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Identification of Box-Jenkins Models for Parameter-Varying Spatially Interconnected Systems

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Abstract— **This paper presents an identification technique to identify models for Parameter-Varying Spatially Interconnected systems. The main focus of the note is the case when there is additive colored noise in the output of the data generating system. A Refined Instrumental Variable method is proposed to identify parameter-varying spatially interconnected models with Box-Jenkins structure. The technique allows identification of models for general multi-dimensional systems, which may be separable or non-separable, causal, semi-causal (spatially interconnected systems) or non-causal. The effectiveness of the method is shown with application to simulation example.**

I. INTRODUCTION

Distributed control of complex engineering systems has attracted the attention of researchers for several decades. These systems fall under the category of multi-dimensional (m-D) systems and are composed of similar subsystems that interact with their closest neighbours. Examples of such systems include vehicle platoons, automated highway systems, spatially distributed flexible structures, fluid flow as well as systems that are governed by partial differential equations.

Methods for controller design of such systems exist see e.g. [1] and [2] and the references therein. A sufficiently accurate model is needed to synthesize optimal and/or robust controllers for such systems. One way is to obtain such a model analytically from a physical description that govern such systems, followed by estimating the physical parameters, see e.g.[3]. In many practical situations however, such physical models are either not available or are too complex.

An alternative is to employ system identification. Few results are available in this regard, however. In [4], identification of a two-dimensional (2-D) causal system transfer function identification from input output data is presented. The identification of non-causal multidimensional systems is presented in [5], [6]. Subspace based methods to identify 2-D state-space models for separable-in-denominator causal filters are discussed in [7] and [8]. Methods to identify spatially distributed interconnected systems are proposed in [9] and [10]. We refer to these methods as decentralized subspace identification of spatially interconnected systems, where the authors identified each subsystem as multi input and single output (MISO) systems. For spatially invariant m-D systems a basic IV method has been introduced in [11], which provides unbiased estimates that are non optimal in term of minimum variance.

As practically distributed parameter systems may have spatially and temporally varying parameters, recently, distributed gain scheduling control approaches have been proposed [12],[13]. An identification method for identifying parameter varying models for spatially interconnected systems has been recently proposed in [14]. However this method does not consider noise corrupted data which is usually the case in practice.

In this paper identification of models using noise corrupted data is considered. Basic Instrumental Variable (IV) methods can provide unbiased estimates but not optimal in terms of minimum variance. By considering a Box-Jenkins model structure for Parameter-Varying Spatially Interconnected (PVSI) systems (which is the case in may practical applications), statistically optimal IV estimates can be achieved. In this context we extend the Refined Instrumental Variable (RIV) and the simplified RIV (SRIV) methods presented in [15] for linear parameter varying systems to m-D PVSI systems for obtaining the parameters estimates in terms of minimum bias and variance in the presence of colored noise. The method is fairly general in the sense that it makes no assumption on separability, further it can equally be applied to causal, semi-causal (spatially interconnected) and non-causal 2-D systems. Moreover an additional benefit is that one can easily include various boundary conditions.

The paper is organized as follows: in section II the method to identify parameter-varying models for 2-D spatially varying systems is described. Identification of Box-Jenkins models for parameter-varying spatially interconnected systems is given in Section III. Section IV presents the application of the proposed method on a simulation example, and conclusions are drawn in section V.

II. P

In this note for notational simplicity we are considering 2-D systems, but the method is valid for m-D systems.

Let $u(n_1,n_2)$ be the two-dimensional discrete input signal to a SISO linear PVSI 2-D system. Then its output $y(n_1,n_2)$ can be represented as follows, and this forms our data generating system:

$$\begin{aligned} A_0(p(n_1,n_2),q_1,q_2)x_0(n_1,n_2) &= B_0(p(n_1,n_2),q_1,q_2)u(n_1,n_2) \\ y(n_1,n_2) &= x_0(n_1,n_2)+v_0(n_1,n_2) \end{aligned} \tag{1}$$

