 = \int_0^\infty \boldsymbol{U}^T R_2 \boldsymbol{U} dt \quad (13)$$

where $R_2 = w_c{}^2 I_n \otimes I_m$ is positive definite and w_c is the weighting parameter.

The following lemma is introduced to derive our main result.

Lemma 4.1: [16] Consider the nonlinear controlled dynamical system

$$\dot{\hat{\boldsymbol{X}}}(t) = f(\hat{\boldsymbol{X}}(t), \boldsymbol{U}(t)), \quad \hat{\boldsymbol{X}}(0) = \hat{\boldsymbol{X}}_0, \quad t \geq 0 \quad (14)$$

with $f(\boldsymbol{0},\boldsymbol{0})=\boldsymbol{0}$ and a cost functional given by

$$J(\hat{\boldsymbol{X}}_0, \boldsymbol{U}(t)) \triangleq \int_0^\infty T(\hat{\boldsymbol{X}}(t), \boldsymbol{U}(t))dt \quad (15)$$

where $\boldsymbol{U}(t)$ is an admissible control. Let $D \subseteq \mathbb{R}^n$ be an open set and $\Omega \subseteq \mathbb{R}^m$. Assume that there exists a continuously differentiable function $V: D \to \mathbb{R}$ and a control law $\phi: D \to \Omega$ such that

$$V(\boldsymbol{0}) = 0 \quad (16)$$

$$V(\hat{\boldsymbol{X}}) > 0, \quad \hat{\boldsymbol{X}} \in D, \quad \hat{\boldsymbol{X}} \neq \boldsymbol{0} \quad (17)$$

$$\phi(\boldsymbol{0}) = \boldsymbol{0} \quad (18)$$

$$V'(\hat{\boldsymbol{X}}) f(\hat{\boldsymbol{X}}, \phi(\hat{\boldsymbol{X}})) < 0, \quad \hat{\boldsymbol{X}} \in D, \quad \hat{\boldsymbol{X}} \neq \boldsymbol{0} \quad (19)$$

$$H(\hat{\boldsymbol{X}}, \phi(\hat{\boldsymbol{X}})) = 0, \quad \hat{\boldsymbol{X}} \in D \quad (20)$$

$$H(\hat{\boldsymbol{X}}, \boldsymbol{U}) \geq 0, \quad \hat{\boldsymbol{X}} \in D, \quad \boldsymbol{U} \in \Omega \quad (21)$$

where $H(\hat{\boldsymbol{X}}, \boldsymbol{U}) \triangleq T(\hat{\boldsymbol{X}}, \boldsymbol{U}) + V'(\hat{\boldsymbol{X}}) f(\hat{\boldsymbol{X}}, \boldsymbol{U})$ is the Hamiltonian function. The superscript ' denotes partial differentiation with respect to $\hat{\boldsymbol{X}}$.

Then, with the feedback control

$$\boldsymbol{U}(t) = \phi(\hat{\boldsymbol{X}}(t)) \quad (22)$$

the solution $\hat{\boldsymbol{X}}(t) \equiv 0$ of the closed-loop system is locally asymptotically stable and there exists a neighborhood of the origin $D_0 \subseteq D$ such that

$$J(\hat{\boldsymbol{X}}_0, \phi(\hat{\boldsymbol{X}}(t))) = V(\hat{\boldsymbol{X}}_0), \quad \hat{\boldsymbol{X}}_0 \in D_0 \quad (23)$$

In addition, if $\hat{\boldsymbol{X}}_0 \in D_0$ then the feedback control (22) minimizes $J(\hat{\boldsymbol{X}}_0, \boldsymbol{U}(t))$ in the sense that

$$J(\hat{\boldsymbol{X}}_0, \phi(\hat{\boldsymbol{X}}(t))) = \min_{\boldsymbol{U}(t) \in S(\hat{\boldsymbol{X}}_0)} J(\hat{\boldsymbol{X}}_0, \boldsymbol{U}(t)) \quad (24)$$

where $S(\hat{\boldsymbol{X}}_0)$ denotes the set of asymptotically stabilizing controllers for each initial condition $\hat{\boldsymbol{X}}_0 \in D$. Finally, if $D = \mathbb{R}^n, \Omega = \mathbb{R}^m$, and

$$V(\hat{\boldsymbol{X}}) \to \infty \text{ as } \left\|\hat{\boldsymbol{X}}\right\| \to \infty \quad (25)$$

the solution $\hat{\boldsymbol{X}}(t) \equiv 0$ of the closed-loop system is globally asymptotically stable.

