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Symmetries and first integrals for nonlinear discrete-time systems

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***Abstract*— In this paper, the concepts of Lie symmetry and of first integral for discrete-time nonlinear systems are discussed. Some results that hold for continuous-time nonlinear systems are extended to discrete-time ones. First, the strong relation between symmetries and first integrals is explored. Then the two concepts are studied separately, illustrating some applications of Lie symmetries and giving a computational procedure for the computation of first integrals.**

I. INTRODUCTION

The concept of (orbital) symmetry of a differential equation was introduced by S. Lie [1], [2] in the second half of the 19-th century and was primarily used for the solution in closed form of differential equations. Modern reference on the subject can be found in many books, among which we mention [3]-[7].

Recently, it has been proved that Lie symmetries are also useful to give elegant geometric conditions for the linearization of nonlinear systems by state transformation [8], [9] and state immersion [10]-[12], and that Lie symmetries, and, more in general, orbital symmetries, can be used to compute efficiently semi-invariants, and, as a special case, first integrals [13].

It is well known that some strong properties of continuous-time systems (briefly, CT-systems) do not hold for discrete-time systems (briefly, DT-systems); as an example, it seems to be difficult if not impossible to extend fully the concept of orbital symmetry to discrete-time systems. Nevertheless, in the paper [14], Lie symmetries have been studied for discrete-time systems, and their usefulness for linearization by state transformation and state immersion has been shown, obtaining results analogous to those in [11], but slightly weaker.

First integrals are very useful in the analysis of dynamical systems: traditionally they have been used, especially in conservative systems, to design control laws that exploit the physical properties of the system; more recently, in [15], they have found new application in stability analysis of switched systems.

Aim of this paper is to extend to the discrete-time case some results about Lie symmetries and first integrals. After a brief review of notation and standard background, in Section III the definition and some properties of Lie symmetries for DT-systems are recalled from [14]; in Sections IV and V the relation between symmetries and first integrals is studied; in Section VI some more results about usefulness of DT symmetries are reported; finally, in Section VII, a computational result about first integrals is given.

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II. NOTATION AND BACKGROUND

Given an open and connected $\mathcal{U} \subseteq \mathbb{R}^n$, call $\mathcal{A}_n$ the set of all analytic functions $\alpha(x) : \mathcal{U} \to \mathbb{R}$, and $\mathcal{K}_n$ the set of meromorphic functions $\alpha = \frac{a}{b}$, $a, b \in \mathcal{A}_n$, $b_i \neq 0$ (see [16]).

Consider a vector function $F(x) \in \mathbb{R}^n$ and the associated discrete-time system described by:

$$x(t+1) = F(x(t)), \quad x \in \mathbb{R}^n, \, t \in \mathbb{Z}, \tag{1}$$

where $x = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}^\top$ is the *state vector*. For the sake of simplicity, it is assumed that all functions are meromorphic on some open and connected set $\mathcal{U}$ of $\mathbb{R}^n$ and, therefore, that they are analytic on $\mathcal{U}^*$, with $\mathcal{U}^*$ being some open and connected set of $\mathcal{U}$. Hence, system (1) has unique *maximal solution* $x(t) = \Psi_F(t, x_0)$, $t \in \mathbb{Z}$, t sufficiently small, from the admissible initial condition $x_0 \in \mathcal{U}^*$ at time $t = 0$; Ψ_F is the discrete-time *flow* (briefly, the *DT-flow*) associated with F.

Consider a vector function $g(x) \in \mathbb{R}^n$ and the corresponding continuous-time system (from now on, the dependencies on times t, τ are omitted, if not necessary):

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = g(x), \quad x \in \mathbb{R}^n, \, \tau \in \mathbb{R}. \tag{2}$$

Since g is meromorphic on $\mathcal{U}$ and, therefore, is analytic on some $\mathcal{U}^*$, one concludes that system (2) has a unique maximal solution $x(\tau) = \Phi_g(\tau, x_0)$, $\tau \in \mathbb{R}$, τ sufficiently close to 0, from the initial condition $x_0 \in \mathcal{U}^*$ at time $\tau = 0$: Φ_g is the continuous-time *flow* (briefly, the *CT-flow*) associated with g.