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where x_0 is noise-free output, v_0 is additive colored noise, n_1 and n_2 represent the discrete instances of the two dimensions respectively, q_1 and q_2 are the shift-operators in both dimensions. As we are considering linear PVSI systems, the model coefficients are assumed to be functions of measurable temporal or spatial signals, the so called scheduling variable $p : \mathbb{Z} \to \mathbb{P}$, where $\mathbb{Z}$ represents the set of integers. The compact set $\mathbb{P} \subseteq \mathbb{R}^{n_p}$ denotes the admissible region in the scheduling space. A_0 and B_0 are given as

$$A_0(p(n_1,n_2),q_1,q_2) = 1 + \sum_{(i_1,i_2)\in M^y\setminus(0,0)} a^0_{i_1,i_2}(p(n_1,n_2))q_1^{-i_1}q_2^{-i_2} \tag{2a}$$

$$B_0(p(n_1,n_2),q_1,q_2) = \sum_{(i_1,i_2)\in M^u} b^0_{i_1,i_2}(p(n_1,n_2))q_1^{-i_1}q_2^{-i_2} \tag{2b}$$

where M^y and M^u denote the support regions (masks) for output and input terms, respectively, and n_1 and n_2 are independent variables (usually time and space). The support region (mask) is defined as a subset of the two-dimensional space in which the indices of the coefficients of input and output terms in the difference equation lie. For more details see [14].

A general support region for 2-D systems lies in the 2-D plane. For causal systems the support region is a subset of the first quadrant of the 2-D plane, for semi-causal systems it is a subset of the right half plane, and for non-causal systems it lies in all four quadrants of the 2-D plane. Let M represent a general support region (mask) for a 2-D system consisting of a union of intervals

$$M = \bigcup_{i_1=k_1}^{l_1} m(i_1) \tag{3}$$

$$m(i_1) = \{(i_1,i_2) : k_2(i_1) \le i_2 \le l_2(i_1)\} \tag{4}$$

where

$$l_1 = max\{i_1 : (i_1,i_2) \in M\}$$
$$k_1 = min\{i_1 : (i_1,i_2) \in M\}$$
$$l_2(i_1) = max\{i_2 : (i_1,i_2) \in m(i_1)\}$$
$$k_2(i_1) = min\{i_2 : (i_1,i_2) \in m(i_1)\}$$

With the above notation, we shall represent support regions for output and input as M^y and M^u respectively. Also $v_0(n_1,n_2) = H_0(q_1,q_2)e(n_1,n_2)$ where H_0 is a 2-D linear filter and $e(n_1,n_2)$ is a 2-D zero-mean random white-noise process with normal distribution.

A two-dimensional linear PVSI-ARX model structure for SISO systems can be defined as

$$y(n_1,n_2) = -\sum_{(i_1,i_2)\in M^y\setminus(0,0)} a_{i_1,i_2}(p(n_1,n_2))y(n_1-i_1,n_2-i_2) + \sum_{(i_1,i_2)\in M^u} b_{i_1,i_2}(p(n_1,n_2))u(n_1-i_1,n_2-i_2) + e(n_1,n_2) \tag{5}$$

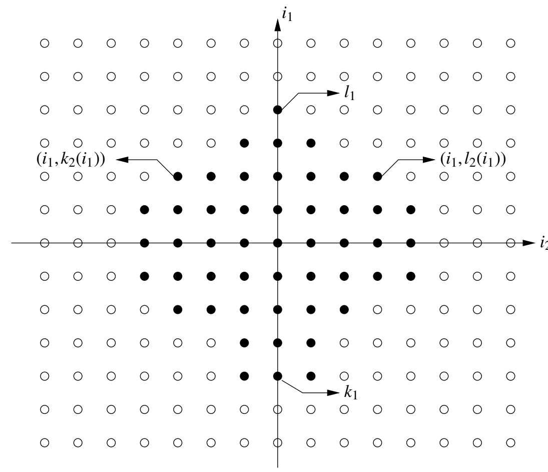

Fig. 1. Two-dimensional lattice, filled circles show the mask M

The coefficient functions a_{i_1,i_2}, b_{i_1,i_2} have static dependence on $p(n_1,n_2)$, i.e. dependence on the instantaneous values of p at the discrete indices n_1 and n_2 of the two independent variables, and are parameterized as

$$a_{i_1,i_2}(p(n_1,n_2)) = a_{i_1,i_2,0} + \sum_{j=1}^{s} a_{i_1,i_2,j}\psi_{i_1,i_2,j}(p(n_1,n_2)) \tag{6}$$

$$b_{i_1,i_2}(p(n_1,n_2)) = b_{i_1,i_2,0} + \sum_{j=1}^{s} b_{i_1,i_2,j}\psi_{i_1,i_2,j}(p(n_1,n_2)) \tag{7}$$

Here $\psi_{i_1,i_2,j}(p(.)) : \mathbb{P} \to \mathbb{R}$ are user-defined functions of the scheduling variables and they are non-singular on $\mathbb{P}$. s represents the number of functions, which can be different for different coefficients, but here we are assuming that all coefficients depend on same number of functions.

