2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Hybrid Control of Rigid-Body Attitude with Synergistic Potential Functions*

Christopher G. Mayhew♯ and Andrew R. Teel†

***Abstract*— Achieving global asymptotic stabilization of rigid-body attitude is impossible using smooth feedback due to topological obstructions. In this paper, we propose a hybrid feedback that coordinates a "synergistic" family of potential functions and their natural feedbacks to achieve global asymptotic stability of a desired attitude with robustness to small perturbations including measurement noise. We illustrate the results via simulation.**

I. Introduction

Control of rigid-body attitude has been a longstanding staple of nonlinear control with a vast literature spanning many decades [1]–[5] and applications in aerospace and underwater vehicles [6], [7]. As described in [8], global stabilization of a desired attitude is obstructed by the topology of the underlying state space, $\mathsf{SO}(3)$. Being a compact manifold without boundary, a result of degree theory yields that $\mathsf{SO}(3)$ does not possess the topological property of contractibility [9, Ex. 2.4.6]. As shown in [8], [10], this precludes the existence of a continuous feedback that globally asymptotically stabilizes a desired rigid-body attitude.

The best existing results for smooth feedback control of rigid-body attitude are "almost global," where the basin of attraction of the desired attitude excludes a nowhere dense set of measure zero. Such results are shown in [11] using total energy as a Lyapunov function and have been extended over the years in [12]–[14]. The aforementioned designs all rely on an appropriate "error function" on $\mathsf{SO}(3)$ and all use the same "modified trace function." This error function is used to create an artificial potential energy for the system through feedback, which, when paired with a feedback component that removes energy, leads the system to converge to critical points of the artificial potential energy [11], [13]. For the modified trace function, this leads to control laws that render the desired attitude (almost globally) asymptotically stable, but create unstable saddle equilibria at 180° rotations about the eigenvectors of a specified matrix [15]–[17].

Smooth feedback strategies for attitude control that are based on a single potential energy are inherently limited by the topology of $\mathsf{SO}(3)$. By a theorem of Lusternik and Schnirelmann [18], [19], there exist at least four critical points of any smooth function on $\mathsf{SO}(3)$, guaranteeing the existence of unwanted equilibria induced by energy-based strategies like those of [11]–[13]. However, these topological obstructions can be overcome by coordinating a "synergistic" family of potential functions and their associated feedbacks with the hybrid strategy proposed in this work, which is an extension of the authors' results for planar rotations [20], spherical orientation [21], and the 3D pendulum [22].

Synergism, as defined in this work, is a condition on a family of potential functions requiring that at each critical point (that is not the desired attitude) of each potential function in the family, there exists another potential function in the family of lower value. As we show through Lyapunov theory and invariance analysis, the synergism condition yields a hybrid controller guaranteeing robust global asymptotic stability of a desired attitude. Unfortunately, as we have shown in the companion paper [17], the class of modified trace functions is not wide enough to yield a synergistic family, despite the fact that one can construct many modified trace functions whose only common critical point is the desired attitude. In this paper, we use the construction proposed in [17] to generate a synergistic family of potentials. Finally, we note that the synergism concept and hybrid feedback have connections with the use of multiple Lyapunov functions [23], the "min-switch" strategy of [24], [25], and [26].

The remainder of this paper is organized as follows. Section II provides a summary of notation used in this paper and outlines the control objective. Section III proposes the hybrid controller and establishes an equivalence between global asymptotic stability of the desired attitude and synergism of a family of potentials. In Section IV, we summarize the main results of the companion paper [17]. In Section V, we provide a brief simulation study of the proposed hybrid controller and illustrate how it eliminates performance constraints of smooth controllers. Finally, we provide some concluding remarks in Section VI.

II. Preliminaries

We denote the special orthogonal group of order three as

$$\mathsf{SO}(3) = \{R \in \mathbb{R}^{3\times 3} : R^\top R = I,\ \det(R) = 1\}.$$

Given two vectors $y, z \in \mathbb{R}^3$, their cross product can be represented by a matrix multiplication: $y \times z = [y]_\times z$, where

$$[y]_\times = \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix}$$

constitutes an isomorphism between $\mathbb{R}^3$ and $\mathfrak{so}(3) = \{S \in \mathbb{R}^{3\times 3} : S^\top = -S\}$, the Lie algebra of $\mathsf{SO}(3)$. We denote

♯mayhew@ieee.org, Robert Bosch Research and Technology Center, 4005 Miranda Ave., Palo Alto, CA 94304.

†teel@ece.ucsb.edu, Center for Control Engineering and Computation, Electrical and Computer Engineering Department, University of California, Santa Barbara, CA 93106-9560.

*Research partially supported by the National Science Foundation under grant ECCS-0925637 and grant CNS-0720842, and by the Air Force Office of Scientific Research under grant FA9550-09-1-0203.

the inverse operation of $[\cdot]_\times$ as $\mathrm{vec}_\times : \mathfrak{so}(3) \to \mathbb{R}^3$, defined implicitly as

$$\mathrm{vec}_\times [y]_\times = y,$$

for all $y \in \mathbb{R}^3$. By defining skew : $\mathbb{R}^{3\times3} \to \mathfrak{so}(3)$ as the map $\mathrm{skew}\, A = (A - A^\top)/2$, we can extend the definition of $\mathrm{vec}_\times$ to all of $\mathbb{R}^{3\times3}$ by taking its composition with skew [11]. In this direction, we define $\psi : \mathbb{R}^{3\times3} \to \mathbb{R}^3$ as

$$\psi(A) = \mathrm{vec}_\times(\mathrm{skew}\, A) = \frac{1}{2}\begin{bmatrix} A_{32} - A_{23} \\ A_{13} - A_{31} \\ A_{21} - A_{12} \end{bmatrix}.$$