Proof: Omitted. Refer to [16]. ■

The main result of this paper is presented in the following theorem.

Theorem 4.1: For a multi-agent system (1) with an *undirected* and *connected* interaction graph, there always exist a large enough w_v, small enough w_p and w_c, such that the feedback control law

$$\boldsymbol{U} = \phi(\boldsymbol{X}) = -\frac{w_p}{w_c}(L \otimes I_m)\boldsymbol{p} - \frac{w_v}{w_c}(L \otimes I_m)\boldsymbol{v} - \frac{1}{2w_c^2} g_v{}'(\boldsymbol{X}) \quad (26)$$

is an optimal control law for the consensus problem (8) and the closed-loop system is globally asymptotically stable. $h(\hat{\boldsymbol{X}})$ in the obstacle avoidance cost function (12) is

$$\begin{aligned} h(\hat{\boldsymbol{X}}) &= \frac{w_p}{w_c} \hat{\boldsymbol{v}}^T (L \otimes I_m)(G_p \otimes I_m)(L \otimes I_m)\hat{\boldsymbol{p}} \\ &+ \frac{w_v}{w_c} \hat{\boldsymbol{v}}^T (L \otimes I_m)(G_p \otimes I_m)(L \otimes I_m)\hat{\boldsymbol{v}} - g_p{}'^T(\hat{\boldsymbol{X}})\hat{\boldsymbol{v}} \\ &+ \frac{1}{4w_c{}^2} \hat{\boldsymbol{v}}^T (L \otimes I_m)(G_p^2 \otimes I_m)(L \otimes I_m)\hat{\boldsymbol{v}} \end{aligned} \quad (27)$$

where $g_v{}'(\boldsymbol{X})$ and $g_p{}'^T(\hat{\boldsymbol{X}})$ in (26) and (27) are derived from the obstacle avoidance potential function defined by

$$g(\hat{\boldsymbol{X}}) = \frac{1}{2} \hat{\boldsymbol{v}}^T (G_p \otimes I_m)(L \otimes I_m)\hat{\boldsymbol{v}} \quad (28)$$

with $G_p = diag\begin{pmatrix} m(\boldsymbol{p}_1) & m(\boldsymbol{p}_2) & \cdots & m(\boldsymbol{p}_n) \end{pmatrix}$ and

$$m(\boldsymbol{p}_i) = \begin{cases} 0 & \mathrm{R} < \|\boldsymbol{p}_i - O_b\| \\ \left(\dfrac{R^2 - \|\boldsymbol{p}_i - \boldsymbol{O}_b\|^2}{\|\boldsymbol{p}_i - \boldsymbol{O}_b\|^2 - \mathrm{r}^2} \right)^2 & \mathrm{r} < \|\boldsymbol{p}_i - O_b\| \leq \mathrm{R} \\ \text{not defined} & \|\boldsymbol{p}_i - O_b\| \leq \mathrm{r} \end{cases} \quad i = 1, \dots, n \quad (29)$$

$$g'(\hat{\boldsymbol{X}}) = \begin{bmatrix} g_p{}'^T(\hat{\boldsymbol{X}}) & g_v{}'^T(\hat{\boldsymbol{X}}) \end{bmatrix}^T$$

$$g_p{}'(\hat{\boldsymbol{X}}) = \begin{bmatrix} \left(\dfrac{\partial g(\hat{\boldsymbol{X}})}{\partial \hat{\boldsymbol{p}}_1} \right)^T & \left(\dfrac{\partial g(\hat{\boldsymbol{X}})}{\partial \hat{\boldsymbol{p}}_2} \right)^T & \cdots & \left(\dfrac{\partial g(\hat{\boldsymbol{X}})}{\partial \hat{\boldsymbol{p}}_n} \right)^T \end{bmatrix}^T \quad (30)$$