Assuming a general support region (see [14]) the difference equation (5) can be written as

$$y(n_1,n_2) = -\sum_{i_1=k_1^y}^{l_1^y} \sum_{i_2=k_2^y(i_1)}^{l_2^y(i_1)} a_{i_1,i_2}(p(n_1,n_2))y(n_1-i_1,n_2-i_2) + \sum_{i_1=k_1^u}^{l_1^u} \sum_{i_2=k_2^u(i_1)}^{l_2^u(i_1)} b_{i_1,i_2}(p(n_1,n_2))u(n_1-i_1,n_2-i_2) + e(n_1,n_2) \tag{8}$$

where for the first term of (8) on the right hand side we have the condition that $(i_1,i_2) \neq (0,0)$. The coefficients a_{i_1,i_2}, b_{i_1,i_2} are given in (6) and (7). The data column vectors and the corresponding system parameter vectors which are associated with the support region M^y and M^u are constructed as follows:

Output data corresponding to M^y is

$$\forall i_1 \in \left[k_1^y, l_1^y\right]$$

$$
\begin{aligned}
y_{m^y(i_1)}(n_1,n_2) &= \\
&-\left[y(n_1-i_1,n_2-i_2)\otimes\begin{bmatrix}1\\ \psi_{i_1,i_2,j}(.)\end{bmatrix}_{j=1:s}\right]_{i_2=k_2^y(i_1),\dots,l_2^y(i_1)} \\
a_{m^y(i_1)} &= \left[\left[a_{i_1,i_2,j}\right]_{j=0:s}\right]_{i_2=k_2^y(i_1),\dots,l_2^y(i_1)}
\end{aligned}
\tag{9}
$$

Here we exclude on the right hand side the terms corressponding to point (0,0). Finally

$$
Y_{M^y}(n_1,n_2) = \left[y_{m^y(i_1)}(n_1,n_2)\right]_{i_1=k_1^y,\dots,l_1^y} \tag{10}
$$

$$
A_{M^y} = \left[a_{m^y(i_1)}\right]_{i_1=k_1^y,\dots,l_1^y}
$$

Input data corresponding to M^u is

$$
\forall i_1 \in \left[k_1^u, l_1^u\right]
$$

$$
\begin{aligned}
u_{m^u(i_1)}(n_1,n_2) &= \\
&\left[u(n_1-i_1,n_2-i_2)\otimes\begin{bmatrix}1\\ \psi_{i_1,i_2,j}(.)\end{bmatrix}_{j=1:s}\right]_{i_2=k_2^u(i_1),\dots,l_2^u(i_1)} \\
b_{m^u(i_1)} &= \left[\left[b_{i_1,i_2,j}\right]_{j=0:s}\right]_{i_2=k_2^u(i_1),\dots,l_2^u(i_1)}
\end{aligned}
\tag{11}
$$

Finally

$$
U_{M^u}(n_1,n_2) = \left[u_{m^u(i_1)}(n_1,n_2)\right]_{i_1=k_1^u,\dots,l_1^u} \tag{12}
$$

$$
B_{M^u} = \left[b_{m^u(i_1)}\right]_{i_1=k_1^u,\dots,l_1^u}
$$

Then (5) can be written in linear regression form as

$$
y(n_1,n_2) = \varphi^\top(n_1,n_2)\theta + e(n_1,n_2) \tag{13}
$$

where the regressor vector φ is given as

$$
\varphi(n_1,n_2) = \begin{bmatrix} Y_{M^y}(n_1,n_2) \\ U_{M^u}(n_1,n_2) \end{bmatrix} \tag{14}
$$