The attitude of a rigid body, denoted by $R \in \mathsf{SO}(3)$, represents a rotation of coordinates that express a vector in a body-fixed frame to coordinates in an inertial frame. The attitude of a rigid body has dynamics

$$\dot{R} = R\,[\omega]_\times \tag{1a}$$

$$J\dot{\omega} = [J\omega]_\times\,\omega + \tau, \tag{1b}$$

where $\omega \in \mathbb{R}^3$ is the body-fixed angular velocity, $J = J^\top > 0$ is the inertia matrix, and $\tau \in \mathbb{R}^3$ is a vector of external torques available for control. For compactness, we define

$$x = (R, \omega) \in \mathsf{SO}(3) \times \mathbb{R}^3$$

$$f(x, \tau) = \left(R\,[\omega]_\times,\, J^{-1}\left([J\omega]_\times\,\omega + \tau\right)\right)$$

and shorten (1) to

$$\dot{x} = f(x, \tau). \tag{2}$$

The projection from $\mathsf{SO}(3) \times \mathbb{R}^3$ to $\mathsf{SO}(3)$ is defined as

$$\pi(R, \omega) = R.$$

For the remainder of this paper, we will be concerned with stabilizing the identity element at zero angular velocity, $(I, 0) \in \mathsf{SO}(3) \times \mathbb{R}^3$. We note that a solution to this problem leads to a solution of the tracking problem by a common change of coordinates and the addition of a feedforward torque (see [14], for example).

The n-dimensional unit sphere embedded in $\mathbb{R}^{n+1}$ is

$$\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : x^\top x = 1\}.$$

Then, given a rotation angle $\theta \in \mathbb{R}$ and an axis $u \in \mathbb{S}^2$, it follows that $e^{\theta[u]_\times} \in \mathsf{SO}(3)$ and

$$\mathcal{R}(\theta, u) := e^{\theta[u]_\times} = I + \sin\theta\,[u]_\times + (1 - \cos\theta)\,[u]_\times^2,$$

where for some $A \in \mathbb{R}^{n\times n}$, e^A denotes the matrix exponential of A. This is commonly known as the angle-axis parametrization of $\mathsf{SO}(3)$, or the Rodrigues formula. One can recover the angle and axis (nonuniqely) as

$$u\sin\theta = \psi(\mathcal{R}(\theta, u)) \quad 2\cos\theta = \mathrm{trace}(\mathcal{R}(\theta, u)) - 1.$$

Given vectors $x, y \in \mathbb{R}^n$ and matrices $A, B \in \mathbb{R}^{m\times n}$, their inner products are defined as $\langle x, y\rangle := x^\top y$ and $\langle A, B\rangle := \mathrm{trace}(A^\top B)$, respectively. The 2-norm of a vector $y \in \mathbb{R}^n$ is $|y| = \sqrt{\langle y, y\rangle}$ and the Frobenius norm of a matrix $A \in \mathbb{R}^{n\times m}$ is $\|A\|_F = \sqrt{\langle A, A\rangle}$. The closed unit ball in $\mathbb{R}^n$ is $\mathbb{B} = \{x \in \mathbb{R}^n : |x| \le 1\}$.

Given differentiable functions $h : \mathbb{R}^n \to \mathbb{R}$, and $k : \mathbb{R}^{m\times n} \to \mathbb{R}$, we define the symbols

$$\nabla h(x) = \begin{bmatrix} \frac{\partial h(x)}{\partial x_1} \\ \vdots \\ \frac{\partial h(x)}{\partial x_n} \end{bmatrix} \quad \nabla k(x) = \begin{bmatrix} \frac{\partial k(x)}{\partial x_{11}} & \cdots & \frac{\partial k(x)}{\partial x_{1n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial k(x)}{\partial x_{m1}} & \cdots & \frac{\partial k(x)}{\partial x_{mn}} \end{bmatrix}.$$

Let $y : \mathbb{R} \to \mathbb{R}^n$ and $z : \mathbb{R} \to \mathbb{R}^{m\times n}$ be differentiable functions. Then, when defining the compositions $\alpha = h \circ y$ and $\beta = k \circ z$, the matrix calculus by vectorization [27] yields the consistent notation

$$\dot{\alpha}(t) = \nabla h(y(t))^\top \dot{y}(t) = \langle \nabla h(y(t)), \dot{y}(t)\rangle$$

$$\dot{\beta}(t) = \mathrm{trace}(\nabla k(z(t))^\top \dot{z}(t)) = \langle \nabla k(z(t)), \dot{z}(t)\rangle.$$

III. Hybrid Controller from Synergistic Potentials

For any continuously differentiable function $V : \mathsf{SO}(3) \to \mathbb{R}$, there will exist at least four critical points[1] where the shape of $\mathsf{SO}(3)$ interacts with ∇V to eliminate infinitesimal change of V with respect to $R \in \mathsf{SO}(3)$. We note that

$$\dot{V}(R(t)) = \left\langle \nabla V(R), R\,[\omega]_\times \right\rangle = 2\left\langle \omega, \psi(R^\top \nabla V(R))\right\rangle, \tag{3}$$

where we have used the property that $\mathrm{trace}(A\,[y]_\times) = 2y^\top \psi(A^\top)$ for any $A \in \mathbb{R}^{3\times3}$ and $y \in \mathbb{R}^3$. Thus, no matter the value of ω, when $\psi(R^\top \nabla V(R)) = 0$ ($R^\top \nabla V(R)$ is symmetric), there is no infinitesimal change in V. Thus, the set of critical points of some $V : \mathsf{SO}(3) \to \mathbb{R}$ is

$$\mathrm{Crit}\, V = \{R \in \mathsf{SO}(3) : \psi(R^\top \nabla V(R)) = 0\}.$$

Definition 1. A continuously differentiable function $V : \mathsf{SO}(3) \to \mathbb{R}_{\ge 0}$ is a *potential function on* $\mathsf{SO}(3)$ *(with respect to I)* if $V(R) > 0$ for all $R \in \mathsf{SO}(3) \setminus \{I\}$ and $V(I) = 0$. The class of potential functions on $\mathsf{SO}(3)$ is denoted $\mathscr{P}$.