$$g_v{}'(\hat{\boldsymbol{X}}) = (G_p \otimes I_m)(L \otimes I_m)\hat{\boldsymbol{v}} = (G_p \otimes I_m)(L \otimes I_m)\boldsymbol{v} = g_v{}'(\boldsymbol{X})$$

where $g_p{}'(\hat{\boldsymbol{X}})$ and $g_v{}'(\boldsymbol{X})$ represent the partial differentiation of $g(\hat{\boldsymbol{X}})$ with respect to the position error $\hat{\boldsymbol{p}}$ and the velocity error $\hat{\boldsymbol{v}}$ respectively.

Proof: Specific to this optimal consensus problem, we have the following equations corresponding to *Lemma 4.1*:

$$T(\hat{\boldsymbol{X}}, \boldsymbol{U}) = \hat{\boldsymbol{X}}^T R_1 \hat{\boldsymbol{X}} + h(\hat{\boldsymbol{X}}) + \boldsymbol{U}^T R_2 \boldsymbol{U} \quad (31)$$

$$f(\hat{\boldsymbol{X}}, \boldsymbol{U}) = A\hat{\boldsymbol{X}} + B\boldsymbol{U} \quad (32)$$

A candidate Lyapunov function $V(\hat{\boldsymbol{X}})$ is chosen to be

$$V(\hat{\boldsymbol{X}}) = \hat{\boldsymbol{X}}^T P \hat{\boldsymbol{X}} + g(\hat{\boldsymbol{X}}) \quad (33)$$

where P is the solution of a Riccati equation, which will be shown afterwards.

In order for the function $V(\hat{\boldsymbol{X}})$ in (33) to be a valid Lyapunov function, it must be continuously differentiable with respect to $\hat{\boldsymbol{X}}$ or equivalently $g(\hat{\boldsymbol{X}})$ must be continuously differentiable with respect to $\hat{\boldsymbol{X}}$. From the equations (28) and (29), it suffices to show that $m(\boldsymbol{p}_i)$ is continuously differentiable in the safety region $\{\boldsymbol{p} \| \|\boldsymbol{p}_i - O_b\| > \mathrm{r}\}$. In fact, this is true if $m(\boldsymbol{p}_i)$ and $\frac{dm(\boldsymbol{p}_i)}{d\boldsymbol{p}_i}$ are continuous at $\|\boldsymbol{p}_i - O_b\| = \mathrm{R}$. Since Eq. (29) implies that $\lim_{\|\boldsymbol{p}_i - O_b\| \to R^-} m(\boldsymbol{p}_i) = 0 = \lim_{\|\boldsymbol{p}_i - O_b\| \to R^+} m(\boldsymbol{p}_i)$, $m(\boldsymbol{p}_i)$ is continuous at $\|\boldsymbol{p}_i - O_b\| = \mathrm{R}$ and thus continuous over the safety region. Similarly, it can be easily shown that $\frac{dm(\boldsymbol{p}_i)}{d\boldsymbol{p}_i}$ is continuous over the safety region. Therefore, $g(\hat{\boldsymbol{X}})$ and the Lyapunov function $V(\hat{\boldsymbol{X}})$ are continuously differentiable with respect to $\hat{\boldsymbol{X}}$ in the safety region.