and the parameter vector θ is given accordingly as

$$
\theta = \begin{bmatrix} A_{M^y} \\ B_{M^u} \end{bmatrix} \tag{15}
$$

If measured data of size $N_1 \times N_2$ is available as input, output and scheduling signal, then output and regressors are respectively constructed as

$$
Y = \begin{bmatrix} y(1,1) & \dots & y(N_1,1) & y(1,2) & \dots & \dots & y(N_1,N_2) \end{bmatrix}^\top \tag{16}
$$

$$
\Phi = \begin{bmatrix} \varphi(1,1) & \dots & \varphi(N_1,1) & \varphi(1,2) & \dots & \dots & \varphi(N_1,N_2) \end{bmatrix}^T \tag{17}
$$

and the parameter vector is computed using the Least Squares (LS) method as

$$
\theta = (\Phi^\top\Phi)^{-1}\Phi^\top Y. \tag{18}
$$

Equation (18) gives consistent estimates of the parameter if $v_0(n_1,n_2) = A_0^\dagger(p(n_1,n_2),q_1,q_2)e(n_1,n_2)$ (where '†' denotes pseudoinverse) in (1) that is data generating system has ARX structure; if this is not the case, then there is colored noise in generated output instead of $e(n_1,n_2)$, thus using the LS method leads to bias in the estimates. To get unbiased estimates we have to consider different model structures other than the ARX one; this is the main contribution of this paper. If we have colored noise $\tilde{v}(n_1,n_2)$ in the output data then (13) can be written as

$$
y(n_1,n_2) = \varphi^\top(n_1,n_2)\theta + \tilde{v}(n_1,n_2) \tag{19}
$$

where

$$
\tilde{v}(n_1,n_2) = A(p(n_1,n_2),q_1,q_2)v(n_1,n_2) \tag{20}
$$

where v is colored noise. Next the Box Jenkins model structure which has been used in LTI system identification, is extended to PVSI systems.

III. B -J M I P -V S I S

A. Non optimality of the IV approach and dynamic dependence

The LS approach has been extended for spatially varying interconnected systems in [14]. In this case an ARX type model has been assumed. On the other hand, if the model of disturbances acting on the system is misspecified, the Instrumental Variable(IV) method can be used to provide consistent estimates; this has been extended recently to 2-D systems in [11]. Nevertheless, it is generally known that the estimates obtained through IV methods are not optimal. Therefore the Box-Jenkins model structure should be considered to obtain such optimal estimates in terms of minimum variance in the estimated parameters.

Another problem appears when PVSI systems are considered, namely the non-commutativity of its parameter-varying coefficients with the shift operators, e.g. $q_1 a(p(n_1,n_2)) \neq a(p(n_1,n_2))q_1$ but $q_1 a(p(n_1,n_2)) = a(p(n_1+1,n_2))q_1$. This problem has been discussed for LPV systems in [16]. This problem prevents the identification problem to be written in linear regression form if a model type other than ARX is considered. Here an approach which has been proposed recently in [16] to handle such problems in the LPV case is extended to multidimensional interconnected systems. To handle the non-commutativity, the model equations in (21) are reformulated, and to provide estimates with the minimum variance an RIV method is proposed.

B. Box-Jenkins Model

A Box-Jenkins model for parameter-varying spatially interconnected system can be represented as:

$$
\begin{aligned}
A(p(n_1,n_2),q_1,q_2,\theta)x(n_1,n_2) &= B(p(n_1,n_2),q_1,q_2,\theta)u(n_1,n_2) \\
v(n_1,n_2) &= H(q_1,q_2,\eta)e(n_1,n_2) \\
y(n_2,n_2) &= x(n_1,n_2) + v(n_1,n_2)
\end{aligned}
\tag{21}
$$

where

$$H(q_1,q_2,\eta) = \frac{C(q_1,q_2)}{D(q_1,q_2)} \tag{22}$$

is a 2-D filter and

$$\begin{aligned} C(q_1,q_2) &= 1 + \sum_{(i_1,i_2)\in M^c,(i_1,i_2)\neq(0,0)} c_{i_1,i_2} q_1^{-i_1} q_2^{-i_2} \\ D(q_1,q_2) &= 1 + \sum_{(i_1,i_2)\in M^d,(i_1,i_2)\neq(0,0)} d_{i_1,i_2} q_1^{-i_1} q_2^{-i_2} \end{aligned} \tag{23}$$

