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Simplified controller design for distributed parameter systems using mobile actuator with augmented vehicle dynamics

Michael A. Demetriou

Abstract— **A way to mitigate the effects of spatiotemporally varying disturbances in systems governed by partial differential equations is to employ actuating devices that are capable of moving throughout the spatial domain. In this work, the guidance of such a moving actuator, that is able to move throughout the spatial domain and dispense the control signal at any spatial location, is considered. By assuming a specific structure of the controller architecture, namely that of a collocated input-output in which the control signal is proportional to the state evaluated at the spatial location of the actuating device, the problem under consideration simplifies to that of obtaining the guidance of the moving actuator. By incorporating the dynamics of a moving vehicle, a Lyapunov stability argument is made in order to obtain the requisite control torque of the vehicle. Extensive numerical simulations for a diffusion equation are included to verify the effectiveness of a such a mobile actuator-sensor pair in suppressing the effects of spatially varying disturbances.**

I. Introduction

The use of mobile and scheduled actuators and/or sensors for the improved estimation and control of PDEs has been receiving renewed attention in the last decade [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13]. The enabling guidance scheme was based on either the optimal regulation or optimal filtering of the spatially distributed process. The realization of such an optimal policy required the solution to large scale Riccati equations. A way to circumvent this computational demand is to address the implementational optimality, by considering a simpler structure of the controller or filter, and be concerned with the guidance or scheduling of the actuating and sensing devices.

This work considers a spatially distributed process in which the actuating device is affixed on a vehicle that is capable of moving throughout the interior of the spatial domain and dispense the control signal at any point within the spatial domain. By assuming a specific structure of the controller architecture, namely that of a static output feedback, the main concern becomes that of the actuator guidance. Using a Lyapunov argument, the guidance of the mobile actuator is derived. Two different Lyapunov functions are considered, each providing a different guidance policy for the moving actuator. The proposed scheme is tested numerically in a 1D diffusion PDE with a mobile actuator attached to a vehicle with 2nd order dynamics.

M. A. Demetriou is with Worcester Polytechnic Institute, Dept of Mechanical Engineering, Worcester, MA 01609, USA, mdemetri@wpi.edu. The author gratefully acknowledges financial support from the AFOSR, grant FA9550-09-1-0469.

II. Problem statement

We consider the state regulation of the diffusion PDE

$$\begin{aligned}\frac{\partial x(t,\xi)}{\partial t} &= a\frac{\partial^2 x(t,\xi)}{\partial \xi^2} + b(\xi)u(t),\\ x(t,0) &= x(t,\ell) = 0, \quad x(0,\xi) = x_0(\xi),\end{aligned} \tag{1}$$

where the function $b(\xi)$ denotes the spatial distribution of the actuating device and the temporal signal $u(t)$ denotes the control signal. It is desired to use an actuating device that is capable of moving within the interior of the spatial domain $\Omega = [0,\ell]$. Such an ability can be realized via the use of a mobile agent, on which the actuating device will be attached to. The reason for such a moving actuator is to be able to effectively address the effects of spatiotemporally varying disturbances (spatially moving disturbances).

It is assumed that $b(\xi)$ takes the form of a spatial delta function with centroid at the location $\theta \in \Omega$ and given by $b(\xi) = \delta(\xi - \theta)$. For the case of a moving actuating device, the centroid will now be time varying. The dependence of the actuating device on the time varying centroid is made explicitly with $b(\xi) = b(\xi;\theta(t))$ and therefore

$$b(\xi;\theta(t)) = \delta(\xi - \theta(t)). \tag{2}$$

The control tasks associated with the above infinite dimensional system with a time varying centroid of the actuator spatial distribution (mobile actuator) are summarized:

1) how to choose the controller architecture
2) how to choose the actuator guidance

As was similarly proposed in [14], [15], [16] a way to address the first task is to use a static output feedback in which case a sensing device is collocated to the actuating device. Such a controller structure takes the form

$$u(t) = -\kappa x(t,\theta(t)) = -\kappa\int_0^\ell \delta(\xi-\theta(t))x(t,\xi)\,d\xi \tag{3}$$

where $\kappa > 0$ is the feedback (scalar) gain and $x(t,\theta)$ is interpreted as the value of the state at the spatial location $\theta(t)$ that coincides with the centroid of the actuator spatial distribution. With the above, one then must only be concerned with the guidance of the collocated actuator-sensor pair within the spatial domain Ω.

