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Semistability of Retarded Functional Differential Equations

Qing Hui

Abstract— **In this paper, we develop a semistability analysis framework for retarded functional differential equations (RFDE) having a continuum of equilibria with time-varying parameters and delays with applications to stability analysis of multiagent dynamic networks with consensus protocols in the presence of unknown heterogeneous time-varying delays and parameters along the communication links. We show that for such a retarded functional differential equation, if the system asymptotically converges to an autonomous functional differential inclusion with constant time-delays and this new system is semistable, then the original retarded functional differential equation system is semistable, provided that the delays are just bounded, not necessarily differentiable. In proving our results, we extend the limiting equation approach to the retarded functional differential equation systems and also develop some new convergence results for functional differential equations and differential inclusions.**

I. Introduction

Delays are unavoidable in communication, where information has to be transmitted over a physical distance. Unfortunately, very little research has been done to investigate the effect of delays on stability of consensus of multiagent networks. To accurately describe the evolution of networked cooperative systems, it is necessary to include in any mathematical model of the system dynamics some information about the past system states. In this case, the state of the system at a given time involves a portion of trajectories in the space of continuous functions defined on an interval of the state space, which leads to (infinite-dimensional) delay dynamical systems [1].

Previously, most of the reported work has either explicitly or implicitly employed the assumption that delays are known and continuously differentiable. Under such an assumption, one can use the delayed state of an agent in its own local control law to match the delays of the states from the neighboring agents [2]–[6], i.e. agent i can use a delayed version of its own state, $x_i(t-\tau_{ij}(t))$. Under that assumption, the control law is

$$u_i = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t-\tau_{ij}) - x_i(t-\tau_{ij})), \tag{1}$$

where $\mathcal{N}_i$ denotes the set of all other agents having a communication with agent i. If the delays are constant and uniform, $\tau_{ij}=\tau$ for all ij, then the network dynamics are of the form of time-delayed linear systems with the system matrix being the Laplacian, $\dot{x} = Lx(t-\tau)$, for which various analysis tools for linear systems with delays can be applied [3], [5], [6]. Additionally, the control law in (1) allows one to utilize disagreement dynamics, in which the disagreement $x_j(t-\tau_{ij}) - x_i(t-\tau_{ij})$ is the delayed version of the disagreement $x_j(t)-x_i(t)$. Because of the preceding property, one can study the behavior of the networks using disagreement dynamics or reduced disagreement dynamics in a similar fashion to the case without delays (the reduced disagreement dynamics are asymptotically stable). However, if the delays are unknown, time-varying, and not uniform over the communication links, the assumption that agent i has access to the delayed state $x_i(t-\tau_{ij}(t))$ raises a practical concern. If agent i does not have $x_i(t-\tau_{ij}(t))$ to use in the control protocol (in which case we say that the delays are asymmetric), the control law actually becomes

$$u_i = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t-\tau_{ij}) - x_i(t)). \tag{2}$$

Because $x_j(t-\tau_{ij}) - x_i(t)$ is no longer the delayed version of the disagreement $x_j(t)-x_i(t)$, the derivatives of the disagreements are not functions of the disagreements only), and hence, the approaches in [3], [5], [6] are not applicable to networks with the protocol (2). Stability of dynamic networks in such a situation has only recently been addressed [7]–[9], most of which are limited to the case of constant time delays. In particular, the authors in [7] have shown that dynamic networks with consensus protocols in the presence of heterogeneous delays are stable for arbitrary constant delays. Another closely related work is [10], where the authors consider networks with different arrival times for communication and with zero-order hold control laws, which leads to discrete-time dynamic networks formulation without time-delays for the overall closed loop. At the same time the authors in [11] explores the time-varying delays for network consensus. However, a major deficiency in [11] is that one has to assume the delays are sufficiently heterogeneous so that the closed-loop system does not have a limit cycle. This assumption is extremely difficult to verify for a nonlinear time delay system and hence, the results in [11] are not really useful. Left open is the problem of stability and convergence of time-varying consensus dynamic networks in the presence of unknown asymmetric non-uniform time-varying delays, which turns out to be a consequence of the more general results in this paper.