Since $\mathsf{SO}(3)$ is a Lie group, one easily translate potential functions. That is, if V is a potential function on $\mathsf{SO}(3)$ with respect to I, then $U(R) = V(R_d^\top R)$ is a potential function on $\mathsf{SO}(3)$ with respect to $R_d \in \mathsf{SO}(3)$.

Definition 2. Let $Q \subset \mathbb{Z}$ be a finite index set with cardinality N and define $\mu : \mathscr{P}^N \to \mathbb{R}_{\ge 0}$ such that, for each family of potential functions $\mathscr{V} = \{V_q\}_{q\in Q} \in \mathscr{P}^N$,

$$\mu(\mathscr{V}) = \min_{\substack{q\in Q \\ R \in \mathrm{Crit}\, V_q \setminus \{I\}}} \max_{p \in Q} V_q(R) - V_p(R). \tag{4}$$

The family $\mathscr{V}$ is *synergistic* if there exists $\delta > 0$ such that

$$\mu(\mathscr{V}) > \delta, \tag{5}$$

where we say that $\mathscr{V}$ is synergistic *with gap exceeding* δ.

[1]The fact that there exist at least four critical points of any smooth function on $\mathsf{SO}(3)$ follows from the calculation of its Lusternik-Schnirelmann category, defined as the minimum number of contractible sets needed to cover $\mathsf{SO}(3)$. We refer the reader to [18] for the original work by Lusternik and Schnirelmann, and to [19] for a more modern and thorough treatment. The Lusternik-Schnirelmann category of $\mathsf{SO}(3)$ is listed in [28].

For the controller specification and closed-loop analysis, we work within the recent hybrid systems framework of [29], [30]. Let $\rightrightarrows$ denote a set-valued mapping. Given a state $\xi \in \mathbb{R}^n$, we write a hybrid system $\mathcal{H}$ as

$$\mathcal{H} = \begin{cases} \dot{\xi} \in F(\xi) & \xi \in C \\ \xi^+ \in G(\xi) & \xi \in D, \end{cases}$$

where the *flow map*, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, governs continuous flow of ξ, the *flow set*, $C \subset \mathbb{R}^n$, defines where continuous flow is permitted, the *jump map*, $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, governs discrete jumps of ξ, and the *jump set*, $D \subset \mathbb{R}^n$, defines where discrete jumps are permitted.

We design the hybrid feedback so that the closed-loop hybrid system satisfies the regularity properties [30, (A1-A3)]. In particular, both C and D will be closed sets, the map F will be a continuous function on C, and the map G will be nonempty, locally bounded, and outer semicontinuous (i.e., the set $\{(x, y) : y \in G(x)\}$ is closed) on D. The satisfaction of these properties ensures robustness of asymptotic stability to small perturbations including measurement noise [29].

Definition 3. A continuous function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be *class-*$\mathcal{K}$ if $\gamma(0) = 0$ and it is strictly increasing.

Definition 4. A continuous function $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is *strongly passive* if $\Psi(0) = 0$ and there exists a class-$\mathcal{K}$ function γ such that $\gamma(|\omega|) \leq \langle \omega, \Psi(\omega) \rangle$ for all $\omega \in \mathbb{R}^n$.

Given a family of potential functions $\mathscr{V} = \{V_q\}_{q \in Q}$, we define the function $\rho : \mathsf{SO}(3) \to \mathbb{R}_{\geq 0}$ as the minimum of V_q over Q and the argmin of V_q over Q, $g : \mathsf{SO}(3) \rightrightarrows Q$, as

$$\rho(R) = \min_{q \in Q} V_q(R) \tag{6a}$$

$$g(R) = \{q \in Q : V_q(R) = \rho(R)\}. \tag{6b}$$

Given $\delta > 0$, we define the sets $C, D \subset \mathsf{SO}(3) \times \mathbb{R}^3 \times Q$ as

$$C = \{(R, \omega, q) : V_q(R) - \rho(R) \leq \delta\} \tag{6c}$$

$$D = \{(R, \omega, q) : V_q(R) - \rho(R) \geq \delta\}, \tag{6d}$$

where inclusion in D or C, respectively, indicates whether or not there exists a potential function that is lower than the current potential function by at least δ. Note that $C \cup D = \mathsf{SO}(3) \times \mathbb{R}^3 \times Q$. Finally, given a strongly passive function $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ and $c > 0$, we define the feedback

$$\kappa(R, \omega, q) = -2c\psi(R^\top \nabla V_q(R)) - \Psi(\omega). \tag{6e}$$

Definition 5. Let $Q \subset \mathbb{Z}$ be a finite set. Given a family of potential functions $\mathscr{V} = \{V_q\}_{q \in Q}$, positive numbers δ, c, and a strongly passive function $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$, the tuple $\mathscr{H} = (\mathscr{V}, \delta, c, \Psi)$ is said to *generate* the dynamic system

$$\begin{aligned} \dot{q} &= 0 & (u, q) \in C \\ q^+ &\in g(\pi(u)) & (u, q) \in D \end{aligned} \tag{7a}$$

with measured input $u \in \mathsf{SO}(3) \times \mathbb{R}^3$ and torque output

$$y = \kappa(u, q). \tag{7b}$$

The dynamic system (7) is the *hybrid attitude controller generated by* $\mathscr{H}$.