The *directional derivative* $L_f h \in \mathbb{R}$ of a scalar function $h(x) \in \mathbb{R}$ by $f(x) \in \mathbb{R}^n$ is $L_f h := \frac{\partial h}{\partial x} f$, where $\frac{\partial h}{\partial x}$ is the gradient of h ($L_f h$ is often called the *Lie directional derivative*); the *directional derivative* $L_f g \in \mathbb{R}^n$ of g by f is the vector having $L_f g_i$ as i-th entry, with g_i being the i-th entry of g, i.e., $L_f g := \frac{\partial g}{\partial x} f$, where $\frac{\partial g}{\partial x}$ is the Jacobian matrix of g; the *CT-Lie bracket* $[f, g] \in \mathbb{R}^n$ of f and g is [17], [18]

$$[f, g] := \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = L_f g - L_g f,$$

and the *DT-Lie bracket* $\lfloor F, g \rfloor \in \mathbb{R}^n$ of F and g is [19]

$$\lfloor F, g \rfloor := g(F) - \frac{\partial F}{\partial x} g = g \circ F - L_g F,$$

where $\frac{\partial g}{\partial x}, \frac{\partial f}{\partial x}$ and $\frac{\partial F}{\partial x}$ are the Jacobian matrices of g, f and F, respectively, and $\circ$ denotes function composition. If no confusion can arise between the continuous-time and

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discrete-time cases, the simpler nomenclature *Lie bracket* is used instead of CT and DT-Lie brackets.

Remark 1: In general, the DT-Lie bracket is not skew symmetric; it has such a property when both F and g are linear. ■

Given a diffeomorphism $y = \varphi(x)$, a scalar function $h(x) \in \mathbb{R}$ and a vector function $g(x) \in \mathbb{R}^n$ associated with a continuous-time system (2) (respectively, a vector function $F(x) \in \mathbb{R}^n$ associated with a discrete-time system (1)), the *push-forward* of h by φ and the *push-forward* of g (respectively, F) by φ are ([20])

$$\begin{aligned}\varphi_* h(y) &= h \circ \varphi^{-1}(y),\\ \varphi_* g(y) &= \left(\frac{\partial \varphi}{\partial x} g\right) \circ \varphi^{-1}(y), \quad \text{if} \quad t \in \mathbb{R},\\ \varphi_* F(y) &= \varphi \circ F \circ \varphi^{-1}(y), \quad \text{if} \quad t \in \mathbb{Z}.\end{aligned}$$

Given a scalar function $\tilde{h}(y) \in \mathbb{R}$ and a vector function $\tilde{g}(y) \in \mathbb{R}^n$ in the continuous-time case (respectively, $\tilde{F}(y) \in \mathbb{R}^n$ in the discrete-time case), the *pull-back* of $\tilde{h}$ by φ and the *pull-back* of $\tilde{g}$ (respectively, $\tilde{F}$) by φ are ([20])

$$\begin{aligned}\varphi^* \tilde{h}(x) &= \varphi_*^{-1} \tilde{h}(x) = \tilde{h} \circ \varphi(x),\\ \varphi^* \tilde{g}(x) &= \varphi_*^{-1} \tilde{g}(x) = \left(\frac{\partial \varphi^{-1}}{\partial y} \tilde{g}\right) \circ \varphi(x),\\ \varphi^* \tilde{F}(x) &= \varphi_*^{-1} \tilde{F}(x) = \varphi^{-1} \circ \tilde{F} \circ \varphi(x).\end{aligned}$$

A scalar function $I(x) \in \mathbb{R}$ is a *first integral* of the discrete-time system (1) (a *DT-first integral* associated with F) if $I \circ F(x) = I(x)$, $\forall x \in \mathcal{U}^*$, with $\mathcal{U}^*$ being an open and connected subset of $\mathcal{U}$; similarly, $I(x)$ is a *first integral* associated with the continuous-time system $\frac{dx}{d\tau} = g(x)$ (a *CT-first integral* associated with g) if $L_g I(x) = 0$, $\forall x \in \mathcal{U}^*$.

III. Lie Symmetries of Discrete-time Nonlinear Systems

In this section, some preliminary results from the paper [14] are briefly reviewed.

For any *admissible* τ (to be considered as a constant parameter),

$$x = \Phi_g(\tau, y) \tag{3}$$

qualifies as a local analytic diffeomorphism, with inverse

$$y = \Phi_g(-\tau, x); \tag{4}$$

system (1) is transformed, according to such a diffeomorphism, as follows:

$$\Delta y = \Phi_g(-\tau, \cdot) \circ F \circ \Phi_g(\tau, y). \tag{5}$$

Definition 1: The diffeomorphism (3) is a *symmetry* of the discrete-time system (1) and the continuous-time system (2) is its *infinitesimal generator* if

$$\Phi_g(-\tau, \cdot) \circ F \circ \Phi_g(\tau, y) = F(y), \quad \forall (\tau, y) \in \mathcal{V}, \tag{6}$$

with $\mathcal{V}$ being an open and connected subset of $\mathbb{R} \times \mathbb{R}^n$ including $\{0\} \times \mathcal{U}$. When (6) holds, by abuse of notation, also the infinitesimal generator (2) is called a *symmetry* of the discrete-time system (1) (briefly, g is called a *DT-symmetry* of F).