In this paper, we develop a general framework for semistability analysis of retarded functional differential equations having a continuum of equilibria and time-varying delays in which the delays are unknown and continuous with respect to time, not necessarily continuously differentiable. Here

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The author is with the Department of Mechanical Engineering, Texas Tech University, Lubbock, TX 79409-1021, USA (qing.hui@ttu.edu).

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semistability is the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. The basic assumption for the main result in this paper involves the idea of *limiting equations* [12] by assuming that the original retarded functional differential equation system asymptotically converges to an autonomous functional differential inclusion system with constant delays. Using these results, next we present stability analysis of time-varying consensus dynamic networks in the presence of time-varying parameters and unknown asymmetric non-uniform time-varying delays. The main feature of the proposed framework is that the assumption on continuous differentiability of the time delays is considerably weakened by use of a limiting function assumption, which is more natural and useful in practical systems. The proposed new results can be viewed as a generalization of network consensus with constant time delays in [7].

II. Mathematical Preliminaries

Let $\mathbb{R}^n$ denote the real Euclidean space of n-dimensional column vectors and let $\|x\|$ denote the norm of the vector x in $\mathbb{R}^n$. Let $r \geq 0$ be given and let $\mathcal{C} = \mathcal{C}([-r, 0], \mathbb{R}^n)$ denote the space of continuous functions that map the interval $[-r, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence and designated norm given by $\|\phi\| := \sup_{-r\leq\theta\leq 0} \|\phi(\theta)\|$ for $\phi \in \mathcal{C}$. Even though double bars are used for norms in different spaces, no confusion should arise. If $x : [-r, \infty) \to \mathbb{R}^n$ be continuous, then for any $t \geq 0$, $x_t \in \mathcal{C}$ is defined by $x_t(s) = x(t + s)$, $-r \leq s \leq 0$.

Consider a *retarded functional differential equation* (RFDE) [1] on $\mathcal{C}$ given by

$$\dot{x}(t) = f(t, x_t), \tag{3}$$

where $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ satisfies the *Carathéodory condition* [1, p. 58] and maps closed and bounded sets into bounded sets. Define the equilibrium set of (3) as $\mathcal{E} := \{\phi \in \mathcal{C} : f(t, \phi) = 0, \ \forall\, t \in \mathbb{R}\}$. Given $\phi \in \mathcal{C}$ and $\tau > 0$, a function $x(\phi)$ is said to be a solution to (3) on $[-r, \tau)$ with initial condition ϕ if $\phi \in \mathcal{C}([-r, \tau), \mathbb{R}^n)$, $x_t \in \mathcal{C}$, $x(t)$ satisfies (3) for $t \in [0, \tau)$ and $x(\phi)(0) = \phi$, where $x(\phi)(\cdot)$ denotes the solution through $(0, \phi)$.

Throughout this paper, we make the following standing assumption on (3).

Assumption 2.1: $\mathcal{E}$ is a connected set.

Recall that a set $\mathcal{E} \subseteq \mathcal{C}$ is *connected* if every pair of open sets $\mathcal{U}_i \subseteq \mathcal{C}$, $i = 1, 2$, satisfying $\mathcal{E} \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$ and $\mathcal{U}_i \cap \mathcal{E} \neq \varnothing$, $i = 1, 2$, has a nonempty intersection. Assumption 2.1 implies that (3) has a continuum of equilibria. In other words, the equilibria of (3) are *not* isolated equilibrium points. This situation occurs in many practical problems such as compartmental modeling of biological systems [13], thermodynamic systems [14], multiagent coordinated networks [5], [7], [11], and synchronization of coupled oscillators [8].

Example 2.1: Consider a special case of (3) where $f(t, x_t) = E(t)x(t) + \sum_{k=1}^{m} F_k(t)x(t - \tau_k(t))$ and $E(t), F_k(t) \in \mathbb{R}^{n\times n}$ are matrix functions, $k = 1, \ldots, m$. If $E(t) + \sum_{k=1}^{m} F_k(t)$ is singular for all $t \in \mathbb{R}$, then $\mathcal{E}$ is a connected set, i.e., (3) has a continuum of equilibria. A relevant example for this case is the consensus problem with time-varying delays given by the consensus protocol

$$\dot{x}(t) = E(t)x(t) + \sum_{k=1}^{m} F_k(t)x(t - \tau_k(t)), \tag{4}$$

where $0 \leq \tau_k(t) \leq r$ and $E(t) + \sum_{k=1}^{m} F_k(t)$ is a Laplacian matrix function. ▲

Recall that a point $z \in \mathcal{C}$ is a positive limit point of a solution $x(t)$ to (3) with $x(s) = \phi(s)$, $-h \leq s \leq 0$, if there exists a sequence $\{t_n\}_{n=1}^{\infty}$ with $t_n \to +\infty$ and $x(t_n) \to z$ as $n \to +\infty$. The set $\omega(\phi)$ of all such positive limit points is the positive limit set of $x_0 = \phi \in \mathcal{C}$ [1, p. 102]. Motivated by Lemma 2.2 of [15], we have the following result.