The proposed hybrid controller provides a simple strategy for coordinating a family of potential-based control laws. The variable q becomes a memory state that selects which control law to use, while each control law creates an artificial potential energy V_q and removes energy via Ψ. The hybrid controller switches to the control law corresponding to the minimal potential function when a decrease in potential energy of at least δ can be obtained.

We now construct the closed-loop system by connecting the system (2) in feedback with the hybrid attitude controller generated by $\mathscr{H}$ defined in (7). Before completing this feedback interconnection, we inflict measurement noise and actuation error on the controller. In particular, we set

$$u = x_\epsilon := (e^{[\epsilon_1]_\times} R, \omega + \epsilon_2) \qquad \tau = \kappa(x_\epsilon, q) + \epsilon_3,$$

where $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) \in \alpha\mathbb{B} \subset \mathbb{R}^9$. Noting that $x \in \mathsf{SO}(3) \times \mathbb{R}^3$ experiences only continuous evolution, the closed-loop system takes the form

$$\underbrace{\begin{aligned} \dot{x} &= f(x, \kappa(x_\epsilon, q) + \epsilon_3) \\ \dot{q} &= 0 \end{aligned}}_{(x_\epsilon, q) \in C} \qquad \underbrace{\begin{aligned} x^+ &= x \\ q^+ &\in g(\pi(x_\epsilon)) \end{aligned}}_{(x_\epsilon, q) \in D.} \tag{8}$$

In the extended state space, $\mathsf{SO}(3) \times \mathbb{R}^3 \times Q$, the goal of the hybrid controller is to globally and asymptotically stabilize the compact set

$$\mathcal{A} = \{(I, 0)\} \times Q.$$

To make $\mathcal{A}$ globally attractive, the hybrid controller must switch the current control law as R approaches a "bad" critical point of the current potential function. That is, the set $\mathcal{X} \subset \mathsf{SO}(3) \times \mathbb{R}^3 \times Q$ defined as

$$\mathcal{X} = \{(R, \omega, q) : R \in \operatorname{Crit} V_q \setminus \{I\}\}$$

should be wholly contained in D. The following lemma gives an equivalent condition for this.

Lemma 6. *Let $Q \subset \mathbb{Z}$ be a finite set, let $\mathscr{V} = \{V_q\}_{q \in Q}$ be a family of potential functions on $\mathsf{SO}(3)$, and let $\delta > 0$. Then, $\mathscr{V}$ is synergistic with gap exceeding δ if and only if $\mathcal{X} \cap C = \emptyset$, or equivalently, $\mathcal{X} \subset D \setminus C$.*

Proof. Recalling the definition of C in (6c), it follows that $C \cap \mathcal{X}$ is empty if and only if, for every $(R, q) \in \mathsf{SO}(3) \times Q$ such that $R \in \operatorname{Crit} V_q \setminus \{I\}$, we have $V_q(R) - \rho(R) > \delta$. This condition is equivalent to

$$\begin{aligned} \min_{(R,q) \in \mathcal{X}} V_q(R) - \rho(R) &= \min_{\substack{q \in Q \\ R \in \operatorname{Crit} V_q \setminus \{I\}}} \max_{p \in Q} V_q(R) - V_p(R) \\ &= \mu(\mathscr{V}) > \delta. \end{aligned}$$

This proves the claim. ☐

Definition 7. A continuous function $\eta : \mathsf{SO}(3) \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ is a proper indicator for a compact set $M \subset \mathsf{SO}(3) \times \mathbb{R}^3$ if $\eta(x) > 0$ for all $x \in \mathsf{SO}(3) \times \mathbb{R}^3$, $\eta(M) = 0$, and for any $r \geq 0$, the set $\{x \in \mathsf{SO}(3) \times \mathbb{R}^3 : \eta(x) \leq r\}$ is compact.

Definition 8. A continuous function $\beta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be *class-*$\mathcal{KL}$ if for each fixed r, the map $s \mapsto \beta(s, r)$

is class-$\mathcal{K}$ and if for each fixed s, the map $r \mapsto \beta(s,r)$ is decreasing and $\lim_{r\to\infty}\beta(s,r) = 0$.

With Lemma 6 in hand, we now state the main result.

Theorem 9. *Let $\mathscr{H} = (\mathscr{V}, \delta, c, \Psi)$ generate a hybrid attitude controller. If $\epsilon = 0$, the set $\mathcal{A} = \{(I,0)\} \times Q$ is globally asymptotically stable for the closed-loop system* (8) *if and only if $\mathscr{V}$ is synergistic with gap exceeding δ.*

Furthermore, given any proper indicator $\eta : \mathsf{SO}(3) \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ for $\{(I,0)\}$, there exists a class-$\mathcal{KL}$ function β such that for every compact set $K \subset \mathbb{R}^3$ and every $\gamma > 0$, there exists $\alpha > 0$ such that for every measurable $\epsilon : [0,\infty) \to \alpha\mathbb{B}$, any solution (x,q) to (8) *with initial condition in $\mathsf{SO}(3) \times K \times Q$ satisfies*

$$\eta(x(t,j)) \leq \beta(\eta(x(0,0)), t+j) + \gamma \qquad \forall (t,j) \in \operatorname{dom} x.$$

Proof. Consider the Lyapunov function

$$V(R,\omega,q) = cV_q(R) + \tfrac{1}{2}\omega^\top J\omega.$$

Since each $V_q \in \mathscr{V}$ is a potential function on $\mathsf{SO}(3)$ it follows that $V(R,\omega,q) \geq 0$ for all $(x,q) \in \mathsf{SO}(3) \times \mathbb{R}^3 \times Q$ and $V(x,q) = 0$ if and only if $(x,q) \in \mathcal{A}$. Moreover, the sub-level sets of V are compact.