The Hamiltonian function can be written as:

$$\begin{aligned} H(\hat{\boldsymbol{X}}, \boldsymbol{U}, V'^T(\hat{\boldsymbol{X}})) &= T(\hat{\boldsymbol{X}}, \boldsymbol{U}) + V'^T(\hat{\boldsymbol{X}}) f(\hat{\boldsymbol{X}}, \boldsymbol{U}) \\ &= \hat{\boldsymbol{X}}^T R_1 \hat{\boldsymbol{X}} + h(\hat{\boldsymbol{X}}) + \boldsymbol{U}^T R_2 \boldsymbol{U} + [2\hat{\boldsymbol{X}}^T P + g'^T(\hat{\boldsymbol{X}})][A\hat{\boldsymbol{X}} + B\boldsymbol{U}] \end{aligned} \tag{34}$$

Setting $(\partial/\partial \boldsymbol{U}) H(\hat{\boldsymbol{X}}, \boldsymbol{U}, V'^T(\hat{\boldsymbol{X}})) = \boldsymbol{0}$ yields the optimal control law:

$$\boldsymbol{U}^* = \phi(\hat{\boldsymbol{X}}) = -\frac{1}{2} R_2^{-1} B^T V'(\hat{\boldsymbol{X}}) = -R_2^{-1} B^T P \hat{\boldsymbol{X}} - \frac{1}{2} R_2^{-1} B^T g'(\hat{\boldsymbol{X}}) \tag{35}$$

With (35) it follows that

$$\begin{aligned} &V'^T(\hat{\boldsymbol{X}}) f(\hat{\boldsymbol{X}}, \phi(\hat{\boldsymbol{X}})) = \hat{\boldsymbol{X}}^T (A^T P + PA - 2PSP) \hat{\boldsymbol{X}} \\ &- \hat{\boldsymbol{X}}^T PSg'(\hat{\boldsymbol{X}}) + g'^T(\hat{\boldsymbol{X}})(A - SP)\hat{\boldsymbol{X}} - \frac{1}{2} g'^T(\hat{\boldsymbol{X}}) Sg'(\hat{\boldsymbol{X}}) \end{aligned} \tag{36}$$

where $S \triangleq BR_2^{-1}B^T$. Using (35) and (36) into (34) yields

$$\begin{aligned} &H(\hat{\boldsymbol{X}}, \phi(\hat{\boldsymbol{X}}), V'^T(\hat{\boldsymbol{X}})) = \hat{\boldsymbol{X}}^T (A^T P + PA + R_1 - PSP) \hat{\boldsymbol{X}} \\ &+ g'^T(\hat{\boldsymbol{X}})(A - SP)\hat{\boldsymbol{X}} + h(\hat{\boldsymbol{X}}) - \frac{1}{4} g'^T(\hat{\boldsymbol{X}}) Sg'(\hat{\boldsymbol{X}}) \end{aligned} \tag{37}$$

In order to prove that the control law (35) is an optimal solution to the consensus problem (8) using the *Lemma 4.1*, the conditions (16)-(21) need to be verified.

Since $B = \begin{bmatrix} 0_{n\times n} \\ I_n \end{bmatrix} \otimes I_m$, it can be seen that $\phi(\hat{\boldsymbol{X}}) = -R_2^{-1} B^T P \hat{\boldsymbol{X}} - \frac{1}{2} R_2^{-1} g_v'(\hat{\boldsymbol{X}})$. From the form of $g_v'(\hat{\boldsymbol{X}})$ in (30), the condition (18), i.e. $\boldsymbol{\phi}(\boldsymbol{0}) = \boldsymbol{0}$, is satisfied.

In order to satisfy the condition (20) in *Lemma 4.1* or let Eq. (37) be zero, we can let

$$A^T P + PA + R_1 - PSP = 0 \tag{38}$$

and require that

$$g'^T(\hat{\boldsymbol{X}})(A - SP)\hat{\boldsymbol{X}} + h(\hat{\boldsymbol{X}}) - \frac{1}{4} g'^T(\hat{\boldsymbol{X}}) Sg'(\hat{\boldsymbol{X}}) = 0 \tag{39}$$

With (35), (38), and (39), it can be shown that

$$H(\hat{\boldsymbol{X}}, \boldsymbol{U}, V'^T(\hat{\boldsymbol{X}})) = [\boldsymbol{U} - \phi(\hat{\boldsymbol{X}})]^T R_2 [\boldsymbol{U} - \phi(\hat{\boldsymbol{X}})] \geq 0 \tag{40}$$

Therefore, the condition (21) is satisfied.