with $C(q_1,q_2)$ and $D(q_1,q_2)$ being stable polynomials. M^c and M^d here correspond to the support region for numerator and denominator of the 2-D filter, respectively. The associated parameters of $H(q_1,q_2)$, i.e. $c_{i_1,i_2}, \forall i_1,i_2 \in M^c$ and $d_{i_1,i_2}, \forall i_1,i_2 \in M^d$, are stacked columnwise in the parameter vector $\eta \in \mathbb{R}^{n_\eta}$. Let $\mathcal{H} = \{\mathsf{H}_\eta \mid \eta \in \mathbb{R}^{n_\eta}\}$ denote the collection of all noise models in the form

$$\mathsf{H}_\eta : (H(q_1,q_2,\eta)) \tag{24}$$

Now let the process model be denoted as $\mathcal{G}_\theta$, and $\mathcal{G} = \{\mathcal{G}_\theta | \theta \in \mathbb{R}^{n_\theta}\}$ be the set of all process models. The parameters corresponding to a given process and noise model $(\mathcal{G}_\theta, \mathcal{H}_\eta)$ can be collected as

$$\rho = [\theta^\top \quad \eta^\top]^\top \tag{25}$$

Let $\mathcal{M}_\rho$ denote the model. Parametrized independently with process $(\mathcal{G}_\theta)$ and noise $(\mathcal{H}_\eta)$ model, the model set is

$$\mathcal{M}_\rho = \{(\mathcal{G}_\theta, \mathcal{H}_\eta) \mid col(\theta,\eta) = \rho \in \mathbb{R}^{n_\theta+n_\eta}\} \tag{26}$$

This set corresponds to the set of candidate models in which we seek our model corresponding to the data generating system.

C. Reformulation of the model equations

Based on the above discussion, in order to write the model (21) in a linear regressor form to be able to use the IV or LS method, the relation in (21) is rewritten as

$$\begin{aligned} &\underbrace{x(n_1,n_2) + \sum_{(i_1,i_2)\in M^y\setminus(0,0)} a_{i_1,i_2,0} x(n_1-i_1,n_2-i_2)}_{F(q_1,q_2)x(n_1,n_2)} \\ &+ \sum_{(i_1,i_2)\in M^y\setminus(0,0)} \sum_{j=1}^{s} a_{i_1,i_2,j} \underbrace{\psi_{i_1,i_2,j}(p(n_1,n_2))x(n_1-i_1,n_2-i_2)}_{x_{i_1,i_2,j}(n_1,n_2)} \\ &= \sum_{(i_1,i_2)\in M^u} \sum_{j=0}^{s} b_{i_1,i_2,j} \underbrace{\psi_{i_1,i_2,j}(p(n_1,n_2))u(n_1-i_1,n_2-i_2)}_{u_{i_1,i_2,j}(n_1,n_2)} \\ &\qquad v(n_1,n_2) = H(q_1,q_2)e(n_1,n_2) \\ &\qquad y(n_1,n_2) = x(n_1,n_2) + v(n_1,n_2) \end{aligned} \tag{27}$$

where $F(q_1,q_2) = 1 + \sum_{(i_1,i_2)\in M^y\setminus(0,0)} a_{i_1,i_2,0} q_1^{-i_1} q_2^{-i_2}$. The Box-Jenkins model of PVSI system is written in this way as a *Multi-Input Single-Output* (MISO) system. As the polynomial operator commutes in this representation, (27) can be rewritten as

$$\begin{aligned} y(n_1,n_2) = &- \sum_{(i_1,i_2)\in M^y\setminus(0,0)} \sum_{j=1}^{s} \frac{a_{i_1,i_2,j}}{F(q_1,q_2)} x_{i_1,i_2,j}(n_1,n_2) \\ &+ \sum_{(i_1,i_2)\in M^u} \sum_{j=0}^{s} \frac{b_{i_1,i_2,j}}{F(q_1,q_2)} u_{i_1,i_2,j}(n_1,n_2) + H(q_1,q_2)e(n_1,n_2) \end{aligned} \tag{28}$$

which is an invariant spatially interconnected (ISI) system representation. equations (28) and (21) are equivalent to each other, where the former is nothing more than a MISO-ISI system representation of the latter. Based on the formulation given in (28), optimal prediction error minimization (PEM) is possible to be achieved by using linear regression and extending the *Refined Instrumental Variable* (RIV) approach of the LTI identification framework. A similar approach has been taken in [15] for 1-D parameter varying systems.