The following theorem (for a proof see [21] and [14]) gives necessary and sufficient conditions for a vector function g to be a DT-symmetry of F.

Theorem 1: Vector function g is a DT-symmetry of F (i.e., (5) holds) if and only if $\lfloor F, g \rfloor = 0$.

The following theorem, also proved in [14], shows that the property of g to be a DT-symmetry of F is independent of the local coordinates chosen to represent g and F, although, in general, the DT-Lie bracket is not invariant with respect to diffeomorphisms.

Theorem 2: Let $y = \varphi(x)$ be a diffeomorphism meromorphic on $\mathcal{U}$. Let $\varphi_* F = \varphi \circ F \circ \varphi^{-1}$ and $\varphi_* g = \left(\frac{\partial \varphi}{\partial x} g\right) \circ \varphi^{-1}$. Then, $\varphi_* g$ is a DT-symmetry of $\varphi_* F$ if and only if g is a DT-symmetry of F, i.e.,

$$\lfloor \varphi_* F, \varphi_* g \rfloor = 0 \quad \Longleftrightarrow \quad \lfloor F, g \rfloor = 0.$$

For the application of the results in this paper, it will often be assumed that one or more Lie symmetries of a given system are known. Computing Lie symmetries may be difficult; one of the tools for solving such a problem is the parametrization, reported in [14], of all the discrete-time systems that admit a given g as a symmetry.

IV. Symmetries and First integrals: the Scalar Case

The following theorem (which is inspired by [21]) gives a necessary and sufficient condition for a scalar discrete-time nonlinear system (system (1) with $n = 1$) to be diffeomorphic to the special form

$$y(t+1) = y(t) + c, \tag{7}$$

with $c \in \mathbb{R}$.

Theorem 3: Let $F(x) \in \mathbb{R}$. There exists a diffeomophism $y = \varphi(x)$ such that

$$\varphi_* F(y) = y + c,$$

where $c \in \mathbb{R}$ is a constant, if and only if there exists a DT-symmetry $g(x) \in \mathbb{R}$, $g \neq 0$, of $F(x)$, $\lfloor F, g \rfloor = 0$. In such a case, there exists a DT-first integral $I(x)$ associated with $F(x)$.

Proof: Assume that $\varphi_* F(y) = y + c$. Let $\tilde{g}(y) = 1$; clearly, $\lfloor \varphi_* F, \tilde{g} \rfloor = 0$, and therefore, by Theorem 2, one has $\lfloor F, g \rfloor = 0$, with $g = \varphi^* \tilde{g} \neq 0$. Conversely, assume that $\lfloor F, g \rfloor = 0$, with $g \neq 0$. Let $y = \varphi(x)$, with

$$\varphi(x) = \int_0^x \frac{1}{g(\xi)} \mathrm{d}\xi,$$

which is well defined in a neighborhood of any regular point of g. Hence, $\varphi_* g(y) = \left(\frac{1}{g(x)} g(x)\right) \circ \varphi^{-1}(y) = 1$. By Theorem 2, condition $\lfloor F, g \rfloor = 0$ implies $\lfloor \varphi_* F, \varphi_* g \rfloor = 0$. Now, since

$$\left\lfloor \tilde{F}(y), \varphi_* g(y) \right\rfloor = 1 - \frac{\partial \tilde{F}(y)}{\partial y},$$

for any $\tilde{F}(y) \in \mathbb{R}$, condition $\lfloor \varphi_* F, \varphi_* g \rfloor = 0$ implies that $\frac{\partial \varphi_* F(y)}{\partial y} = 1$, i.e., $\varphi_* F(y) = y + c$. Note that if $c = 0$ in (7), then $\varphi(x)$ is a DT-first integral associated with F; conversely, if $\varphi(x)$ is a non-constant DT-first integral associated with F, then $y = \varphi(x)$ is a diffeomorphism such that $\varphi_* F(y) = y$. For $c \neq 0$, a first integral of the discrete-time system (7) is

$$\tilde{I}(y) = \sin\left(\frac{2\pi}{c} y\right),$$

whence $I = \varphi^* \tilde{I}$ is a DT-first integral associated with F. As a matter of fact, letting $\tilde{F}(y) = y + c$, one has $\tilde{I} \circ \tilde{F}(y) = \sin\left(\frac{2\pi}{c}(y + c)\right) = \sin\left(\frac{2\pi}{c} y + 2\pi\right) = \sin\left(\frac{2\pi}{c} y\right) = \tilde{I}(y)$. ■

Remark 2: Theorem 3 gives a complete picture of scalar discrete-time systems admitting a symmetry, which can be summarized by saying that the following statements are equivalent:

(2.1) the scalar discrete-time system admits a symmetry $g(x)$;

(2.2) the scalar discrete-time system is diffeomorphic by $y = \varphi(x)$ to form (7), for some $c \in \mathbb{R}$;

(2.3) the scalar discrete-time system admits a non-constant first integral $I(x)$.