Lemma 2.1: Assume that the solutions of (3) are bounded and let $x(\cdot)$ be a solution of (3) with $x_0 = \phi \in \mathcal{C}$. If $z \in \omega(x_0)$ is a Lyapunov stable equilibrium point of (3), then $z = \lim_{t\to\infty} x(t)$ and $\omega(\phi) = \{z\}$.

Definition 2.1: i) An equilibrium point $x \in \mathcal{E}$ is *semistable* if there exists an open set $\mathcal{U} \subseteq \mathcal{C}$ containing x such that for every initial condition in $\mathcal{U}$, the trajectory of (3) converges, that is, $\lim_{t\to\infty} x(t)$ exists, and every equilibrium point in $\mathcal{U}$ is Lyapunov stable. The system (3) is *semistable* if every equilibrium point in $\mathcal{E}$ is semistable.

ii) An equilibrium point $x \in \mathcal{E}$ is *uniformly semistable* if there exists an open set $\mathcal{U} \subseteq \mathcal{C}$ containing x such that for every initial condition in $\mathcal{U}$, the trajectory of (3) uniformly converges, that is, $\lim_{t\to\infty} x(t)$ converges uniformly in the initial time instant, and every equilibrium point in $\mathcal{U}$ is uniformly Lyapunov stable. The system (3) is *uniformly semistable* if every equilibrium point in $\mathcal{E}$ is uniformly semistable.

III. Main Results

A. Nonlinear RFDE Systems

In this section, we use a limiting system approach to study the asymptotic behavior of the non-autonomous RFDE (3). Here our limiting system becomes a *retarded functional differential inclusion* (RFDI), which is more general and flexible than a differential equation always considered in the literature. To this end, we first introduce the notion of weak asymptotic autonomy for (3). This notion has been introduced in [16] for finite-dimensional non-autonomous differential equations.

Definition 3.1: Let $\mathbb{U}$ denote the class of set-valued maps $z \mapsto \mathcal{F}(z) \subset \mathbb{R}^n$, defined on $\mathcal{C}$, that are upper semicontinuous at each $z \in \mathcal{C}$ and take non-empty convex compact values. The vector field $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ of (3) is said to be *weakly asymptotically autonomous* (WAA) if there exists $\mathcal{F} \in \mathbb{U}$ such that for all compact $\mathcal{D} \subset \mathcal{C}$ and all $\varepsilon > 0$, there exists $T \geq 0$ such that

$$\operatorname*{ess\,sup}_{t\geq T} \operatorname{dist}(f(t, z), \mathcal{F}(z)) < \varepsilon, \quad \forall z \in \mathcal{D}. \tag{5}$$

If (5) holds with $\mathcal{F}$ singleton-valued (i.e., $\mathcal{F} : z \mapsto \{f^*(z)\}$) for some completely continuous function $f^* : \mathcal{C} \to \mathbb{R}^n$, then f is said to be *asymptotically autonomous* (AA).

Remark 3.1: Condition 5 implies that $f(t, z)$ essentially approaches $\mathcal{F}(z)$ locally uniformly with respect to z as $t \to \infty$.

Definition 3.2: Suppose all the conditions in Definition 3.1 holds. Then the RFDI

$$\dot{x}(t) \in \mathcal{F}(x_t), \tag{6}$$

is called a *limiting system* of (3).

Remark 3.2: The idea of the limiting equation approach was originally from [12] and has been extended to various finite-dimensional dynamical systems by changing the definition of limiting functions [16]–[18]. Our definition extends this approach to infinite-dimensional time-varying dynamical systems and gives a new definition of limiting systems for time-delay systems.

Lemma 3.1: If f is WAA, then the limiting system (6) has the same equilibrium set as (3).