Revised notation: The motion of the actuator centroid $\theta(t)$ within the spatial domain Ω will affect the state of the closed loop system. Therefore in order to emphasize the dependence of the state on the actuator centroid $\theta(t)$, we explicitly state

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this dependence as $x(t, \xi; \theta(t))$. In this case, the closed loop system is re-written as

$$\Sigma_1 \begin{cases} \dfrac{\partial x(t,\xi;\theta(t))}{\partial t} = a\dfrac{\partial^2 x(t,\xi;\theta(t))}{\partial \xi^2} + \delta(\xi-\theta(t))u(t), \\ x(t,0;\theta(t)) = x(t,\ell;\theta(t)) = 0, \\ x(0,\xi;\theta(t)) = x_0(\xi), \\ u(t) = -\kappa \displaystyle\int_0^\ell \delta(\xi-\theta(t))x(t,\xi;\theta(t))\,d\xi. \end{cases}$$

We are now left with addressing the second task, namely that of centroid guidance; this is essentially the derivation of the variation of $\theta(t)$, either in terms of explicit assignment of $\theta(t)$ to a function, or by assigning the time derivative (velocity) $\dot{\theta}$ to a function. In earlier works the velocity $\dot{\theta}(t)$ was derived via the aid of a Lyapunov function. In [17], [15], [18] using an MPC-like scheme, the position θ of a collocated actuator-sensor pair was proposed for the containment of a moving source. Using a Lyapunov scheme in [16], [19], the velocity $\dot{\theta}(t)$ of a massless mobile agent was expressed in terms of the state measurement and the spatial gradient at the current location $\theta(t)$.

In the above works on the guidance of the mobile actuator, the simplified assumption of a massless and inertialess agent was made. Inclusion of either kinematics or dynamics of the mobile agent that carries the collocated actuator-sensor pair would certainly alter the (idealized) performance of the controller. A way to account for agent dynamics, is to augment the Lyapunov function, used in the stability and guidance derivation, by the energy of the motion dynamics of the mobile actuator-sensor.

The dynamics of the mobile actuator will be assumed here and which are given by

$$m\ddot{\theta}(t) + d\dot{\theta}(t) + k\theta(t) = f(t),\ \theta(0) = \theta_0,\ \dot{\theta}(0) = 0, \quad (4)$$

where $\theta(t)$ denotes the position of the mobile actuator (the location of the centroid of the actuator's spatial distribution) within the spatial domain $\Omega = [0, \ell]$, and $f(t)$ denotes the control force. In the above, it was assumed that the mobile actuator will start from rest (i.e. $\dot{\theta}(0) = 0$).

III. Problem formulation

Using a static collocated feedback (3) addresses task #1. The control objective then becomes that of choosing the actuator guidance so that the closed loop system is stable. By incorporating the vehicle dynamics this then translates to choosing the control force input $f(t)$ so that the following system is stable

$$\Sigma_2 \begin{cases} \dfrac{\partial x(t,\xi;\theta(t))}{\partial t} = a\dfrac{\partial^2 x(t,\xi;\theta(t))}{\partial \xi^2} + \delta(\xi-\theta(t))u(t), \\ x(t,0;\theta(t)) = x(t,\ell;\theta(t)) = 0,\ x(0,\xi;\theta(t)) = x_0(\xi), \\ u(t) = -\kappa \displaystyle\int_0^\ell \delta(\xi-\theta(t))x(t,\xi;\theta(t))\,d\xi, \\ m\ddot{\theta}(t) + d\dot{\theta}(t) + k\theta(t) = f(t),\ \theta(0) = \theta_0,\ \dot{\theta}(0) = 0. \end{cases}$$

Using Lyapunov stability arguments, one can derive an expression for the force control and indirectly obtain the guidance of the vehicle that carries the actuator. Completely different force control laws, and indirectly different guidance laws, can be obtained with different Lyapunov functions. Two choices of Lyapunov functions are proposed and which provide different force expressions.

While the well-posedness of the two guidance schemes will be summarized below, we consider an abstract notation that will simplify the derivation of the guidance laws.

Let $\mathcal{X}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding induced norm $|\cdot|$. Let $\mathcal{V}$ be a reflexive Banach space with norm denoted by $\|\cdot\|$, and assume that $\mathcal{V}$ is embedded densely and continuously in $\mathcal{X}$, [20], [21]. Let $\mathcal{V}^*$ denote the conjugate dual of $\mathcal{V}$ (in other words, the space of continuous conjugate linear functionals on $\mathcal{V}$) and $\|\cdot\|_*$ denote the usual uniform operator norm on $\mathcal{V}^*$. It follows $\mathcal{V} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}^*$, with both embeddings dense and continuous [22], [23], and as a consequence we have $|\phi| \le c\|\phi\|$, $\phi \in \mathcal{V}$, for some positive constant c, [21]. The notation $\langle \cdot, \cdot \rangle$ will also be used to denote the duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$ induced by the continuous and dense embeddings given above; that is, for $\phi \in \mathcal{V}^*$, and $\psi \in \mathcal{V}$, $\langle \phi, \psi \rangle$ denotes the action of the bounded linear functional ϕ on the vector ψ. This quantity reduces to $\langle \phi, \psi \rangle$ if $\phi \in \mathcal{X}$, i.e. the value of ϕ acting on ψ is equal to the $\mathcal{X}$ inner product of ϕ and ψ.