If g is a DT-symmetry of F, then

$$\varphi(x) = \int_0^x \frac{1}{g(\xi)} d\xi;$$

a first integral $I(x)$ associated with F is

$$I(x) = \begin{cases} \varphi(x), & \text{if } c = 0 \\ \sin(\frac{2\pi}{c}\varphi(x)), & \text{if } c \neq 0. \end{cases} \tag{8}$$

If $y = \varphi(x)$ is the diffeomorphism such that $\varphi_* F(y) = y + c$, then

$$g(x) = \left(\frac{\partial \varphi(x)}{\partial x}\right)^{-1}$$

is a DT-symmetry of F, from which the same formula (8) is obtained.

If I is a DT-first integral associated with F, then

$$g(x) = \left(\frac{\partial I(x)}{\partial x}\right)^{-1}$$

is a DT-symmetry of F and the diffeomorphism $y = \varphi(x)$, with $\varphi = I$, is such that $\varphi_* F(y) = y$. ■

Example 1: Let

$$F(x) = \frac{ax + b}{cx + d},$$

where $a, b, c, d \in \mathbb{R}$, and look for a DT-symmetry of F of the form

$$g(x) = \alpha x^2 + \beta x + \gamma;$$

from

$$\begin{aligned} g \circ F(x) &= \alpha \frac{(ax + b)^2}{(cx + d)^2} + \beta \frac{ax + b}{cx + d} + \gamma, \\ \frac{\partial F(x)}{\partial x} g(x) &= \frac{ad - cb}{(cx + d)^2} \left(\alpha x^2 + \beta x + \gamma\right), \end{aligned}$$

one has that $\lfloor F, g \rfloor = 0$ if and only if the following algebraic system has a real solution in the unknowns α, β, γ

$$\left(ad - cb - a^2\right)\alpha - ac\beta - c^2\gamma = 0, \tag{9a}$$
$$-2ab\alpha - 2bc\beta - 2cd\gamma = 0, \tag{9b}$$
$$-b^2\alpha - bd\beta + \left(-d^2 - cb + ad\right)\gamma = 0. \tag{9c}$$

In particular, one of the solutions of (9) is $\alpha = -c, \beta = a - d, \gamma = b$, which yields the DT-symmetry of F

$$g(x) = -cx^2 + (a - d)x + b.$$

For the sake of simplicity, consider the case $a = 3$, $b = 1$, $c = -1$ and $d = 1$,

$$\begin{aligned} F(x) &= \frac{3x + 1}{1 - x}, \\ g(x) &= x^2 + 2x + 1. \end{aligned}$$

The resulting diffeomorphism, which is well defined in a neighborhood of $x = 0$, is $y = \varphi(x)$, with

$$\begin{aligned} \varphi(x) &= \int_0^x \frac{1}{\xi^2 + 2\xi + 1} \mathrm{d}\xi \\ &= \frac{x}{x + 1}, \end{aligned}$$

with inverse

$$\varphi^{-1}(y) = \frac{y}{1 - y}.$$

It is easy to verify that

$$\begin{aligned} \varphi_* F(y) &= \varphi \circ F \circ \varphi^{-1}(y) \\ &= \left(\left(\frac{F}{F + 1} \right)\bigg|_{F = \frac{3x+1}{1-x}} \right)\bigg|_{x = \frac{y}{1-y}} \\ &= y + \frac{1}{2}. \end{aligned}$$

Since a DT-first integral associated with $\varphi_* F$ is $\sin(4\pi y)$, a DT-first integral associated with F is

$$I(x) = \sin\left(4\pi \frac{x}{x + 1}\right);$$

as a matter of fact, one can check

$$\begin{aligned} I \circ F(x) &= \sin\left(4\pi \frac{3x + 1}{(1 - x)\left(\frac{3x+1}{1-x} + 1\right)}\right) \\ &= \sin\left(4\pi \frac{x}{x + 1} + 2\pi\right) \\ &= \sin\left(4\pi \frac{x}{x + 1}\right) \\ &= I(x). \end{aligned}$$

Now, consider the case $a = 1, b = -3, c = 1$ and $d = 1$,

$$\begin{aligned} F(x) &= \frac{x - 3}{x + 1}, \\ g(x) &= -x^2 - 3. \end{aligned}$$