The following technical result demonstrates compactness of trajectories for RFDIs. To state this result, however, we need to define some notions first. Let $\mathcal{A}$ be a separable Banach space. A *multifunction* $\Gamma : \mathcal{A} \to \mathcal{C}$ is a mapping from $\mathcal{A}$ to the subsets of $\mathcal{C}$. If D is a subset of $\mathcal{A}$, we say that Γ is closed, compact, convex, or nonempty on D if for each $x \in D$, the set $\Gamma(x)$ has that particular property. A multifunction $\Gamma : D \to \mathcal{C}$ is said to be *measurable* if for all $x \in D$, the nonnegative-valued function $\omega \mapsto \mathrm{dist}(x, \Gamma(\omega)) := \inf\{\|x - y\| : y \in \Gamma(\omega)\}$ is measurable [19]. Consider the case where $D = [a, b]$. We say that Γ is *integrably bounded* on $[a, b]$ if there is an integrable function $\kappa(t)$ such that for all t in $[a, b]$ and all γ in $\Gamma(t)$, $\|\gamma\| \leq \kappa(t)$. Let $S, \mathcal{K}_t$ be the sets defined by $S := \{t : (t, x) \in \mathcal{K} \text{ for some } x \text{ in } \mathcal{C}\}$ and $\mathcal{K}_t := \{x \in \mathcal{C} : (t, x) \in \mathcal{K}\}$. $\mathcal{K}$ is called a *tube* if the set S is an interval (say, $[a, b]$) and there exist a continuous function $w(t)$ and a continuous positive function $\varepsilon(t)$ on $[a, b]$ such that $\mathcal{K}_t = \mathcal{B}_{\varepsilon(t)}(w(t))$, where $\mathcal{B}_r(s)$ denotes the open ball centered at s with radius r.

Let Γ be a multifunction defined on a tube $\mathcal{K}$ on $[a, b]$. We assume that Γ is integrably bounded by κ on $\mathcal{K}$ and Γ is nonempty, compact, and convex on $\mathcal{K}$. Furthermore, we assume that there exist a multifunction $X : [a, b] \to \mathcal{C}$ and a positive-valued function $\varepsilon(t)$ with the following properties:

1) For all $t \in [a, b]$, $\mathcal{B}_{\varepsilon(t)}(X(t)) \subset \mathcal{K}_t$.
2) For every $t \in [a, b]$ and every $x \in \mathcal{B}_{\varepsilon(t)}(X(t))$, the multifunction $\phi \mapsto \Gamma(t, \phi)$ is upper semicontinuous at x.
3) For every (t, x) in the interior of $\mathcal{K}$, the multifunction $t' \mapsto \Gamma(t', x)$ is measurable.

An *arc* is a function $x(\cdot)$ having a derivative at t denoted by $\dot{x}(t)$ for almost all $t \in [a, b]$ and which is the integral of its derivative. A *trajectory* for Γ is an arc x such that for almost all $t \in [a, b]$, $\dot{x}(t)$ belongs to the set $\Gamma(t, x(t))$, i.e., $\dot{x}(t) \in \Gamma(t, x(t))$ a.e. and "a.e." denotes almost everywhere.

Lemma 3.2: Let $\{x_j\}$ be a sequence of arcs on $[a, b]$ satisfying

i) $x_j(t) \in X(t)$ and $\|\dot{x}_j(t)\| \leq \kappa(t)$ for almost all t in $[a, b]$.

ii) $\dot{x}_j(t) \in \mathcal{B}_{r_j(t)}(\Gamma(t, x_j(t) + y_j(t)))$ for $t \in A_j$, where $\{y_j\}, \{r_j\}$ are sequences of measurable functions on $[a, b]$ which converge uniformly to 0, and $\{A_j\}$ is a sequence of measurable subsets of $[a, b]$ such that the measure of A_j converges to $(b - a)$.

iii) The sequence $\{x_j(a)\}$ is bounded.

Then there exists a subsequence of $\{x_j\}$ which converges uniformly to an arc x which is a trajectory for Γ.

Based on the notion of limiting systems, we have the following convergence result.

Lemma 3.3: Consider the RFDE (3). Assume the trajectories of (3) are bounded. Furthermore, assume (6) is a limiting system of (3), that is, f is WAA. Let $x(t)$ be a global solution to (3). Then $x(t) \to \mathcal{F}^{-1}(0)$ as $t \to \infty$ and $\omega(x) \subseteq \mathcal{F}^{-1}(0)$.