Let $\epsilon = 0$. We calculate change in V along flows as

$$\dot{V}(R,\omega,q) = c\dot{V}_q(R) + \omega^\top \left([J\omega]_\times \omega + \kappa(R,\omega,q)\right).$$

Since $[J\omega]_\times = -[J\omega]_\times^\top$, it follows that $\omega^\top [J\omega]_\times \omega = 0$. Applying (3), the control law κ from (6e), and the fact that Ψ is strongly passive, we have

$$\begin{aligned}\dot{V}(R,\omega,q) &= 2c\omega^\top \psi(R^\top \nabla V_q(R)) \\ &\quad + \omega^\top \left(-2c\psi(R^\top \nabla V_q(R)) - \Psi(\omega)\right) \\ &= -\omega^\top \Psi(\omega) \leq -\gamma(|\omega|) \leq 0.\end{aligned}$$

Thus, V is nonincreasing along flows of (8) on C. Moreover, it follows that for any $(R,\omega,q) \in D$ and any $p \in g(R)$, that

$$\begin{aligned}V(R,\omega,q) - V(R,\omega,p) &= c(V_q(R) - V_p(R)) \\ &= c(V_q(R) - \rho(R)) \\ &\geq c\delta > 0,\end{aligned}$$

so that V is strictly decreasing over jumps of (8). By [31, Theorem 7.6], it follows that $\mathcal{A}$ is stable. It remains to establish the global attractivity of $\mathcal{A}$.

Since $\dot{V}(R,\omega,q) \leq 0$ for all $(R,\omega,q) \in C$, $\dot{V}(R,\omega,q) = 0$ if and only if $\omega = 0$, and $V(R,\omega,p) - V(R,\omega,q) < 0$ for all $(R,\omega,q) \in D$ and $p \in g(R)$, it follows from [31, Theorem 4.7] that (R,ω,q) must converge to the largest invariant set contained in $C \cap \Omega$, where $\Omega = \{(R,\omega,q) : \omega = 0\}$. Since $\Psi(0) = 0$, if $\omega \equiv 0$, it follows that $\dot{\omega} = 0$. From (8), $\omega = \dot{\omega} = 0$ implies that $0 = 2c\psi(R^\top \nabla V_q(R))$, or equivalently, $R \in \operatorname{Crit} V_q$. Thus, solutions must converge to the set $\Omega \cap C \cap (\mathcal{A} \cup (\mathcal{X} \cap \Omega))$. Clearly $\Omega \subset \mathcal{A} \subset C$, so solutions must converge to $\mathcal{A} \cup (C \cap \mathcal{X} \cap \Omega)$. Hence, $\mathcal{A}$ is globally attractive if and only if $C \cap \mathcal{X} \cap \Omega = \emptyset$. By definition of C, $\mathcal{X}$, and Ω this is the case if and only if $C \cap \mathcal{X} = \emptyset$. By Lemma 6, this is equivalent to the condition that $\mathscr{V}$ is synergistic with gap exceeding δ.

Finally, we note that the existence of the class-$\mathcal{KL}$ function β follows from [29, Theorem 6.6]. □

A commonly-used potential function is the *modified trace function*, which, given $A = A^\top > 0$, is defined as

$$P_A(X) = \operatorname{trace}(A(I-X)) = \tfrac{1}{2}\langle I - X, A(I-X)\rangle. \quad (9)$$

Taking inspiration from [15], P_A was first introduced for attitude control in [11] and later used by [12] and others. We recall our results in [17], which show that it is impossible to construct a synergistic family of modified trace functions. In what follows, let $\mathscr{E}(A) = \{v \in \mathbb{S}^2 : \exists \lambda\ Av = \lambda v\}$ denote the set of unit eigenvectors of a symmetric matrix $A \in \mathbb{R}^{3\times 3}$.

Lemma 10. *Let $A \in \mathbb{R}^{3\times3}$ be symmetric and positive definite. Then, the function P_A satisfies*

$$\operatorname{Crit} P_A = \{I\} \cup \mathcal{R}(\pi, \mathscr{E}(A)). \quad (10)$$

When A has distinct eigenvalues, P_A obtains the minimal number of critical points possible. However, any finite family of modified trace functions is not synergistic.

Fair questions come to mind. Does there exist a synergistic family of potentials? If so, what is the minimum number N such that there exists a synergistic family of N potential functions? In the following section, we give the answers: yes and two, respectively, by the construction of [17].

IV. Synergistic Potentials by Angular Warping

Let $\mathscr{C}^1(\mathsf{SO}(3))$ denote the set of continuously differentiable real-valued functions on $\mathsf{SO}(3)$ and let

$$\mathscr{C}^1_I(\mathsf{SO}(3)) = \{P \in \mathscr{C}^1(\mathsf{SO}(3)) : P(I) = 0\}. \quad (11)$$

Then, define the function $\mathcal{T} : \mathsf{SO}(3) \to \mathsf{SO}(3)$ as

$$\mathcal{T}(R,k,P,u) = e^{kP(R)[u]_\times} R, \quad (12)$$

where $k \in \mathbb{R}$, $P \in \mathscr{C}^1_I(\mathsf{SO}(3))$, and $u \in \mathbb{S}^2$ are fixed parameters. At each $R \in \mathsf{SO}(3)$, $\mathcal{T}$ applies a rotation in the amount of $kP(R)$ to R about the axis u.

Definition 11. A map $h : X \to Y$ is a *diffeomorphism* if it is bijective, differentiable, and has a differentiable inverse.