The resulting diffeomorphism, which is well defined in a neighborhood of $x = 0$, is $y = \varphi(x)$, where

$$\begin{aligned}\varphi(x) &= \int_0^x \frac{1}{-\xi^2 - 3}\mathrm{d}\xi \\ &= -\frac{1}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right),\end{aligned}$$

with inverse

$$\varphi^{-1}(y) = -\sqrt{3}\tan\left(\sqrt{3}y\right).$$

It is easy to verify that

$$\varphi_* F(y) = y + \frac{\sqrt{3}\pi}{9}.$$

Finally, it is easy to check that the diffeomorphism

$$y = \frac{\left(3 + x^2\right)^3}{\left(1 + x\right)^2\left(x - 1\right)^2}$$

transforms system

$$x(t + 1) = \frac{x(t) - 3}{x(t) + 1}$$

into the linear system

$$y(t + 1) = y(t).$$

Hence, a DT-symmetry $g(x)$ of $F(x)$ can be computed as follows:

$$\begin{aligned}g(x) &= \left(\frac{\partial\varphi(x)}{\partial x}\right)^{-1} \\ &= \frac{\left(1 + x\right)^3\left(x - 1\right)^3}{2\left(3 + x^2\right)^2\left(x^2 - 9\right)x};\end{aligned}$$

it is left to the reader to show that $\lfloor F, g\rfloor = 0$. ■

V. Symmetries and First integrals: Higher Order Systems

Theorem 3 is extended to the case $n > 1$ by the following theorem, whose proof can be found in [22].

Theorem 4: Let $F(x) \in \mathbb{R}^n$. There exists a diffeomophism $y = \varphi(x)$ such that

$$\varphi_* F(y) = y + c,$$

where $c \in \mathbb{R}^n$ is a constant, if and only if there exist n symmetries $g_i(x) \in \mathbb{R}$ of $F(x)$, $\lfloor F, g_i\rfloor = 0$, $i = 1, ..., n$, such that $[g_i, g_j] = 0$, for all $i, j \in \{1, ..., n\}$, and $\det\left(\left[\begin{array}{ccc} g_1 & ... & g_n\end{array}\right]\right) \neq 0$. In such a case, there exist n functionally independent DT-first integrals $I_i(x)$, $i = 1, ..., n$, associated with $F(x)$.

Example 2: Consider the discrete-time system described by the vector function $F(x)$ having as entries $F_1(x) = -4x_2^4 - 8x_1x_2^2 + 4x_2^3 - 4x_1^2 + 4x_1x_2 - 4x_2^2 - 3x_1 + 2x_2$ and $F_2(x) = -2x_2^2 - 2x_1 + x_2 - 1$. Let

$$\begin{aligned}g_1(x) &= \left[\begin{array}{c} 1 + 4x_1x_2 + 4x_2^3 \\ -2x_1 - 2x_2^2\end{array}\right], \\ g_2(x) &= \left[\begin{array}{c} -2x_2 \\ 1\end{array}\right].\end{aligned}$$

It is easy to check that $\lfloor F, g_i\rfloor = 0$, $i = 1, 2$, $[g_1, g_2] = 0$ and $\det\left(\left[\begin{array}{cc} g_1 & g_2\end{array}\right]\right) \neq 0$. Hence the rows of

$$\left[\begin{array}{cc} g_1(x) & g_2(x)\end{array}\right]^{-1} = \left[\begin{array}{cc} 1 & 2x_2 \\ 2x_1 + 2x_2^2 & 1 + 4x_1x_2 + 4x_2^3\end{array}\right]$$

are exact and their integrals yield the diffeomorphism $y = \varphi(x)$, with

$$\begin{aligned}\varphi(x) &= \left[\begin{array}{c} x_1 + x_2^2 \\ x_2 + x_1^2 + 2x_1x_2^2 + x_2^4\end{array}\right], \\ \varphi^{-1}(y) &= \left[\begin{array}{c} y_1 - y_2^2 + 2y_2y_1^2 - y_1^4 \\ y_2 - y_1^2\end{array}\right].\end{aligned}$$

Compute the push-forward

$$\varphi_* F(y) = \left[\begin{array}{c} 1 + y_1 \\ y_2\end{array}\right];$$

the DT-first integrals associated with $\varphi_* F(y)$ are $\tilde{I}_1(y) = \sin(2\pi y_1)$ and $\tilde{I}_2(y) = y_2$. Hence, two functionally independent DT-first integrals associated with $F(x)$ can be computed by the pull-back to the original coordinates,

$$\begin{aligned}I_1(x) &= \varphi^* \tilde{I}_1(x) = \sin\left(2\pi\left(x_1 + x_2^2\right)\right), \\ I_2(x) &= \varphi^* \tilde{I}_2(x) = x_2 + x_1^2 + 2x_1x_2^2 + x_2^4.\end{aligned}$$

■

VI. Reduction and decomposition of Discrete-time Nonlinear Systems

This section describes briefly two other application of symmetries to the analysis of discrete-time systems.