Lemma 3.4: Consider the RFDE (3). Assume the trajectories of (3) are bounded. Furthermore, assume (6) is a limiting system of (3), that is, f is WAA. Then $\omega(\phi)$ is invariant with respect to (6) for every initial condition $x_0 = \phi \in \mathcal{C}$.

Lemma 3.5: Consider (6). If the trajectories of (6) converge, that is, $\lim_{t\to\infty} z_t(\phi)$ exists for every $\phi \in \mathcal{C}$, then the function $\Omega : \mathcal{C} \to \mathcal{C}$ defined by $\Omega(\phi) = \lim_{t\to\infty} z_t(\phi)$, $\phi \in \mathcal{C}$, is an equilibrium point for (6).

Now we have the main result for this paper. For the definition of semistability of differential inclusions, see [20].

Theorem 3.1: Consider the RFDE (3). Assume (3) is Lyapunov stable. Furthermore, assume (6) is a limiting system of (3) and (6) is semistable. Then (3) is semistable. Alternatively, if (3) is uniformly Lyapunov stable, (6) is a limiting system of (3), and (6) is uniformly semistable, then (3) is uniformly semistable.

Remark 3.3: To discuss semistability of (3) using Theorem 3.1, one has to know the information on Lyapunov stability of (3). Note that here we only assume $\tau_k(t)$ is *continuous* for every $k = 1, \ldots, m$. Hence, it is very difficult to use the Lyapunov-Krasovskii functional approach [1], [21] to prove the Lyapunov stability of (3) since it requires the first-order derivative of $\tau_k(t)$. In this case, the Lyapunov stability of (3) may be verified using Razumikhin theorems via Lyapunov-Razumikhin functions [1], [22], [23].

Example 3.1: Consider the scalar time-delay system given by

$$\dot{x}(t) = -a(t)x(t) + b(t)x(t - \tau(t)), \tag{7}$$

where $x(t) \in \mathbb{R}$, $a(\cdot)$, $b(\cdot)$, and $\tau(\cdot)$ are continuous, $|b(t)| \leq a(t)$, and $0 \leq \tau(t) \leq h$ for all $t \in \mathbb{R}$. Consider the Lyapunov-Razumikhin function given by $V(x) = (x - \alpha)^2/2$, where α is an arbitrary constant. Then it follows from Theorem 4.1 of Chapter 5 of [1] that (7) is uniformly Lyapunov stable. See [1, p. 154] for a detailed proof. ▲

Remark 3.4: Consider the RFDE (3) with $f(t, x_t) = F(t, x(t)) + G(t, x(t - \tau_1(t)), \ldots, x(t - \tau_m(t)))$. Suppose the trajectories of (3) are bounded. If $\lim_{t\to\infty} F(t, x) = \bar{f}(x)$, $\bar{f}(\cdot)$ is a continuous function, $G(t, x(t - \tau_1(t)), \ldots, x(t - \tau_m(t))) = \sum_{k=1}^{m} g_k(t, x(t - \tau_k(t)))$, $g_k(t, x)$ is globally Lipschitz continuous with respect to x, $\|g_k(t, 0)\| \leq$

M_k, $\lim_{t\to\infty} g_k(t,x) = \bar{g}_k(x)$, $k = 1,\ldots,m$, and $\lim_{t\to\infty}\tau_k(t) = h_k$ for every $k = 1,\ldots,m$, then (6) is a limiting system of (3). To see this, suppose $\|x(t)\| \le M$. Then from (3), there exists $\epsilon > 0$ such that $\|\dot{x}(t)\| \le \epsilon + \sup_{\|z\|\le M}\|\beta(z)\| + \sum_{k=1}^m (L_k M + M_k) := K$, where L_k is the Lipschitz constant, $k = 1,\ldots,m$. Because $x(t-\tau_k(t)) - x(t-h_k) = \int_{t-h_k}^{t-\tau_k(t)}\dot{x}(s)ds$, it follows that $\|x(t-\tau_k(t)) - x(t-h_k)\| \le K|\tau_k(t)-h_k|$. Hence, $\|g_k(t,x(t-\tau_k(t))) - \bar{g}_k(x(t-h_k))\| \le \|g_k(t,x(t-\tau_k(t))) - g_k(t,x(t-h_k))\| + \|g_k(t,x(t-h_k)) - \bar{g}_k(x(t-h_k))\| \le L_k\|x(t-\tau_k(t)) - x(t-h_k)\| + \|g_k(t,x(t-h_k)) - \bar{g}_k(x(t-h_k))\| \le KL_k|\tau_k(t)-h_k| + \|g_k(t,x(t-h_k)) - \bar{g}_k(x(t-h_k))\|$. Thus, if $\lim_{t\to\infty}\tau_k(t) = h_k$ and $\lim_{t\to\infty} g_k(t,x) = \bar{g}_k(x)$, then $\lim_{t\to\infty}\{f(t,x_t) - [\bar{f}(x) + \sum_{k=1}^m \bar{g}_k(x(t-h_k))]\} = 0$. By definition,