We consider the state space $\mathcal{X} = L_2(\Omega)$ and $x(t, \cdot) = \{x(t, \xi),\ 0 \le \xi \le \ell\}$ denotes the state. The space $\mathcal{V}$ is identified by the Sobolev space $\mathcal{V} = H_0^1(\Omega) = \{\psi \in H^1(\Omega)\,|\,\psi(0) = \psi(\ell) = 0\}$. The system's second order elliptic operator $\mathcal{A}$ and its domain are given by [22]

$$\mathcal{A}\phi = \frac{d}{d\xi}\left(a\frac{d\phi}{d\xi}\right), \quad a > 0, \quad \phi \in \text{Dom}\ (\mathcal{A}),$$

Dom $(\mathcal{A}) = \{\psi \in L_2(\Omega)\,|\,\psi, \psi'$ abs. continuous, $\psi'' \in L_2(\Omega)$ and $\psi(0) = 0 = \psi(\ell)\}$. From (2), the input operator $\mathcal{B} : \mathbb{R}^1 \to \mathcal{V}^*$ is

$$\langle \mathcal{B}(\theta(t))u(t), \phi \rangle = \int_0^\ell b(\xi; \theta(t))\phi(\xi)\,d\xi\,u(t).$$

In the ensuing guidance laws, the state $x(t, \xi)$ at the centroid location $\theta(t)$ will be required. This can be thought of as the "output" of the sensing device that is collocated with the actuator. It can be written as

$$y(t; \theta) = \mathcal{C}(\theta)x(t), \quad (5)$$

where the measurement operator is parameterized by the actuator-sensor location θ and is given by $\mathcal{C}(\cdot) : \mathcal{V} \to \mathbb{R}^1$

$$\mathcal{C}(\theta(t))\phi = \int_0^\ell b(\xi; \theta(t))\phi(\xi)\,d\xi.$$

From the above, we have $\mathcal{C} = \mathcal{B}^*$. Following [14], we have that the operator $\mathcal{A}$ is $\mathcal{V} - \mathcal{V}^*$ bounded, $\mathcal{V}$-coercive and symmetric. When the control signal is assumed to be square integrable, i.e. yielding $\mathcal{B}u \in L_2(0, t, \mathcal{V}^*)$, and $x(0) = x_0 \in \mathcal{X}$, then the initial value problem

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \qquad x_0 \in \mathcal{X}, \quad (6)$$

is well-posed. By a solution to (6), we mean a weak solution [24]; this means a function $x \in L_2(0,t;\mathcal{V})$ with $\frac{d}{dt}x \in L_2(0,t;\mathcal{V}^*)$ for all $t > 0$, that satisfies the above evolution system, [21], [24]. The complexity comes at the well-posedness and possibly the additional arguments for establishing closed-loop stability and guidance laws for the mobile agents. In this case the closed loop operator is $\mathcal{A}_{cl}(\theta) \triangleq \mathcal{A} - \kappa\mathcal{B}(\theta)\mathcal{B}^*(\theta)$ and the well-posedness can be established using the implicit function theorem or analytic semigroup theory on semilinear evolution equations [25].

A. Guidance based on Lyapunov function $V_I(t)$ that includes vehicle dynamics

The following choice of Lyapunov function was similarly used in [14] but without the vehicle dynamics. By augmenting the vehicle dynamics, we now have

$$V_I(t) = -\frac{1}{2}\langle x(t), \mathcal{A}_{cl}(\theta(t))x(t)\rangle + \frac{1}{2}m\dot{\theta}^2(t) + \frac{1}{2}k\theta^2(t).$$

The derivative of $V_I(t)$ along the dynamics of the closed loop system with augmented vehicle dynamics is

$$\begin{aligned}\dot{V}_I(t) = &-\|\mathcal{A}_{cl}(\theta(t))x(t)\|^2 - \langle x(t), \tfrac{\partial \mathcal{A}_{cl}(\theta(t))}{\partial t}x(t)\rangle \\ &-d\dot{\theta}^2(t) + \dot{\theta}(t)f_I(t)\end{aligned}$$

Following [14], the second term is found to be

$$\begin{aligned}-\langle x(t), &\frac{\partial \mathcal{A}_{cl}(\theta(t))}{\partial t}x(t)\rangle = \\ &\dot{\theta}(t)x_\xi(t,\theta(t);\theta(t))x(t,\theta(t);\theta(t)),\end{aligned}$$

where $x_\xi(t,\theta(t);\theta(t))$ denotes the spatial derivative evaluated at the current spatial location of the actuator centroid $\theta(t)$, i.e. it is the spatial gradient of the output signal $y(t;\theta(t))$. Similarly, $x(t,\theta(t);\theta(t))$ denotes the state evaluated at the spatial location $\xi = \theta(t)$, i.e. it is the "output" signal $y(t;\theta(t))$.