First, it is shown that the knowledge of a DT-symmetry $g(x)$ of $F(x)$, with $x \in \mathbb{R}^n$, allows one to project the given system into a system $\xi(t + 1) = F_r(\xi(t))$ having a state space $\xi \in \mathbb{R}^{n-1}$ of smaller dimension.

Let $F(x), g(x) \in \mathbb{R}^n$ be such that $\lfloor F, g\rfloor = 0$. Let $\mathcal{I}_C(g)$ be the set of the CT-first integrals associated with g; note that g may be very simple also for complicated F, e.g., when F is homogeneous w.r.t. an integer dilation. Then, there exist $n-1$ functionally independent elements $J_1, ..., J_{n-1}$ of $\mathcal{I}_C(g)$ that generate the whole $\mathcal{I}_C(g)$, *i.e.*, any $J \in \mathcal{I}_C(g)$ can be expressed as $C(J_1, ..., J_{n-1})$, where C is an arbitrary function of the arguments. Since $J_i \in \mathcal{I}_C(g)$, it follows that $J_i \circ F \in \mathcal{I}_C(g)$: as a matter of fact, taking into account that $\lfloor F, g\rfloor = 0$ implies $F \circ \Phi_g = \Phi_g \circ F$ and that $J_i \in \mathcal{I}_C(g)$ implies $J_i \circ \Phi_g = J_i$, one concludes that:

$$J_i \circ F \circ \Phi_g = J_i \circ \Phi_g \circ F = J_i \circ F,$$

as to be shown. Since $J_i \circ F \in \mathcal{I}_C(g)$, there exists a function C_i such that $J_i \circ F = C_i(J_1, ..., J_{n-1})$. Therefore, by the *projection* $\mathbb{R}^n \to \mathbb{R}^{n-1}$ given by $\xi_i = J_i(x)$, $i = 1, ..., n-1$, a discrete-time nonlinear system, of reduced dimension $n-1$, is found.

Example 3: Consider

$$\begin{aligned}F(x) &= \left[\begin{array}{c} x_1 + x_2 \\ x_2 \\ 3x_3 + a_1x_1^2 + a_2x_1x_2 + a_3x_2^2\end{array}\right], \\ g(x) &= \left[\begin{array}{c} x_1 \\ x_2 \\ 2x_3\end{array}\right];\end{aligned}$$

clearly, $\lfloor F, g \rfloor = 0$. Two functionally independent CT-first integrals associated with g are $J_1(x) = \frac{x_2}{x_1}$ and $J_2(x) = \frac{x_3}{x_1^2}$; then, by the projection $\xi_1 = \frac{x_2}{x_1}$, $\xi_2 = \frac{x_3}{x_1^2}$, taking into account that (with the substitution, $F_1(x) = x_1 + x_2$, $F_2(x) = x_2$ and $F_3(x) = 3x_3 + a_1 x_1^2 + a_2 x_1 x_2 + a_3 x_2^2$)

$$\begin{aligned}\xi_1 \circ F(x) &= \frac{F_2(x)}{F_1(x)} = \frac{\frac{x_2}{x_1}}{1 + \frac{x_2}{x_1}},\\ \xi_2 \circ F(x) &= \frac{F_3(x)}{F_1^2(x)} = \frac{a_1 + 3\frac{x_3}{x_1^2} + a_2\frac{x_2}{x_1} + a_3\frac{x_2^2}{x_1^2}}{1 + 2\frac{x_2}{x_1} + \frac{x_2^2}{x_1^2}},\end{aligned}$$

one obtains $\xi(t+1) = F_r(\xi(t))$, with

$$F_r(\xi) = \begin{bmatrix} \frac{\xi_1}{1+\xi_1} \\ \frac{a_1 + 3\xi_2 + a_2\xi_1 + a_3\xi_1^2}{(1+\xi_1)^2} \end{bmatrix}. \qquad \blacksquare$$

As an extension of the above reasoning, it is now shown that the existence of m DT-symmetries of F, $1 \le m < n$, pairwise commuting, allows system (1) to be decomposed as a block-triangular system in some local coordinates.