Substituting A, B, R_1, R_2 in (38) and assuming $P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \otimes I_m$ yields

$$\begin{bmatrix} -\frac{1}{w_c^2} P_2^2 & P_1 - \frac{1}{w_c^2} P_2 P_3 \\ P_1 - \frac{1}{w_c^2} P_3 P_2 & 2P_2 - \frac{1}{w_c^2} P_3^2 \end{bmatrix} + \begin{bmatrix} w_p^2 L^2 & 0_{n\times n} \\ 0_{n\times n} & w_v^2 L^2 - 2 w_p w_c L \end{bmatrix} = 0 \tag{41}$$

Then, P can be solved in the analytical form

$$P = \begin{bmatrix} w_p w_v L^2 & w_p w_c L \\ w_p w_c L & w_c w_v L \end{bmatrix} \otimes I_m \tag{42}$$

Next, the cost function term $h(\hat{\boldsymbol{X}})$ in J_2 is constructed from solving Eq. (39) and using (42):

$$\begin{aligned} h(\hat{\boldsymbol{X}}) &= \frac{w_p}{w_c} \hat{\boldsymbol{v}}^T (L \otimes I_m)(G_p \otimes I_m)(L \otimes I_m) \hat{\boldsymbol{p}} \\ &+ \frac{w_v}{w_c} \hat{\boldsymbol{v}}^T (L \otimes I_m)(G_p \otimes I_m)(L \otimes I_m) \hat{\boldsymbol{v}} - g_p'^T(\hat{\boldsymbol{X}}) \hat{\boldsymbol{v}} \\ &+ \frac{1}{4 w_c^2} \hat{\boldsymbol{v}}^T (L \otimes I_m)(G_p^2 \otimes I_m)(L \otimes I_m) \hat{\boldsymbol{v}} \end{aligned} \tag{43}$$

which turns out to be (27).

Using (38) and (39), (36) becomes

$$\begin{aligned} &V'^T(\hat{\boldsymbol{X}}) f(\hat{\boldsymbol{X}}, \phi(\hat{\boldsymbol{X}})) \\ &= -[\hat{\boldsymbol{X}}^T R_1 \hat{\boldsymbol{X}} + h(\hat{\boldsymbol{X}}) + (\hat{\boldsymbol{X}}^T P + \frac{1}{2} g'^T(\hat{\boldsymbol{X}})) S (P\hat{\boldsymbol{X}} + \frac{1}{2} g'(\hat{\boldsymbol{X}}))] \end{aligned} \tag{44}$$

It can be seen from (51) that the condition (19) can be met if $h(\hat{\boldsymbol{X}}) \geq 0$ since $\hat{\boldsymbol{X}}^T R_1 \hat{\boldsymbol{X}}$ is positive semi-definite and $(\hat{\boldsymbol{X}}^T P + \frac{1}{2} g'^T(\hat{\boldsymbol{X}})) S (P\hat{\boldsymbol{X}} + \frac{1}{2} g'(\hat{\boldsymbol{X}}))$ is positive definite. By selecting proper values of the weights w_p, w_v, and w_c, one can always make $h(\hat{\boldsymbol{X}}) \geq 0$. Specifically, if all the agents are outside the detection region, $h(\hat{\boldsymbol{X}}) = 0$ by the definition of G_p in (29). $h(\hat{\boldsymbol{X}}) > 0$ can be guaranteed if one choose a large enough w_v, small enough w_p and w_c such that the positive terms $\hat{\boldsymbol{v}}^T (L \otimes I_m)(G_p \otimes I_m)(L \otimes I_m) \hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{v}}^T (L \otimes I_m)(G_p^2 \otimes I_m)(L \otimes I_m) \hat{\boldsymbol{v}}$ in (44) are always greater than other sign-indefinite terms.