Theorem 12. *Let $k \in \mathbb{R}$, $P \in \mathscr{C}^1_I(\mathsf{SO}(3))$, $u \in \mathbb{S}^2$, $V \in \mathscr{P}$, $U = V \circ \mathcal{T}$, and define $\Theta : \mathsf{SO}(3) \to \mathbb{R}^{3\times3}$ as*

$$\Theta(R) = I + 2kR^\top u \psi(\nabla P(R) R^\top)^\top R. \quad (13)$$

If k satisfies

$$\sqrt{2}|k| \max \|\nabla P(\mathsf{SO}(3))\|_F < 1, \quad (14)$$

then $\mathcal{T} : \mathsf{SO}(3) \to \mathsf{SO}(3)$ is a diffeomorphism and satisfies

$$\mathcal{T}(I) = I \quad (15)$$

$$\psi(R^\top \nabla U(R)) = \Theta(R)^\top \psi\left(\mathcal{T}(R)^\top \nabla V(\mathcal{T}(R))\right) \quad (16)$$

$$\dot{\mathcal{T}}(R) = \mathcal{T}(R)\,[\Theta(R)\omega]_\times. \quad (17)$$

$$\operatorname{Crit} U = \mathcal{T}^{-1}(\operatorname{Crit} V) \quad (18)$$

$$U \in \mathscr{P} \quad (19)$$

for all $R \in \mathsf{SO}(3)$.

Example 13. Let $z = [11 \quad 12 \quad 13]^\top$, $A = 3\,\mathrm{diag}(z)/\sum_{i=1}^{3} z_i$, $u = z/|z|$. Then, define $V_1^k(R) = P_A(\mathcal{T}(R,k,u,P_A))$, $V_2^k(R) = P_A(\mathcal{T}(R,-k,u,P_A))$. We note that $\mathcal{T}$ is a diffeomorphism if $|k| < 1/(\sqrt{2}\|A\|_F) \approx 0.4073$. Then, define the family $\mathscr{V}^k = \{V_1^k, V_2^k\}$. It is not difficult to compute $\mu(\mathscr{V}^k)$ for every k satisfying this bound, for which we provide a plot in [17]. As a representative case, when $k = 0.2$, we have $\mu(\mathscr{V}^k) \approx 0.5972$.

V. Simulation Study

In this section, we contrast the proposed hybrid controller with a similar smooth controller by simulation. Here, the hybrid controller is generated by the potential functions in Example 13. In the following simulations (as in Example 13), we let $z = [11 \quad 12 \quad 13]^\top$, $A = 3\,\mathrm{diag}(z)/\sum_{i=1}^{3} z_i$, $J = \mathrm{diag}(200, 300, 150)$, $\Psi(\omega) = K\omega$, and $K = \mathrm{diag}(40, 60, 40)$. The smooth control law is given as

$$\tau_s(R,\omega) = -2\psi(AR) - \Psi(\omega). \tag{20}$$

As in Example 13, we let $u = z/|z|$, $k_1 = -k_2 = 0.2$, and we select $\delta = 0.5 < \mu(\mathscr{V}^k)$. The hybrid control law is given in (6e), where, for $q \in \{1,2\}$,

$$\tau_q(R,\omega) = -2\Theta_q(R)^\top \psi(AT_q(R)) - \Psi(\omega) \tag{21}$$

is used as the torque feedback for mode q and

$$T_q(R) = \exp(k_q P_A(R)\,[u]_\times)R$$
$$\Theta_q(R) = I + 2k_q R^\top u \psi(RA)^\top R.$$

The smooth control law (20) is derived from $P_A(R) = \mathrm{trace}(A(I-R))$ and for each $q \in \{1,2\}$, the torque feedback from the hybrid controller (21) is derived from $V_q = P_A \circ T_q$. Note that the smooth control law τ_s (20) is the midpoint (in a homotopy sense) of the two control laws (21) used by the hybrid controller. As a final notational convenience for this section, we define the function $\theta : \mathsf{SO}(3) \to \mathbb{R}$ as

$$\theta(R) = \cos^{-1}\left(\tfrac{1}{2}(\mathrm{trace}(R) - 1)\right).$$

The simulation in Figures 1 and 2 compare the hybrid and smooth controllers beginning from an initial condition close to a critical point of P_A. In particular, the initial value of R was chosen as $\mathcal{R}(\pi, v)$, where $v = [\sqrt{(1-h^2)} \quad h \quad 0]^\top$ and $h = 0.1$. The initial condition of ω was chosen as zero. For the hybrid controller, the initial condition of q was set to one. In this simulation, the hybrid controller begins from an initial condition that is not a critical point for any of its control laws and in fact, forces the rigid body to converge to the identity rotation without switching q.

Fig. 3 and 4 depict a similar simulation as Fig. 1 and 2; however, the initial condition in Figures 3 and 4 is chosen close to a critical point of $P_A \circ T_1$. In particular, the initial value of R was chosen as $R(0) = T_1^{-1}(\mathcal{R}(\pi, \mathbf{e}_1))$. The initial condition of ω was chosen as zero. For the hybrid controller, the initial condition of q was set to one. In this simulation, the hybrid controller begins from an initial condition that *is* a critical point for its *initial* choice of potential function. Because δ was selected appropriately (i.e., $0 < \delta < \mu(\mathscr{V}^k)$), the hybrid controller *immediately* switches q from 1 to 2 (not pictured). This results in similar performance to the smooth control law. For the remainder of this simulation, q remained constant at 2.