Theorem 5: Let $g_1(x), ..., g_m(x) \in \mathbb{R}^n$ be m linearly independent (over $\mathcal{K}_n$) and pairwise commuting symmetries of F,

$$\begin{aligned}\lfloor F, g_i \rfloor &= 0, \quad i = 1, ..., m, &(10a)\\ \mathrm{rank}_{\mathcal{K}_n}\left(\begin{bmatrix} g_1 & ... & g_m \end{bmatrix}\right) &= m, &(10b)\\ [g_i, g_j] &= 0, \forall i, j. &(10c)\end{aligned}$$

Then, there exist local coordinates $y = \varphi(x)$ such that the nonlinear system (1) can be *decomposed* in the local y-coordinates as

$$\begin{aligned}y_a(t+1) &= \tilde{F}_a(y_a(t), y_b(t)),\\ y_b(t+1) &= \tilde{F}_b(y_b(t)),\end{aligned}$$

where $y_a = \begin{bmatrix} y_1 & ... & y_m \end{bmatrix}^\top$, $y_b = \begin{bmatrix} y_{m+1} & ... & y_n \end{bmatrix}^\top$ and $\tilde{F}^\top = \begin{bmatrix} \tilde{F}_a^\top & \tilde{F}_b^\top \end{bmatrix}$.

Proof: By (10b), (10c), there exists a diffeomorphism $y = \varphi(x)$ such that the push-forward of g_i is straightened $\varphi_* g_i = e_i$, $i = 1, ..., m$ (where e_i is the i-th column of the identity matrix). Then, condition $\lfloor \varphi_* F, \varphi_* g_i \rfloor = 0$ can be rewritten as follows, with $\tilde{F} = \varphi_* F$,

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{\partial \tilde{F}_1}{\partial y_i} \\ \vdots \\ \frac{\partial \tilde{F}_{i-1}}{\partial y_i} \\ \frac{\partial \tilde{F}_i}{\partial y_i} \\ \frac{\partial \tilde{F}_{i+1}}{\partial y_i} \\ \vdots \\ \frac{\partial \tilde{F}_n}{\partial y_i} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad i = 1, ..., m,$$

which shows how the last $n - m$ entries of $\tilde{F}$ do not depend on y_i, $i = 1, ..., m$. $\blacksquare$

VII. Computation of First Integrals as Semi-Invariants

The goal of this section is to introduce a computational procedure for the computation of first integrals, to be possibly used for the computation of symmetries. First, note that first-integrals are a special case (for $\lambda = 1$) of semi-invariants, which are defined as follows (see [22] and [23] for continuous-time systems).

Definition 2: A *semi-invariant* of system (1) is a meromorphic scalar function $\omega(x) \in \mathbb{R}$ such that

$$\omega \circ F = \lambda \omega,$$

with $\lambda(x) \in \mathbb{R}$ being meromorphic and such that there is no zero/pole cancellation between λ and ω; if ω and λ are polynomial, then ω is said to be a *Darboux polynomial*; λ is called the *characteristic function* (respectively, the *characteristic polynomial*) of the semi-invariant (respectively, of the Darboux polynomial).

The following theorem, characterizes the Darboux polynomials associated with F, although some of such properties hold for semi-invariants too, subject to some amendments.

Theorem 6: Assume that F is polynomial.

(6.1) If $I = \frac{\omega_1}{\omega_2}$ is a first integral of system (1), with ω_1 and ω_2 being co-prime polynomials, then ω_1 and ω_2 are Darboux polynomials of system (1), with characteristic polynomials λ_1 and λ_2 such that $\lambda_1 - \lambda_2 = 0$.

(6.2) Let ω_1 and ω_2 be Darboux polynomials of system (1) with respective characteristic polynomials λ_1 and λ_2; then, the product $\omega_1^{n_1}\omega_2^{n_2}$ is a Darboux polynomial of system (1) for any pair $n_1, n_2 \in \mathbb{Z}^{\geq}$, with characteristic polynomial $\lambda_1^{n_1}\lambda_2^{n_2}$.

Remark 3: The proof of Theorem 6 is similar to the proof of Theorem 4.3 of [13], valid for CT-systems. Such a CT result is stronger than Theorem 6 above, in particular for CT-systems the "converse" of statement (6.2) also holds, in the sense that the polynomial factors of a Darboux polynomial are always Darboux polynomials too, whereas for DT-systems this is not always true (counterexamples can be given). $\blacksquare$

To obtain a clear result, assume that F is polynomial, and consider its Darboux polynomials; the algorithm proposed hereafter can be adapted to cover the computation of semi-invariants associated with F, when F is not polynomial, as shown in the subsequent Example 5.

Assume that ω is a Darboux polynomial associated with F, with characteristic polynomial λ, i.e., $\omega \circ F = \lambda\omega$. Assume, in addition, that ω is a linear combination with real and constant coefficients c_i of some functionally independent polynomials p_1, p_2, ..., p_k, for some $k > 0$, $\omega = \sum_{i=1}^{k} c_i p_i$. Consider the square $k \times k$ matrix

$$\Gamma := \begin{bmatrix} p_1 & p_2 & \cdots & p_k \\ \Delta p_1 & \Delta p_2 & \cdots & \Delta p_k \\ \vdots & \vdots & \vdots & \vdots \\ \Delta^{k-1} p_1 & \Delta^{k-1} p_2 & \cdots & \Delta^{k-1} p_k \end{bmatrix}, \qquad (11)$$

where $\Delta p_j = p_j \circ F$, $\Delta^2 p_j = p_j \circ F \circ F$ and so on.