Next we will verify the conditions (16) and (17). Note that

$$\begin{aligned} \hat{\boldsymbol{X}}^T P \hat{\boldsymbol{X}} &= \hat{\boldsymbol{X}}^T \left(\begin{bmatrix} w_p w_v L^2 & w_p w_c L \\ w_p w_c L & w_c w_v L \end{bmatrix} \otimes I_m \right) \hat{\boldsymbol{X}} \\ &= w_p w_v \hat{\boldsymbol{p}}^T (L^2 \otimes I_m) \hat{\boldsymbol{p}} + w_c w_v \hat{\boldsymbol{v}}^T (L \otimes I_m) \hat{\boldsymbol{v}} + 2 w_p w_c \hat{\boldsymbol{p}}^T (L \otimes I_m) \hat{\boldsymbol{v}} \\ &= w_p w_v \boldsymbol{p}^T (L^2 \otimes I_m) \boldsymbol{p} + w_c w_v \boldsymbol{v}^T (L \otimes I_m) \boldsymbol{v} + 2 w_p w_c \boldsymbol{p}^T (L \otimes I_m) \boldsymbol{v} \end{aligned} \tag{45}$$

The last equality is obtained using the property (5).

The Lyapunov function finally turns out to be:

$$V(\hat{\boldsymbol{X}}) = \hat{\boldsymbol{X}}^T P \hat{\boldsymbol{X}} + g(\hat{\boldsymbol{X}})$$

$$= \begin{cases} \hat{\boldsymbol{X}}^T P \hat{\boldsymbol{X}} & \mathrm{R} < \|\boldsymbol{p}_i - O_b\| \\ \hat{\boldsymbol{X}}^T P \hat{\boldsymbol{X}} + \frac{1}{2}\hat{\boldsymbol{v}}^T (G_p \otimes I_m)(L \otimes I_m)\hat{\boldsymbol{v}} & \mathrm{r} < \|\boldsymbol{p}_i - O_b\| \le \mathrm{R} \\ \text{not defined} & \|\boldsymbol{p}_i - O_b\| \le \mathrm{r} \end{cases} \quad (46)$$

It can be seen from (45) and (46) that the condition (16) is satisfied. Moreover, if $\hat{\boldsymbol{X}} \neq \boldsymbol{0}$, i.e. $\boldsymbol{X} \neq \boldsymbol{X}_{cs}$, $\boldsymbol{p}^T (L^2 \otimes I_m)\boldsymbol{p}$ and $\boldsymbol{v}^T (L \otimes I_m)\boldsymbol{v}$ will not be equal to zero but positive according to the property of L, i.e. Eq. (5). Note that $\boldsymbol{p} = \boldsymbol{0}$ and $\boldsymbol{v} = \boldsymbol{0}$ that leads to $\boldsymbol{p}^T (L^2 \otimes I_m)\boldsymbol{p} = 0$ and $\boldsymbol{v}^T (L \otimes I_m)\boldsymbol{v} = 0$ is a special case of $\boldsymbol{p} = \boldsymbol{p}_{cs}$ and $\boldsymbol{v} = \boldsymbol{v}_{cs}$ when $\boldsymbol{p}_{cs} = \boldsymbol{0}$ and $\boldsymbol{v}_{cs} = \boldsymbol{0}$, which implies $\hat{\boldsymbol{X}} = \boldsymbol{0}$ as well. Therefore, the condition (17), $V(\hat{\boldsymbol{X}}) > 0$ when $\hat{\boldsymbol{X}} \neq \boldsymbol{0}$, can be met by selecting a large enough w_v for given w_p and w_c such that the positive terms $w_p w_v \boldsymbol{p}^T (L^2 \otimes I_m)\boldsymbol{p}$ and $w_c w_v \boldsymbol{v}^T (L \otimes I_m)\boldsymbol{v}$ are always greater than the sign-indefinite terms.

Substituting P and $g'(\hat{\boldsymbol{X}})$ into (35) leads to

$$\phi(\hat{\boldsymbol{X}}) = -\frac{w_p}{w_c}(L \otimes I_m)\hat{\boldsymbol{p}} - \frac{w_v}{w_c}(L \otimes I_m)\hat{\boldsymbol{v}} - \frac{1}{2w_c^2} g_v'(\hat{\boldsymbol{X}}) \quad (47)$$

which turns out to be Eq. (26) by substituting $\hat{\boldsymbol{p}} = \boldsymbol{p} - \boldsymbol{p}_{cs}$ and $\hat{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{v}_{cs}$ into (47) and using the property of (5). Note that the optimal control law (26) is only a function of $\boldsymbol{X}$. This is desired because $\boldsymbol{X}_{cs}$ is not known *a priori*.