$$\dot{z}(t) = \bar{f}(z(t)) + \sum_{k=1}^m \bar{g}_k(z(t-h_k)) \tag{8}$$

is a limiting system of (4).

Next, motivated by [20], we present a Lyapunov-type result for semistability of autonomous functional differential inclusions with constant time delays using Lyapunov-Krasovskii functionals. This result will help us determine the semistability of (6) which is required by Theorem 3.1. For the notion of *weakly positive invariance*, see [20] for the details.

Theorem 3.2: Consider the RFDI (6). Assume the trajectories of (6) are bounded and there exists a continuous functional $V : \mathcal{C} \to \mathbb{R}$ such that $\dot{V}$ is defined on $\mathcal{C}$ and $\dot{V}(\phi) \le 0$ for all $\phi \in \mathcal{C}$. If every point in the largest weakly positively invariant set $\mathcal{M}$ of $\dot{V}^{-1}(0)$ is a Lyapunov stable equilibrium point of (6), then (6) is semistable.

As an alternative to Theorem 3.2, we present a Lyapunov-Razumikhin function approach to semistability analysis of RFDIs with constant time delays. Motivated by [24], this result gives a different method to prove semistability of (6) other than Theorem 3.2, which is useful for many cases in that constructing a Lyapunov-Krasovskii functional for (6) may not be an easy task in these cases.

Theorem 3.3: Consider the RFDI (6). Assume the trajectories of (6) are bounded and there exists a continuous function $V : \mathcal{C} \to \mathbb{R}$ such that $\dot{V}$ is defined on $\mathcal{C}$ and $\dot{V}(\phi) \le 0$ for all $\phi \in \mathcal{C}$ such that $V(\phi(0)) = \max_{-h\le s\le 0} V(\phi(s))$. If every point in the largest weakly positively invariant set $\mathcal{M}$ of $\mathcal{R} := \{\phi \in \mathcal{C} : \max_{s\in[-h,0]} V(z_t(\phi)(s)) = \max_{s\in[-h,0]} V(\phi(s)), \forall t \ge 0\}$ is a Lyapunov stable equilibrium point of (6), then (6) is semistable.

B. Specialization to the Consensus Problem

Lemma 3.6: Consider the time-delay system (4). Assume the trajectories of (4) are bounded. If $\lim_{t\to\infty} E(t) = E$, $\lim_{t\to\infty} F_k(t) = F_k$, and $\lim_{t\to\infty}\tau_k(t) = h_k$ for every $k = 1,\ldots,m$, where E, F_k are some constant matrices, then

$$\dot{z}(t) = Ez(t) + \sum_{k=1}^m F_k z(t-h_k) \tag{9}$$

is a limiting system of (4).

Next, we present a Lyapunov stability result for (9). Define $F := \sum_{k=1}^m F_k$. For a matrix $A \in \mathbb{R}^{m\times n}$, we use $A_{(i,j)}$ to denote the (i,j)th element of A.

Lemma 3.7: Consider the time-delay system (9) having the following structure: all the elements in F_k are nonnegative, $k = 1,\ldots,m$,

$$E_{(i,j)} = \begin{cases} -\sum_{k=1}^n a_{ik}, & i = j, \\ 0, & i \ne j, \end{cases} \tag{10}$$

$$F_{(i,j)} = \begin{cases} 0, & i = j, \\ a_{ij}, & i \ne j, \end{cases} \tag{11}$$

$a_{ij} \ge 0$, $i,j = 1,\ldots,n$. Then (9) is Lyapunov stable.