The following theorem can be proved similarly to the analogous one for continuous-time system [24].

Theorem 7: Under the above positions and assumptions, if $\det(\Gamma) \neq 0$, then ω is a factor of $\det(\Gamma)$.

Remark 4: When $\det(\Gamma) \neq 0$, Theorem 7 guarantees that if a Darboux polynomial ω, associated with F, is a linear combination with constant coefficients of $p_1, ..., p_k$, then ω is a factor of $\det(\Gamma)$. But in the application of the theorem, all factors of $\det(\Gamma)$ or of its minors, not only those that are linear combinations of p_1, ..., p_k, are good candidates to be Darboux polynomials associated with F, because Γ could be a minor of another matrix $\check{\Gamma}$ found with an enlarged choice of the polynomials p_1, ..., $p_{\check{k}}$.

Remark 5: When $\det(\Gamma) = 0$, Theorem 7 cannot be applied: in such a case, good candidates to be Darboux polynomials associated with F are the factors of the non-zero minors of Γ. As a matter of fact, one typical reason for $\det(\Gamma)$ to be identically equal to 0 is that two or more different linear combinations, with constant coefficients, of some polynomials p_1, ..., p_k are Darboux polynomials associated with F, with the same characteristic polynomial.

Example 4: Let $F(x) = \begin{bmatrix} x_2 & x_2 + x_2^2 - x_1^2 \end{bmatrix}^\top$. Take as basis polynomials $p_1(x) = x_2$, $p_2(x) = x_1^2$. Then,

$$\Gamma(x) = \begin{bmatrix} x_2 & x_1^2 \\ x_2 + x_2^2 - x_1^2 & x_2^2 \end{bmatrix},$$

with $\det(\Gamma(x)) = \left(x_1^2 - x_2\right)\left(x_1^2 - x_2^2\right)$. Let $\omega(x) = x_1^2 - x_2$; since

$$\Delta\omega(x) = \left[F_1^2 - F_2\right]_{F_1 = x_2, F_2 = x_2 + x_2^2 - x_1^2} = x_1^2 - x_2 = \omega(x),$$

ω is Darboux polynomial associated with F, with characteristic value equal to 1, i.e., ω is a first integral associated with F.

Example 5: Consider the Lyness-type system characterized by $F(x) = \begin{bmatrix} x_2 & \frac{x_2}{x_1} \end{bmatrix}^\top$, which is not polynomial. Take as basis polynomials $p_1(x) = x_1$, $p_2(x) = x_2$, $p_3(x) = x_1^2$, $p_4(x) = x_1 x_2$, $p_5(x) = x_2^2$, $p_6(x) = x_1^3$, $p_7(x) = x_1^2 x_2$, $p_8(x) = x_1 x_2^2$, $p_9(x) = x_2^3$ (i.e., all monomials of degree less than 4, with respect to the standard dilation). Matrix Γ corresponding to such a choice has not full generic rank (its generic rank is 6). Taking the minor $\hat{\Gamma}$, found from Γ deleting the columns 4, 6 and 9 and the rows 7, 8 and 9 (actually, this corresponds to exclude monomials p_4, p_6 and p_9 from the chosen basis), one finds that $\det\left(\hat{\Gamma}\right) = q\omega_1\omega_2$, where $\omega_1(x) = x_1 + x_2 + x_1 x_2^2 + x_1^2 x_2 + x_1^2 + x_2^2$, $\omega_2(x) = \frac{x_1 + x_2 + x_1 x_2^2 + x_1^2 x_2 + x_1^2 + x_2^2}{x_1 x_2}$ and $q(x)$ is another rational function; in particular, ω_1 and ω_2 are semi-invariants associated with F, with respective characteristic functions $\lambda_1(x) = \frac{x_2}{x_1^2}$ and $\lambda_2(x) = 1$: actually, ω_2 is a first integral associated with F.

VIII. Conclusions

In this paper, the notions of Lie symmetry and of first integral for discrete-time nonlinear systems have been discussed, with particular attention to their connections. In particular, it is shown that in the scalar case the existence of a symmetry is equivalent to the existence of a first integral and to the existence of some local coordinates, under which the system behaves linearly. A similar result is given for higher order systems in Theorem 4. To complete the study, two further applications of Lie symmetries are proposed and a computational method is given in order to find first-integrals, or, more generally, semi-invariants.

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