Now, all the conditions (16)-(21) in *Lemma 4.1* can be satisfied by selecting a large enough w_v and small enough w_p and w_c. Furthermore, this rule of weight selection also applies to satisfy the condition (10). Therefore, according to *Lemma 4.1*, the control law (26) is an optimal control law for the problem (8) in the sense of (23) and (24), and the closed-loop system is asymptotically stable. It implies $\boldsymbol{X} = \boldsymbol{X}_{cs}$ and the consensus is achieved.

In addition, it can be easily seen from (46) that $V(\hat{\boldsymbol{X}}) \to \infty$ as $\|\hat{\boldsymbol{X}}\| \to \infty$. Therefore, the closed-loop system is globally asymptotically stable. Note that the globally asymptotic stability region excludes the undefined area $\{\boldsymbol{p} \,|\, \|\boldsymbol{p}_i - O_b\| \le \mathrm{r}\}$, which is physically meaningful because no agent can start from inside the obstacle. ■

Remark 4.2: As can be seen from *Theorem 4.1*, the optimal consensus algorithm is developed from an inverse optimal control approach since the cost function $h(\hat{\boldsymbol{X}})$ is not given *a priori* but constructed from the optimality condition (39).

Remark 4.3: From (26) and $g_v'(\boldsymbol{X})$ in (30), it can be also seen that the optimal control law of each agent only requires the local information based on the information exchange topology since it is a linear function of L.

IV. Simulation Results and Analysis

In this section, two simulation scenarios are used to validate the proposed optimal consensus algorithm. Consider a planar motion in Fig. 1 with 4 agents and thus $m = 2$.

The initial positions are given by (-2, -2), (2, -2), (2, 2) and (-2, 2), respectively. The initial velocities are assumed to be (0.2, 0.4), (-0.4, 0.2), (-0.2, -0.4), and (0.2, -0.2), respectively. The weights in the consensus algorithm are set to $w_p = 0.04$, $w_v = 1.2$, and $w_c = 0.8$.

A. Consensus without obstacles on the trajectories

In this scenario, an obstacle is assumed to appear on (2, 0), which is not on the trajectory of any agent. The radius of the obstacle and the detection region are set to $\mathrm{r} = 0.1$ and $\mathrm{R} = 0.5$. The simulation results of the four agents' motion are shown in Fig. 2. As can be seen, the obstacle avoidance does not take effect since no agent steps into the detection region and the four agents achieve consensus.

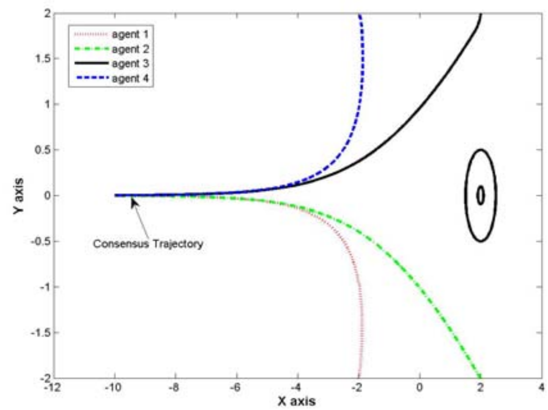

Fig. 2: Trajectories of the four agents without obstacle

Note that $h(\hat{\boldsymbol{X}})$ in the obstacle avoidance cost function is equal to zero since all the agents are outside the detection region, which implies that the problem is just a normal optimal consensus problem. It has the same form as the conventional consensus algorithm for networked double integrator systems [18] except the weighting parameters.