The following proposition regarding semistability of time-varying delay network consensus protocols given by (4) follows directly from Lemmas 3.6 and 3.7, Theorem 3.1, and a result from [7]. To state this result, define $\mathbf{1} = [1,\ldots,1]^{\mathrm{T}} \in \mathbb{R}^n$.

Proposition 3.1: Consider the time-delay system (4). Assume that $\lim_{t\to\infty} E(t) = E$, $\lim_{t\to\infty} F_k(t) = F_k$, and $\lim_{t\to\infty}\tau_k(t) = h_k$ for every $k = 1,\ldots,m$. Furthermore, assume that E and F_k have the structure given by (10) and (11), $(E+F)^{\mathrm{T}}\mathbf{1} = (E+F)\mathbf{1} = 0$, and $\mathrm{rank}(E+F) = n-1$. Then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{1}$ is a semistable equilibrium point of (4). Furthermore, $x(t) \to \alpha^*\mathbf{1}$ as $t \to \infty$, where

$$\alpha^* = \frac{\mathbf{1}^{\mathrm{T}}\phi(0) + \sum_{k=1}^m \int_{-h_k}^0 \mathbf{1}^{\mathrm{T}} F_k \phi(\theta)\mathrm{d}\theta}{n + \sum_{k=1}^m h_k \mathbf{1}^{\mathrm{T}} F_k \mathbf{1}}. \tag{12}$$

Next, we generalize Proposition 3.1 to the nonlinear system given by

$$\dot{x}(t) = a(t)f(x(t)) + \sum_{k=1}^m b_k(t) g_k(x(t-\tau_k(t))), \tag{13}$$

where $f = [f_1,\ldots,f_q]^{\mathrm{T}}$ and $a(\cdot), b_k(\cdot)$ are scalar continuous functions. Using some result from [7], we have the following stability result for the nonlinear network consensus with time-varying delays given by the form of (13). Recall that for a diagonal matrix $\Lambda \in \mathbb{R}^{n\times n}$, the *Drazin inverse* $\Lambda^{\mathrm{D}} \in \mathbb{R}^{n\times n}$ is given by $\Lambda^{\mathrm{D}}_{(i,i)} = 0$ if $\Lambda_{(i,i)} = 0$ and $\Lambda^{\mathrm{D}}_{(i,i)} = 1/\Lambda_{(i,i)}$ if $\Lambda_{(i,i)} \ne 0$, $i = 1,\ldots,n$ [25, p. 227].

Proposition 3.2: Consider the time-delay system (13) where $f(0) = 0$, $g_k(0) = 0$, $k = 1,\ldots,m$, and $f_i(\cdot)$ is strictly decreasing for $f_i \not\equiv 0$, $i = 1,\ldots,n$. Assume (13) is Lyapunov stable, $0 < \lim_{t\to\infty} a(t) = \lim_{t\to\infty} b_k(t) < \infty$, and $\lim_{t\to\infty}\tau_k(t) = h_k$ for every $k = 1,\ldots,m$. Next, assume that $\mathbf{1}^{\mathrm{T}}(f(x) + \sum_{k=1}^m g_k(x)) = 0$ for all $x \in \mathbb{R}^n$ and $f(x) + \sum_{k=1}^m g_k(x) = 0$ if and only if $x = c\mathbf{1}$ for some $c \in \mathbb{R}$. Furthermore, assume that there exist nonnegative diagonal matrices $P_k \in \mathbb{R}^{n\times n}$, $k = 1,\ldots,m$, such that $P := \sum_{k=1}^m P_k > 0$, $P_k^{\mathrm{D}} P_k g_k(x) = g_k(x)$ for every $x \in \mathbb{R}^n$ and $k = 1,\ldots,m$, and

$$\sum_{k=1}^m g_k^{\mathrm{T}}(x) P_k g_k(x) \le f^{\mathrm{T}}(x) P f(x), \quad x \in \mathbb{R}^n, \tag{14}$$

$$\sum_{k=1}^m f^{\mathrm{T}}(x) P P_k^{\mathrm{D}} P f(x) \le f^{\mathrm{T}}(x) P f(x), \quad x \in \mathbb{R}^n. \tag{15}$$