When vehicle dynamics are not included [14], then the choice

$$\dot{\theta}(t) = -\alpha\, x_\xi(t,\theta(t);\theta(t))x(t,\theta(t);\theta(t)), \quad \alpha > 0,$$

with $\alpha > 0$ *any* positive gain, produces

$$\begin{aligned}\dot{V}_I(t) = &-\|\mathcal{A}_{cl}(\theta(t))x(t)\|^2 \\ &-\alpha|x_\xi(t,\theta(t);\theta(t))x(t,\theta(t);\theta(t))|^2.\end{aligned}$$

However, with the inclusion of the vehicle dynamics, the indefinite terms in $\dot{V}_I(t)$ are

$$\begin{aligned}&\dot{\theta}(t)x_\xi(t,\theta(t);\theta(t))x(t,\theta(t);\theta(t)) - d\dot{\theta}^2(t) + \dot{\theta}(t)f_I(t) = \\ &-\dot{\theta}(t)\Big(- x_\xi(t,\theta(t);\theta(t))x(t,\theta(t);\theta(t)) + d\dot{\theta}(t) - f_I(t)\Big).\end{aligned}$$

The following choice with $\gamma \geq 0$ ensures that the indefinite term becomes negative semidefinite:

$$\begin{aligned}f_I(t) &= -x(t,\xi;\theta)\Big|_{\xi=\theta(t)}\, x_\xi(t,\xi;\theta)\Big|_{\xi=\theta(t)} - \gamma\dot{\theta}(t) \\ &= -x(t,\theta;\theta)x_\xi(t,\theta;\theta) - \gamma\dot{\theta}(t).\end{aligned} \tag{7}$$

The gain γ may be chosen as $\gamma = 0$ or any positive value. With this choice, the Lyapunov function becomes

$$\dot{V}_I(t) = -\|\mathcal{A}_{cl}(\theta(t))x(t)\|^2 - (d+\gamma)\dot{\theta}^2(t) \leq 0.$$

We summarize the closed loop equations for the above choice of the Lyapunov function

$$\Sigma_3 \begin{cases} \dfrac{\partial x(t,\xi;\theta(t))}{\partial t} = a\dfrac{\partial^2 x(t,\xi;\theta(t))}{\partial \xi^2} + \delta(\xi-\theta(t))u(t), \\ x(t,0;\theta(t)) = x(t,\ell;\theta(t)) = 0, \\ x(0,\xi;\theta(t)) = x_0(\xi), \\ u(t) = -\kappa\displaystyle\int_0^\ell \delta(\xi-\theta(t))x(t,\xi;\theta(t))\,d\xi, \\ m\ddot{\theta}(t) + d\dot{\theta}(t) + k\theta(t) = f_I(t),\ \theta(0) = \theta_0,\ \dot{\theta}(0) = 0 \\ f_I(t) = -x(t,\theta;\theta)x_\xi(t,\theta;\theta) - \gamma\dot{\theta}(t),\ \ \gamma \geq 0. \end{cases}$$

Remark 1: The control force $f_I(t)$ requires the signals $x(t,\theta;\theta)$, $x_\xi(t,\theta;\theta)$ and $\dot{\theta}(t)$. The signal $x(t,\theta;\theta)$ is the output $y(t;\theta(t))$ and $x_\xi(t,\theta;\theta)$ is the spatial derivative of the output $y(t;\theta(t))$. For compact notation, we adopt $y_\xi(t;\theta(t)) = x_\xi(t,\theta(t);\theta(t))$ with the understanding that

$$y_\xi(t;\theta(t)) = \frac{\partial x(t,\xi;\theta(t))}{\partial \xi}\Big|_{\xi=\theta(t)}.$$

Finally, it is assumed that the vehicle knows its own state $(\theta,\dot{\theta})$ and therefore the velocity $\dot{\theta}(t)$ is assumed to be available. Then using the above notation, the expression for the control force can be compactly written as

$$f_I(t) = -y(t;\theta(t))y_\xi(t;\theta(t)) - \gamma\dot{\theta}(t),$$

and which requires the three scalar signals $y(t;\theta(t)), y_\xi(t;\theta(t))$ and $\dot{\theta}(t)$ to be realized.