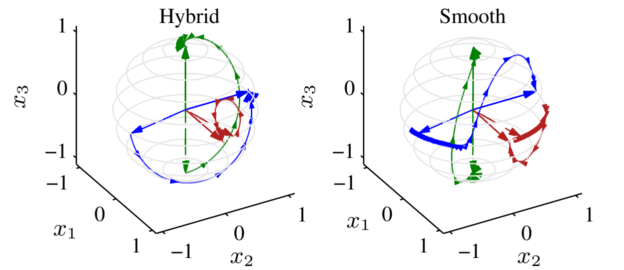

Fig. 1. Comparison of rigid body trajectories for smooth and hybrid controllers starting near a critical point of P_A with zero initial angular velocity. The hybrid controller (left) immediately exerts a torque to steer the rigid body. The smooth controller (right) exerts almost zero torque near the critical points. This significantly delays convergence. The red trajectory is $R(t)\mathbf{e}_1$, the blue is $R(t)\mathbf{e}_2$, and the green is $R(t)\mathbf{e}_3$.

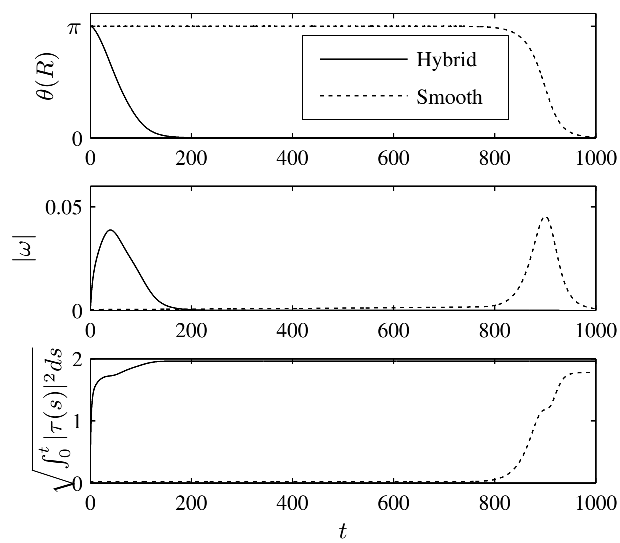

Fig. 2. Comparison of smooth and hybrid controller performance starting near a critical point of P_A with zero initial angular velocity. The hybrid controller (solid) immediately exerts a torque to steer the rigid body. The smooth controller (dashed) has close to zero torque near the critical points. This significantly delays convergence.

VI. Conclusion

The task of global asymptotic stabilization of rigid body attitude is impossible with smooth feedback; however, this obstacle can be overcome if one is willing to allow for a hybrid feedback. Such a hybrid controller achieving global asymptotic stability of a desired attitude is readily available when one can find a synergistic family of at least two potential functions, a property that is not attributable to any family of modified trace functions on $\mathsf{SO}(3)$.

To generate new potential functions capable of forming a synergistic family, we proposed a parametrized diffeomorphism that stretches and compresses $\mathsf{SO}(3)$ while leaving the identity element a fixed point. When composed with an existing potential function, this diffeomorphism is capable

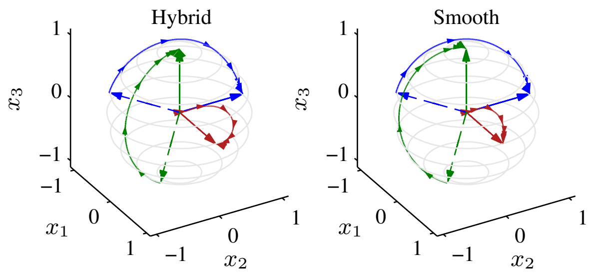

Fig. 3. Comparison of rigid body trajectories for smooth and hybrid controllers starting near a critical point of $P_A \circ T_1$ with zero initial angular velocity. The hybrid controller (left) immediately switches to a new control law and exhibits similar performance to the smooth controller (right). The red trajectory is $R(t)\mathbf{e}_1$, the blue is $R(t)\mathbf{e}_2$, and the green is $R(t)\mathbf{e}_3$.

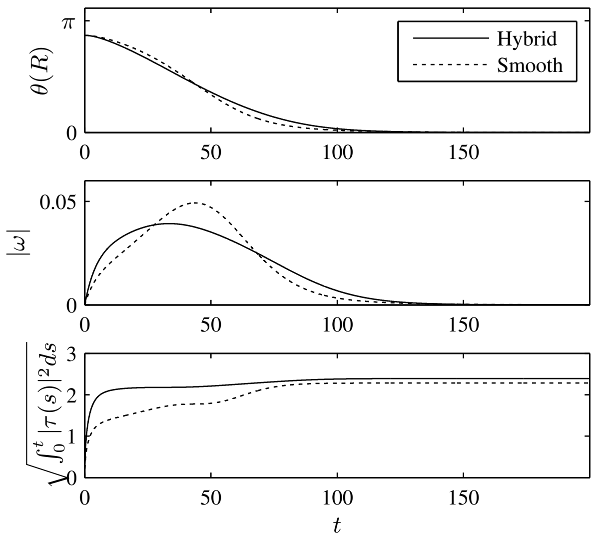

Fig. 4. Comparison of rigid body trajectories for smooth and hybrid controllers starting near a critical point of $P_A \circ T_1$ with zero initial angular velocity. The hybrid controller (solid) immediately switches to a new control law and exhibits similar performance to the smooth controller (dashed).

of relocating critical points. Applying this diffeomorphism with different parameters to the same potential function allows one to construct a synergistic family, paving the way for global asymptotic stabilization of rigid-body attitude by hybrid feedback.

References

[1] R. E. Mortensen, "A globally stable linear attitude regulator," *International Journal of Control*, vol. 8, no. 3, pp. 297–302, 1968.

[2] G. Meyer, "Design and global analysis of spacecraft attitude control systems," NASA, Tech. Rep. R-361, Mar. 1971.

[3] B. Wie and P. M. Barba, "Quaternion feedback for spacecraft large angle maneuvers," *Journal of Guidance, Control, and Dynamics*, vol. 8, no. 3, pp. 360–365, 1985.