B. Consensus with multiple obstacles on the trajectories

In this scenario, one obstacle with the same radius and detection region as Scenario *A* is assumed to appear on (1, 1.3), which is on the trajectory of agent 3. The other obstacle with $\mathrm{r} = 0.2$ and $\mathrm{R} = 0.8$ is assumed to appear on (0.5, 3.2), which is on the trajectories of agent 1 and agent 4.

The simulation results are shown in Figs. 3-5. Fig. 3 demonstrates that all agents avoid the obstacles and reach the final consensus. Fig. 4 presents the time histories of the agents' positions and velocities. The optimal control inputs are shown in Fig. 5. In the bottom two subfigures of Fig. 5, the time histories in the first 50 seconds are shown for better illustrating the transient responses. The velocity response and the control response show that the optimal obstacle avoidance control law does not require large control effort.

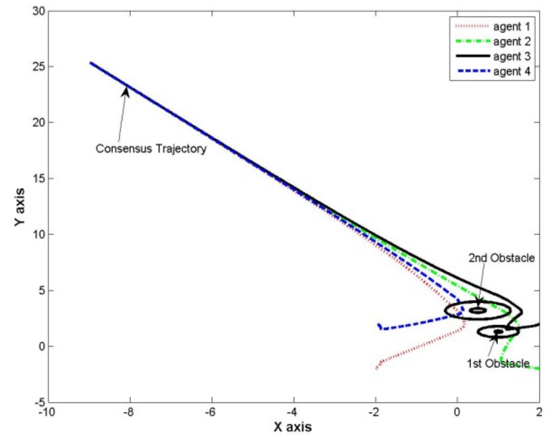

Fig. 3: Trajectories of the four agents with two obstacles

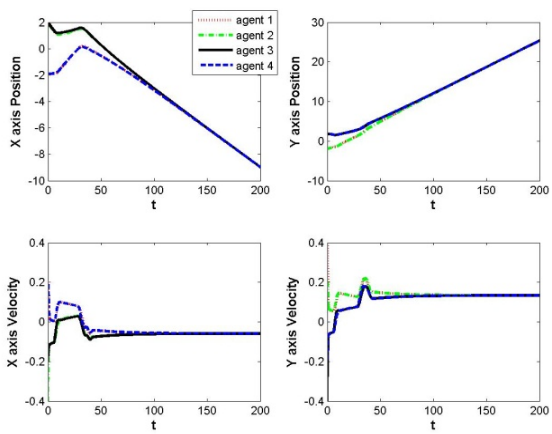

Fig. 4: Time histories of the four agents' positions and velocities

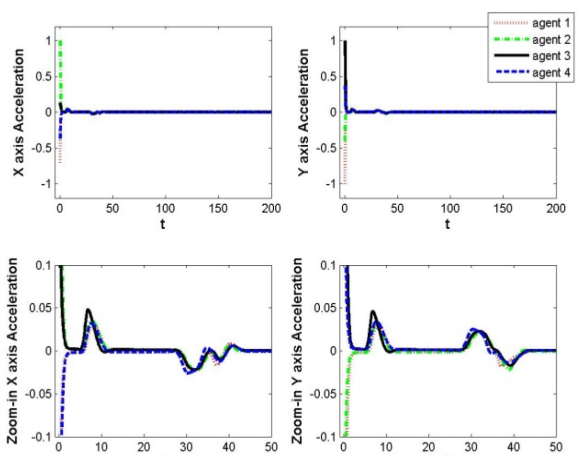

Fig. 5: Time histories of the four agents' optimal control inputs

V. Conclusion

In this paper, a novel optimal control law was developed for multi-agent consensus with obstacle avoidance. The primary contribution is to formulate the consensus problem and obstacle avoidance in a unified optimal control framework. A nonquadratic obstacle avoidance cost function was constructed from an inverse optimal control approach such that the optimal control law can be obtained in an analytical form and was shown to be a linear function of the Laplacian matrix, which indicates that the control law requires only the local information and offers a great implementation advantage. Both globally asymptotic stability and optimality of this algorithm have been proven. The simulation results have demonstrated that the proposed optimal approach is capable of solving the consensus problem under different obstacle avoidance scenarios.

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