D. Optimal PEM for PVSI-BJ models

Using (28), $y(n_1,n_2)$ can be written in the linear regressor form as

$$y(n_1,n_2) = \varphi^\top(n_1,n_2)\theta + \tilde{v}(n_1,n_2) \tag{29}$$

where

$$\tilde{v}(n_1,n_2) = F(q_1,q_2)v(n_1,n_2) \tag{30}$$

It is important to keep in mind that (19) and (29) are not equivalent, as the extended regressor in the latter equation contains noise free terms. Using a conventional PEM approach on (29), the prediction error $\varepsilon(n_1,n_2)$ is given as

$$\begin{aligned} \varepsilon(n_1,n_2) = \frac{D(q_1,q_2)}{C(q_1,q_2)F(q_1,q_2)} \{ &F(q_1,q_2)y(n_1,n_2) \\ &- \sum_{(i_1,i_2)\in M^y\setminus(0,0)} \sum_{j=1}^{s} a_{i_1,i_2,j} x_{i_1,i_2,j}(n_1,n_2) \\ &+ \sum_{(i_1,i_2)\in M^u} \sum_{j=0}^{s} b_{i_1,i_2,j} u_{i_1,i_2,j}(n_1,n_2) \} \end{aligned} \tag{31}$$

where $\frac{D(q_1,q_2)}{C(q_1,q_2)}$ can be seen as the inverse of the ARMA noise model in (21). However, as the system written in (28) is equivalent to an ISI system, the polynomial operators commute and (31) can be written in an alternative form as

$$\begin{aligned} \varepsilon(n_1,n_2) = F(q_1,q_2)&y_f(n_1,n_2) \\ &- \sum_{(i_1,i_2)\in M^y\setminus(0,0)} \sum_{j=1}^{s} a_{i_1,i_2,j} x^f_{i_1,i_2,j}(n_1,n_2) \\ &+ \sum_{(i_1,i_2)\in M^u} \sum_{j=0}^{s} b_{i_1,i_2,j} u^f_{i_1,i_2,j}(n_1,n_2) \end{aligned} \tag{32}$$

where $y_f(n_1,n_2)$, $x^f_{i_1,i_2,j}(n_1,n_2)$ and $u^f_{i_1,i_2,j}(n_1,n_2)$ represent the outputs of the prefiltering operation using the filter

$$Q(q_1,q_2) = \frac{D(q_1,q_2)}{C(q_1,q_2)F(q_1,q_2)} \tag{33}$$

Based on (32), the associated linear-in-the-parameters model takes the form

$$y_f(n_1,n_2) = \varphi_f^\top(n_1,n_2)\theta + \tilde{v}_f(n_1,n_2) \tag{34}$$

where

$$\begin{aligned}\tilde{v}_f(n_1,n_2) = F(q_1,q_2)v_f(n_1,n_2) = \\ F(q_1,q_2)\frac{D(q_1,q_2)}{C(q_1,q_2)F(q_1,q_2)}v(n_1,n_2) \\ = e(n_1,n_2)\end{aligned} \tag{35}$$

and $\varphi_f(n_1,n_2)$ contains associated filtered terms. This means that if the optimal filter (33) is known *a priori*, it is possible to filter the data such that a simple LS algorithm applied to the data pre-filtered with (33) leads to the statistically optimal estimate. Since the filter $Q(q_1,q_2)$ is unknown in advance, the RIV estimates involves an iterative algorithm, in which at each iteration an auxiliary model is used to generate the instrumental variables and the prefilter.