We summarize the above result in the following lemma.

Lemma 1: Consider the system (1) with the control law (3). Assume that the vehicle dynamics that describe the motion of the actuator centroid $\theta(t)$ are described by (4) and that the vehicle knows its own state $(\theta,\dot{\theta})$. Then the proposed Lyapunov-based vehicle+actuator guidance law (7) renders the system Σ_3 stable.

Proof: The proof essentially follows from the previous analysis. However, one must show that $c_1\|\phi\|^2 \leq V_I \leq c_2\|\phi\|^2$. Indeed, from the $\mathcal{V}-\mathcal{V}^*$ boundedness and $\mathcal{V}$ coercivity of $\mathcal{A}$ we have

$$\begin{aligned}\langle \mathcal{A}_{cl}(\theta)\phi,\psi\rangle &\leq \Big|\langle(\mathcal{A}-\mathcal{C}^*(\theta)\Gamma\mathcal{C}(\theta))\phi,\psi\rangle\Big| \\ &= \langle\mathcal{A}\phi,\psi\rangle - \langle\mathcal{C}(\theta)\phi,\Gamma\mathcal{C}(\theta)\psi\rangle \\ &\leq \alpha_1\|\phi\|\|\psi\| + \lambda_{max}(\Gamma)\|\mathcal{C}(\theta)\phi\|\|\mathcal{C}(\theta)\psi\|\end{aligned}$$

and therefore $\langle\mathcal{A}_{cl}(\theta)\phi,\phi\rangle \leq \alpha_1\|\phi\|^2$. Similarly

$$\begin{aligned}-\langle\mathcal{A}_{cl}(\theta)\phi,\phi\rangle &= -\langle(\mathcal{A}-\mathcal{C}^*(\theta)\Gamma\mathcal{C}(\theta))\phi,\phi\rangle \\ &= -\langle\mathcal{A}\phi,\phi\rangle + \langle\mathcal{C}(\theta)\phi,\Gamma\mathcal{C}(\theta)\phi\rangle \\ &\geq \alpha_0\|\phi\|^2 + \lambda_{min}(\Gamma)\|\mathcal{C}(\theta)\phi\|^2\end{aligned}$$

From the above two, we have

$$\alpha_0 \tfrac{1}{2}\|x(t)\|^2 + \tfrac{1}{2}m\dot{\theta}^2(t) + \tfrac{1}{2}d\theta^2(t) \le V_I(t)$$
$$\le \alpha_1 \tfrac{1}{2}\|x(t)\|^2 + \tfrac{1}{2}m\dot{\theta}^2(t) + \tfrac{1}{2}d\theta^2(t).$$

However, one cannot establish that $\dot{V}_I(t) \le -\alpha_3\|x(t)\|^2 - \alpha_4\theta^2(t) - \alpha_5\dot{\theta}^2(t)$ but only $\dot{V}_I(t) = -\|A_{cl}(\theta(t))x(t)\|^2 - (d+\gamma)\dot{\theta}^2(t) \le 0$. Luckily, examining the time variation of the closed loop operator $\mathcal{A} - \mathcal{C}^*(\theta)\Gamma\mathcal{C}(\theta)$ yields a well-posed system with the state converging exponentially to zero. Therefore the term $x(t,\theta;\theta)x_\xi(t,\theta;\theta)$ in $f_I(t)$ is bounded and thus the vehicle system $m\ddot{\theta}(t) + d\dot{\theta}(t) + k\theta(t) = -x(t,\theta;\theta)x_\xi(t,\theta;\theta) - \gamma\dot{\theta}(t)$ is also stable. ■

B. Guidance based on Lyapunov function $V_{II}(t)$ that includes vehicle dynamics

We now consider a second Lyapunov function for the derivation of the centroid guidance. This is essentially the energy of the closed loop system and is given by

$$V_{II}(t) = \frac{1}{2}\|x(t)\|^2 + \frac{1}{2}m\dot{\theta}^2(t) + \frac{1}{2}k\theta^2(t)$$
$$= \frac{1}{2}\int_0^\ell x^2(t,\xi;\theta(t))\,d\xi + \frac{1}{2}m\dot{\theta}^2(t) + \frac{1}{2}k\theta^2(t).$$

Using the state equation, the total derivative is given by

$$\frac{d}{dt}x(t,\xi;\theta(t)) = \frac{\partial}{\partial t}x(t,\xi;\theta(t)) + \frac{d\theta(t)}{dt}\,\frac{\partial}{\partial\zeta}x(t,\xi;\zeta)\Big|_{\zeta=\theta(t)}$$