[4] J. T.-Y. Wen and K. Kreutz-Delgado, "The attitude control problem," *IEEE Transactions on Automatic Control*, vol. 36, no. 10, pp. 1148–1162, Oct. 1991.

[5] C. G. Mayhew, R. G. Sanfelice, and A. R. Teel, "Quaternion-based hybrid control for robust global attitude tracking," *IEEE Transactions on Automatic Control*, 2011.

[6] O. E. Fjellstad and T. I. Fossen, "Singularity-free tracking of unmanned underwater vehicles in 6 DOF," in *Proceedings of the 33rd IEEE Conference on Decision and Control*, vol. 2, 1994, pp. 1128–1133.

[7] R. Cunha, C. Silvestre, and J. Hespanha, "Output-feedback control for stabilization on SE(3)," *Systems & Control Letters*, vol. 57, no. 12, pp. 1013–1022, Dec. 2008.

[8] S. P. Bhat and D. S. Bernstein, "A topological obstruction to continuous global stabilization of rotational motion and the unwinding phenomenon," *Systems & Control Letters*, vol. 39, no. 1, pp. 63–70, Jan. 2000.

[9] V. Guillemin and A. Pollack, *Differential Topology*. Prentice-Hall, Inc., 1974.

[10] E. Sontag, *Mathematical Control Theory*. Springer, 1998.

[11] D. E. Koditschek, "The application of total energy as a Lyapunov function for mechanical control systems," in *Dynamics and Control of Multibody Systems*, ser. Contemporary Mathematics, J. E. Marsden, P. S. Krishnaprasad, and J. C. Simo, Eds. American Mathematical Society, 1989, vol. 97, pp. 131–157.

[12] F. Bullo and R. M. Murray, "Tracking for fully actuated mechanical systems: a geometric framework," *Automatica*, vol. 35, no. 1, pp. 17–34, Jan. 1999.

[13] N. A. Chaturvedi, A. M. Bloch, and N. H. McClamroch, "Global stabilization of a fully actuated mechanical system on a Riemannian manifold: controller structure," in *Proceedings of the American Control Conference*, 2006, pp. 3612–3617.

[14] A. Sanyal, A. Fosbury, N. A. Chaturvedi, and D. S. Bernstein, "Inertia-free spacecraft attitude tracking with disturbance rejection and almost global stabilization," *Journal of Guidance, Control, and Dynamics*, vol. 32, no. 4, pp. 1167–1178, 2009.

[15] D. R. J. Chillingworth, J. E. Marsden, and Y. H. Wan, "Symmetry and bifurcation in three-dimensional elasticity, part I," *Archive for Rational Mechanics and Analysis*, vol. 80, no. 4, pp. 295–331, Dec. 1982.

[16] I. A. Dynnikov and A. P. Veselov, "Integrable gradient flows and Morse theory," *Algebra i Analiz*, vol. 8, no. 3, pp. 78–103, 1996.

[17] C. G. Mayhew and A. R. Teel, "Synergistic potential functions for hybrid control of rigid-body attitude," in *Proceedings of the American Control Conference*, 2011.

[18] L. Lusternik and L. Schnirelmann, "Méthodes topologiques dan les problémes variationnels," *Paris: Hermann*, 1934.

[19] O. Cornea, G. Lupton, J. Opera, and D. Tanré, *Lusternik-Schnirelman Category*, ser. Mathematical Surveys and Monographs. American Mathematical Society, 2003, vol. 103.

[20] C. G. Mayhew and A. R. Teel, "Hybrid control of planar rotations," in *Proceedings of the American Control Conference*, 2010, pp. 154–159.

[21] ——, "Hybrid control of spherical orientation," in *Proceedings of the 49th IEEE Conference on Decision and Control*, 2010, pp. 4198–4203.

[22] ——, "Global asymptotic stabilization of the inverted equilibrium manifold of the 3D pendulum by hybrid feedback," in *Proceedings of the 49th IEEE Conference on Decision and Control*, 2010, pp. 679–684.

[23] M. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 475–482, Apr. 1998.

[24] J. Malmborg, B. Berhardsson, and K. J. Åström, "A stabilizing switching scheme for multi-controller systems," in *Proceedings of the Triennial IFAC World Congress*, vol. F, 1996, pp. 229–234.

[25] J. Malmborg and J. Eker, "Hybrid control of a double tank system," in *Proceedings of the IEEE International Conference on Control Applications*, 1997, pp. 133–138.

[26] A. Leonessa, W. M. Haddad, and V. S. Chellaboina, "Nonlinear system stabilization via hierarchical switching control," *IEEE Transactions on Automatic Control*, vol. 46, no. 1, pp. 17–28, 2001.

[27] J. R. Magnus and H. Neudecker, "Matrix differential calculus with applications to simple, Hadamard, and Kronecker products," *Journal of Mathematical Psychology*, vol. 29, no. 4, pp. 474–492, Dec. 1985.

[28] N. Iwase, M. Mimura, and T. Nishimoto, "Lusternik-Schnirelmann category of non-simply connected compact simple Lie groups," *Topology and its Applications*, vol. 150, no. 1-3, pp. 111–123, May 2005.

[29] R. Goebel and A. Teel, "Solutions to hybrid inclusions via set and graphical convergence with stability theory applications," *Automatica*, vol. 42, no. 4, pp. 573–587, Apr. 2006.

[30] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Hybrid dynamical systems," *IEEE Control Systems Magazine*, vol. 29, no. 2, pp. 28–93, Apr. 2009.

[31] R. G. Sanfelice, R. Goebel, and A. R. Teel, "Invariance principles for hybrid systems with connections to detectability and asymptotic stability," *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2282–2297, Dec. 2007.