E. The Iterative RIV Algorithm

Step 1 *ARX model estimation*
Compute an initial ARX estimate using the LS approach. This gives $\hat{A}^{(0)}$ and $\hat{B}^{(0)}$. Set $\hat{D}^{(0)}(q_1,q_2) = 1, \hat{C}^{(0)}(q_1,q_2) = 1$ and $i = 0$

Step 2 *Generate data*
Compute an estimate of the noise free output $\hat{x}(n_1,n_2)$ by simulating the auxiliary model

$$\begin{aligned}\hat{A}^{(i)}(p(n_1,n_2),q_1,q_2,\hat{\theta}^{(i)})\hat{x}(n_1,n_2) = \\ \hat{B}^{(i)}(p(n_1,n_2),q_1,q_2,\hat{\theta}^{(i)})u(n_1,n_2)\end{aligned}$$

based on the estimated parameters $\hat{\theta}^{(i)}$ of the previous iteration.

Step 3 *Compute the estimated filter*

$$\hat{Q}(q_1,q_2) = \frac{\hat{D}^{(i)}(q_1,q_2)}{\hat{C}^{(i)}(q_1,q_2)\hat{F}^{(i)}(q_1,q_2)}$$

along with obtaining the associated filtered signals $\{u^f_{i_1,i_2,j}(n_1,n_2)\}, y_f(n_1,n_2)$ and $\{x^f_{i_1,i_2,j}(n_1,n_2)\}$.

Step 4 *Build the estimated regressor*
Define

$$U^f(n_1,n_2) = \begin{bmatrix} u^f_{i_1,i_2,0}(n_1,n_2) \\ u^f_{i_1,i_2,1}(n_1,n_2) \\ \vdots \\ u^f_{i_1,i_2,s}(n_1,n_2) \end{bmatrix}_{\forall(i_1,i_2)\in M^u}$$

$$X_1^f(n_1,n_2) = \left[x^f_{i_1,i_2,0}(n_1,n_2)\right]_{\forall(i_1,i_2)\in M^y}$$

$$X_2^f(n_1,n_2) = \begin{bmatrix} x^f_{i_1,i_2,1}(n_1,n_2) \\ \vdots \\ x^f_{i_1,i_2,s}(n_1,n_2) \end{bmatrix}_{\forall(i_1,i_2)\in M^y}$$

$$Y^f(n_1,n_2) = \left[y_f(n_1-i_1,n_2-i_2)\right]_{\forall(i_1,i_2)\in M^y}$$

Proceed to build the filtered estimated regressor as

$$\varphi_f(n_1,n_2) = \begin{bmatrix} Y^f(n_1,n_2) \\ X_2^f(n_1,n_2) \\ U^f(n_1,n_2) \end{bmatrix} \tag{36}$$

with the filtered instrument given as

$$\varsigma_f(n_1,n_2) = \begin{bmatrix} X_1^f(n_1,n_2) \\ X_2^f(n_1,n_2) \\ U^f(n_1,n_2) \end{bmatrix} \tag{37}$$

With measured data size $N_1 \times N_2$, the filtered regressor becomes

$$\hat{\Phi}_f = \left[\varphi_f(1,1) \quad \ldots\varphi_f(N_1,1) \quad \varphi_f(1,2) \quad \ldots \quad \varphi_f(N_1,N_2)\right]^\top \tag{38}$$

The filtered instrument as

$$\hat{\xi}_f = \left[\varsigma_f(1,1) \quad \ldots \quad \varsigma_f(N_1,1) \qquad \varsigma_f(1,2) \quad \ldots \quad \varsigma_f(N_1,N_2)\right]^\top \tag{39}$$

also the filtered output is represented as

$$Y_f = \left[y_f(1,1) \quad \ldots \quad y_f(N_1,1) \quad y_f(1,2) \quad \ldots \quad y_f(N_1,N_2)\right]^\top \tag{40}$$

Step 5 *Compute the IV estimate*
The IV estimate is computed as

$$\hat{\theta}^{(i+1)} = (\hat{\xi}_f^\top\hat{\Phi}_f)^{-1}\hat{\xi}_f^\top Y_f \tag{41}$$

where $\hat{\theta}^{(i+1)}$is the IV estimate of the process model parameter vector at iteration $i+1$ based on prefiltered data.

Step 6 *Noise model estimate*
Estimate the noise signal as

$$\hat{v}(n_1,n_2) = y(n_1,n_2) - \hat{x}(n_1,n_2) \tag{42}$$

Based on this, the noise model parameter vector $\hat{\eta}^{(i+1)}$ is estimated using the ARMA estimation algorithm of the MATLAB identification toolbox. If we take $H(q_1,q_2) = 1$ at this step and avoid noise model estimation, the method is referred as simplified RIV (SRIV).