The time derivative of $V_{II}(t)$ is then given by

$$\begin{aligned}\dot{V}_{II}(t) &= \int_0^\ell \frac{\partial x(t,\xi;\theta(t))}{\partial t}\,x(t,\xi;\theta(t))\,d\xi \\ &\quad + \frac{d\theta(t)}{dt}\int_0^\ell \frac{\partial x(t,\xi;\zeta)}{\partial\zeta}\Big|_{\zeta=\theta(t)} x(t,\xi;\theta(t))\,d\xi \\ &\quad + m\ddot{\theta}(t)\dot{\theta}(t) + k\dot{\theta}(t)\theta(t) \\ &= -a\int_0^\ell \Big(\frac{\partial x(t,\xi;\theta(t))}{\partial\xi}\Big)^2 d\xi - \kappa x^2(t,\theta(t);\theta(t)) \\ &\quad + \dot{\theta}(t)\int_0^\ell \frac{\partial x(t,\xi;\zeta)}{\partial\zeta}\Big|_{\zeta=\theta(t)} x(t,\xi;\theta(t))\,d\xi \\ &\quad + \dot{\theta}(t)f_{II}(t) - d\dot{\theta}^2(t)\end{aligned}$$

The function $S(t,\xi;\theta) \triangleq \frac{\partial x(t,\xi;\theta)}{\partial\theta}$ is the sensitivity function and is satisfies an evolution equation similar to the state equation. To derive the guidance (via the appropriate choice of $f(t)$) we cancel the indefinite terms in the Lyapunov function. The term

$$\dot{\theta}(t)\Big(\int_0^\ell \frac{\partial x(t,\xi;\zeta)}{\partial\zeta}\Big|_{\zeta=\theta(t)} x(t,\xi;\theta(t))\,d\xi + f_{II}(t) - d\dot{\theta}(t)\Big)$$

must be made negative definite. The choice

$$\begin{aligned}f_{II}(t) &= -\int_0^\ell \frac{\partial x(t,\xi;\zeta)}{\partial\zeta}\Big|_{\zeta=\theta(t)} x(t,\xi;\theta)\,d\xi - \gamma\dot{\theta}(t) \\ &= -\langle S(t;\theta(t)), x(t)\rangle - \gamma\dot{\theta}(t)\end{aligned} \tag{8}$$

with $\gamma \ge 0$ any non-negative gain produces

$$\begin{aligned}\dot{V}_{II}(t) &= -a\int_0^\ell \Big(\frac{\partial x(t,\xi;\theta(t))}{\partial\xi}\Big)^2 d\xi - \kappa x^2(t,\theta(t);\theta(t)) \\ &\quad -(\gamma+d)\dot{\theta}^2(t) \\ &= -\langle \mathcal{A}_{cl}(\theta(t))x(t), x(t)\rangle - (\gamma+d)\dot{\theta}^2(t) \le 0.\end{aligned}$$

Remark 2: If one considers the actuator/sensor pair without motion dynamics, then in a similar fashion to [14], the choice

$$\dot{\theta}(t) = -\alpha\,\langle S(t;\theta(t)), x(t)\rangle, \quad \alpha > 0,$$

produces

$$\dot{V}_{II}(t) = -\langle \mathcal{A}_{cl}(\theta(t))x(t), x(t)\rangle - \alpha\langle S(t;\theta(t)), x(t)\rangle^2 < 0$$

and which simplifies the arguments for exponential stability.

Remark 3: The above guidance policy which takes into account the vehicle dynamics, requires both the entire state $x(t,\xi;\theta(t))$ and the sensitivity function $\frac{\partial x(t,\xi;\zeta)}{\partial\zeta}$. In particular, it requires their averaged product through the inner product $\langle S(t;\theta(t)), x(t)\rangle$.

In addition to the guidance law, one must also solve the equation that the sensitivity function satisfies. From the closed loop system

$$\begin{aligned}\frac{\partial x(t,\xi)}{\partial t} &= a\frac{\partial^2 x(t,\xi)}{\partial\xi^2} \\ &\quad -\kappa\delta(\xi-\theta(t))\int_0^\ell \delta(\xi-\theta(t))x(t,\xi;\theta(t))\,d\xi, \\ x(t,0) &= x(t,\ell) = 0, \quad x(0,\xi) = x_0(\xi),\end{aligned} \tag{9}$$