Then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{1}$ is a semistable equilibrium point of (13). Furthermore, $x(t) \to \alpha^*\mathbf{1}$ as $t \to \infty$, where α^* satisfies

$$n\alpha^* + \sum_{k=1}^{m} h_k \mathbf{1}^{\mathrm{T}} g_k(\alpha^*\mathbf{1}) = \mathbf{1}^{\mathrm{T}}\phi(0) + \sum_{k=1}^{m}\int_{-h_k}^{0} \mathbf{1}^{\mathrm{T}} g_k(\phi(\theta))\mathrm{d}\theta. \qquad (16)$$

The inequality conditions (14) and (15) in Proposition 3.2 are implicit constraints in the sense that they are not the conditions in terms of P and P_k alone. Next, we present a sufficient condition on P and P_k to guarantee (15).

Lemma 3.8: Let $P_i = \mathrm{diag}[p_i^1, \ldots, p_i^q] \in \mathbb{R}^{q\times q}$, where $p_i^j \geq 0$, $i, j = 1, \ldots, q$ and $P = \sum_{i=1}^{q} P_i = \mathrm{diag}[p^1, \ldots, p^q]$. For any f_{c}, the equality

$$f_{\mathrm{c}}^{\mathrm{T}}(x)Pf_{\mathrm{c}}(x) = \sum_{i=1}^{q} f_{\mathrm{c}}^{\mathrm{T}}(x)PP_i^{\mathrm{D}}Pf_{\mathrm{c}}(x), \quad x \in \mathbb{R}^q \qquad (17)$$

holds if and only if for any j, there exists only one i_j such that $p^j = p_{i_j}^j$ and all the other $p_i^j = 0$, $i \neq i_j$, or in other words, each diagonal element of P comes from just one single P_i.

Now we have a corollary for Proposition 3.2.

Corollary 3.1: Consider the time-delay system (13) where $f(0) = 0$, $g_k(0) = 0$, $k = 1, \ldots, m$, and $f_i(\cdot)$ is strictly decreasing for $f_i \not\equiv 0$, $i = 1, \ldots, n$. Assume (13) is Lyapunov stable, $0 < \lim_{t\to\infty} a(t) = \lim_{t\to\infty} b_k(t) < \infty$, and $\lim_{t\to\infty} \tau_k(t) = h_k$ for every $k = 1, \ldots, m$. Next, assume that $\mathbf{1}^{\mathrm{T}}(f(x) + \sum_{k=1}^{m} g_k(x)) = 0$ for all $x \in \mathbb{R}^n$ and $f(x) + \sum_{k=1}^{m} g_k(x) = 0$ if and only if $x = c\mathbf{1}$ for some $c \in \mathbb{R}$. Furthermore, assume that there exist nonnegative diagonal matrices $P_k \in \mathbb{R}^{n\times n}$, $k = 1, \ldots, m$, such that $P := \sum_{k=1}^{m} P_k > 0$, each diagonal element of P comes from just one single P_k, $P_k^{\mathrm{D}} P_k g_k(x) = g_k(x)$ for every $x \in \mathbb{R}^n$ and $k = 1, \ldots, m$, and (14) holds. Then for every $\alpha \in \mathbb{R}$, $\alpha\mathbf{1}$ is a semistable equilibrium point of (13). Furthermore, $x(t) \to \alpha^*\mathbf{1}$ as $t \to \infty$, where α^* satisfies (16).

IV. Conclusions

A new general framework concerning semistability of RFDEs having a continuum of equilibria and asymptotically converging to an autonomous RFDI is presented and its applications to stability analysis of multiagent dynamic networks with consensus protocol in the presence of time-varying parameters and unknown heterogeneous time-varying delays are discussed in this paper. Those time delays are not necessarily differentiable and known. We provided conditions, in terms of the limiting system–a RFDI, to guarantee semistability of nonlinear time-varying systems with multiple time-varying delays and applied those stability results to show that multiagent dynamic networks can still achieve consensus in the presence of time-varying parameters and heterogeneous delays, provided that the parameters and the delays converge to their limits asymptotically.

There are many future research directions regarding semistability theory of RFDEs. For example, the existence of converse Lyapunov theorems for semistability of RFDEs remains an open problem. For finite-dimensional nonlinear systems, this has been proved by [26].

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