Step 7 *Stopping criteria*
If convergence has occured or the maximum number of iterations reached then stop, else set $i = i+1$ and go to Step 2.

Remarks: The above algorithm gives optimal estimates if the noise filter is known and if the algorithm converges. As the filter is unknown in practice, optimality can not be guaranteed. However, our experience shows that the algorithm converges most of the time.

IV. I E

As a practical example for the approach discussed in the previous section, a spatially-varying interconnected system is considered. It is heat flow in a rod having thermal conductivity of the material varying with the spatial dimension. Such a system is governed by the equation

$$\rho c\frac{\partial y(t,x)}{\partial t} = \frac{\partial}{\partial x}K(x)\frac{\partial y(t,x)}{\partial x} + u(t,x), \tag{43}$$

TABLE I

E SNR

Method		15dB	10dB	5dB
LS	Bias Norm	0.0979	0.2225	0.3308
	Variance Norm	0.0001	0.0002	0.0006
SRIV	Bias Norm	0.0002	0.0005	0.0012
	Variance Norm	6×10^{-6}	2×10^{-5}	5×10^{-5}
RIV	Bias Norm	0.0001	0.0002	0.0005
	Variance Norm	3.6×10^{-6}	1.3×10^{-5}	3.6×10^{-5}

where y is the temperature, u is a linear heat source, ρ is the density, c is the specific heat and K is the thermal conductivity of the material. Here we are assuming that thermal conductivity varies linearly along the spatial dimension (x) as

$$K(x) = K_0(1 - \epsilon\frac{x}{L})$$

where L is the length of the rod, and we take K_0, ρ and c as unity. ϵ is a constant having value between zero and one; we take it here as 0.5. The rod is divided spatially into 10 subsystem. The scheduling parameter, which only depends on the spatial dimension, is defined as

$$p(x) = \frac{2x}{L} - 1$$

or in discrete domain

$$p(n_2) = \frac{2n_2}{L} - 1$$

The scheduling function is taken as

$$\psi_{i_1,i_2,1}(.) = \psi_1(.) = p(n_2) = \frac{2n_2}{L} - 1$$

The model for the mentioned system is identified using the approach given in [14], without considering any additive output noise. The parameters of this identified model are considered as the true values. The sampling time is 0.001s. Noisy data is generated from this system by considering a structure as in (1) and (2) with $H_0(q_1,q_2) = \frac{1}{1+d_1q_1^{-1}+d_2q_1^{-2}}$. The true parameter values are given in Table **??**. The inputs $u(n_1,n_2)$ and $e(n_1,n_2)$ are taken as 2-D zero-mean normally distributed white-noise. The data used for identification is of size 4000×10. The model structure is given as in (21). Monte-Carlo simulations of 100 runs are carried out at different signal-to-noise ratios (SNR). True values of filter parameters are $d_1 = -0.4$ and $d_2 = 0.8$. Mean of both the parameters is $d_1 = -0.404$ and $d_2 = 0.7996$ when RIV method is employed at an SNR of 10 dB. Similarly the standard deviation is $d_1 = 0.011$ and $d_2 = 0.0115$ and shows that the algorithm identifies the filter in an efficient method. Table I shows the comparison of estimated parameters, bias norm ($\|\theta_0 - E[\hat{\theta}]\|_2$) and variance norm ($\|E[(\hat{\theta} - E[\hat{\theta}])^2]\|_2$), where θ_0 is the true and $\hat{\theta}$ the estimated parameter vector. The results show that RIV gives the estimate with a minimum bias and variance. Further SRIV also gives reasonable results for this example.

V. CONCLUSIONS

A method to identify linear PVSI input-output model of multi-dimensional parameter-varying spatially interconnected system is proposed. The method identifies Box-Jenkins models for such systems. The method can give estimates with minimum bias and variance. It can be used for the identification of a general class of multi-dimensional linear PVSI systems which may be separable or non-separable. The method is equally valid for causal, semi-causal and non-causal systems. Furthermore, boundary conditions can be easily included in the algorithm. Simulation example demonstrates the effectiveness of the method.

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