we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\Big(\frac{\partial}{\partial\theta}x(t,\xi;\theta)\Big) = a\frac{\partial^2}{\partial\xi^2}\Big(\frac{\partial}{\partial\theta}x(t,\xi;\theta)\Big) \\ &\quad -\kappa\frac{\partial}{\partial\theta}\Big(\delta(\xi-\theta(t))\int_0^\ell \delta(\xi-\theta(t))x(t,\xi;\theta(t))\,d\xi\Big) \\ &= a\frac{\partial^2}{\partial\xi^2}S(t,\xi;\theta) - \kappa\frac{\partial\delta(\xi-\theta)}{\partial\theta}x(t,\theta(t);\theta(t)) \\ &\quad +\kappa\delta(\xi-\theta)\frac{\partial x(t,\xi;\theta)}{\partial\xi}\Big|_{\xi=\theta} - \kappa\delta(\xi-\theta)S(t,\theta(t);\theta(t), \\ &S(t,0) = S(t,\ell) = 0, \;\; S(0,\xi) = 0,\end{aligned}$$

where $\frac{\partial\delta(\xi-\theta)}{\partial\theta}$ is understood in the distributional sense [26]. Viewing the above in weak form we have

$$\begin{aligned}\langle S_t, \phi\rangle &= a\langle S_{\xi\xi}, \phi\rangle + \kappa\phi_\xi(\theta)y(t;\theta) + \kappa\phi(\theta)x_\xi(t,\theta;\theta) \\ &\quad -\kappa\phi(\theta)S(t,\theta(t);\theta(t)), \;\; S(0) = 0.\end{aligned}$$

Similar to the previous case, we summarize the closed loop system

$$\Sigma_4 \begin{cases} \dfrac{\partial x(t,\xi;\theta(t))}{\partial t} = a\dfrac{\partial^2 x(t,\xi;\theta(t))}{\partial \xi^2} + \delta(\xi-\theta(t))u(t), \\ x(t,0;\theta(t)) = x(t,\ell;\theta(t)) = 0, \\ x(0,\xi;\theta(t)) = x_0(\xi), \\ u(t) = -\kappa \displaystyle\int_0^\ell \delta(\xi-\theta(t))x(t,\xi;\theta(t))\,d\xi, \\ m\ddot{\theta}(t) + d\dot{\theta}(t) + k\theta(t) = f_{II}(t),\ \theta(0)=\theta_0,\ \dot{\theta}(0)=0 \\ f_{II}(t) = -\displaystyle\int_0^\ell S(t,\xi;\theta(t))\,x(t,\xi;\theta(t))\,d\xi - \gamma\dot{\theta}(t), \\ \dfrac{\partial S(t,\xi;\theta(t))}{\partial t} = a\dfrac{\partial^2 S(t,\xi;\theta(t))}{\partial \xi^2} \\ \quad -\kappa\dfrac{\partial \delta(\xi-\theta(t))}{\partial \theta}x(t,\theta(t);\theta(t)) \\ \quad +\kappa\delta(\xi-\theta)\Big(\dfrac{\partial x(t,\xi;\theta(t))}{\partial \xi}\Big|_{\xi=\theta(t)} - S(t,\theta(t);\theta(t))\Big), \\ S(t,0;\theta) = S(t,\ell;\theta) = 0,\ \ S(0,\xi;\theta) = 0. \end{cases}$$

IV. Numerical results

The PDE system in (1) was simulated using 80 linear elements [27] in the interval $\Omega = [0,1]$ with an initial condition $x(0,x) = \sin(\pi\xi/L)e^{-7\xi^2}$. The coefficient of the Laplacian operator was $a = 0.005$. The moving source was taken as a spatial delta function with constant intensity and a moving centroid $\xi_s(t)$

$$d(t,\xi) = 2\times 10^{-3}\delta(\xi-\xi_s(t)),\ \xi_s(t) = 0.3\ell(\cos(\frac{2\pi t}{t_f})+2).$$

The vehicle parameters were $m = 1, k = 1, d = \sqrt{2}$ with initial condition $\theta(0) = 0.25\ell, \dot{\theta}(0) = 0$. The static feedback gain was chosen as $\kappa = 100$ and implementing the policy $f_I(t) = \alpha y(t;\theta(t))y_\xi(t;\theta(t)) - \gamma\dot{\theta}(t)$ with $\alpha = 1,\ \gamma = 0.05 - d$. Such a weighted expression results from the weighted Lyapunov function $V_I(t) = 0.5\alpha\langle x(t), A_{cl}(\theta(t))x(t)\rangle + 0.5m\dot{\theta}^2(t) + 0.5k\theta^2(t)$ The closed loop system was simulated in the time interval $[t_0, t_f] = [0, 20]s$.

Figure 1 depicts the time evolution of the state $L_2(0,1)$ norm for the uncontrolled case, the closed loop system with a fixed actuator and the closed loop system with a moving actuator and an actuator guidance given by (7). It is evident that the mobile actuator can improve state regulation. The spatial distribution of the closed loop state at four time instances $x(0,\xi), x(5,\xi), x(10,\xi)$ and $x(15,\xi)$ are depicted in Figure 2. The trajectory of the mobile actuator along with the motion of the moving source are depicted in Figure 3.

V. Conclusions

A stability-based scheme for the guidance of a mobile actuator used for performance enhancement of a class of partial differential equations was proposed. The Lyapunov

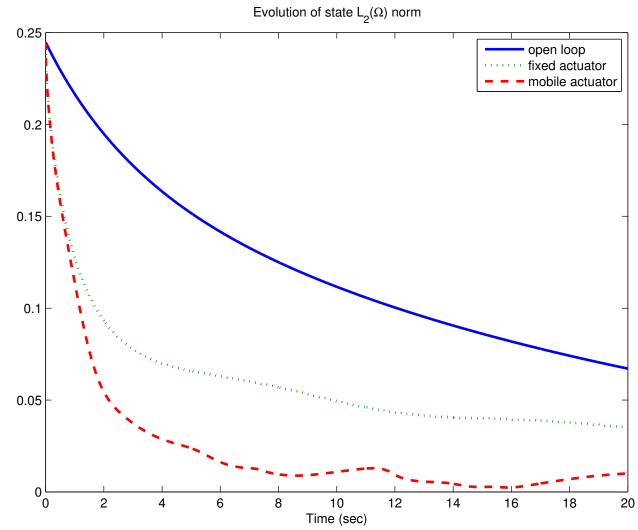

Fig. 1. Evolution of $L_2(\Omega)$ norms.

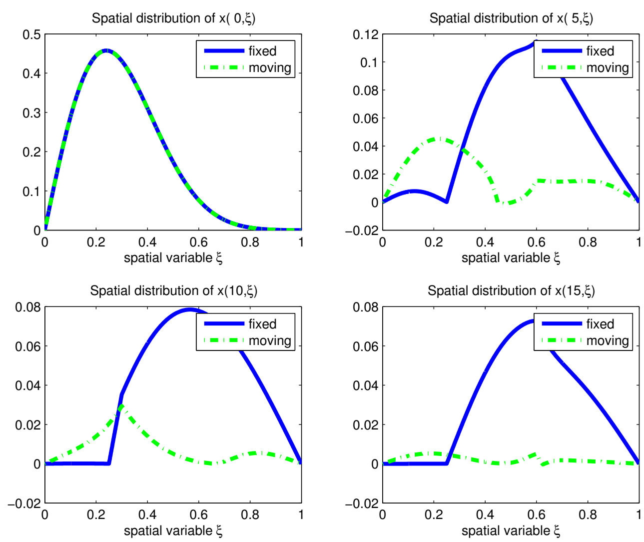

Fig. 2. Closed loop state vs spatial variable at different time instances.

scheme, which included the mobile agent dynamics, provided an analytical expression for the motion of the centroid of the moving actuator. This was achieved because of the specific structure of the controller architecture which required a static output feedback policy with the additional condition of a pointwise-in-space spatial distribution of the actuating device, i.e. a spatial delta function.

An immediate extension, assuming the specific structure of the controller architecture and the actuator spatial distribution, involves the use of multiple mobile actuators. In this case, especially when migrating to a 2D spatial domain, one must be concerned with the motion coordination of multiple vehicles with collision avoidance modifications and localization algorithms for estimating the state of each vehicle. Added to the above challenges, one must also account for modifications that allow for boundary conditions other than

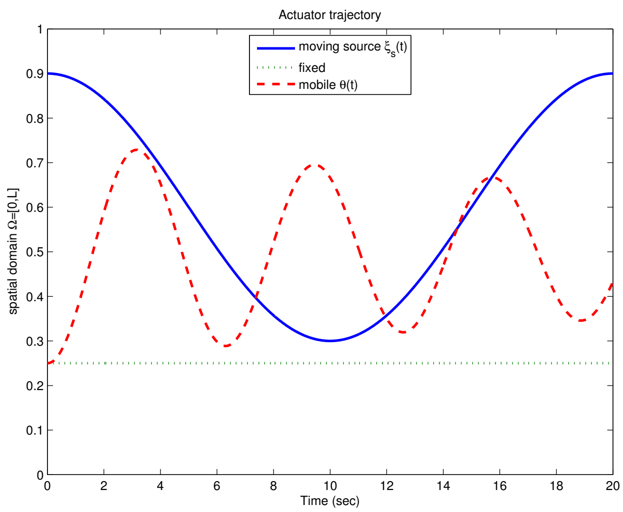

Fig. 3. Evolution of actuator and disturbance trajectories.

Dirichlet conditions. These are currently under study by the author and will appear in a forthcoming publication.

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