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On Tracking for the PVTOL Model with Bounded Feedbacks

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Abstract— **We study a class of feedback tracking problems for the planar vertical takeoff and landing (PVTOL) aircraft dynamics, which is a benchmark model in aerospace engineering. After a survey of the literature on the model, we construct new feedback stabilizers for the PVTOL tracking dynamics. The novelty of our contribution is in the boundedness of our feedback controllers and their applicability to cases where the velocity measurements may not be available, coupled with the uniform global asymptotic stability and uniform local exponential stability of the closed loop tracking dynamics, the generality of our class of trackable reference trajectories, and the input-to-state stable performance of the closed loop tracking dynamics with respect to actuator errors. Our proofs are based on a new bounded backstepping result. We illustrate our work in a tracking problem along a circle.**

I. Introduction

Since its introduction in [13], the PVTOL aircraft model has become a benchmark model in aerospace engineering, and it is of continuing ongoing research interest [2], [5], [10]. The model is

$$\left\{\begin{array}{rcl} \ddot{x} & = & -\bar{u}_1\sin(\theta)+\varepsilon u_2\cos(\theta) \\ \ddot{y} & = & \bar{u}_1\cos(\theta)+\varepsilon u_2\sin(\theta)-g \\ \ddot{\theta} & = & u_2, \end{array}\right. \tag{1}$$

where (x, y) gives the lateral and vertical coordinates of the center of mass of the aircraft, θ is the roll angle relative to the horizon, the control $\bar{u}_1$ is the thrust directed out of the bottom, g is the gravitational constant, the control u_2 is the rolling moment, and the constant ε gives the coupling between the roll moment and the lateral force [2]. It is a simplified model with the minimal number of states and inputs that has the main features needed to design controllers for real aircraft.

The coordinates $z_1 = x - \varepsilon\sin(\theta)$, $z_2 = \dot{x} - \varepsilon\dot{\theta}\cos(\theta)$, $w_1 = y + \varepsilon(\cos(\theta) - 1)$, $w_2 = \dot{y} - \varepsilon\dot{\theta}\sin(\theta)$, $\xi_1 = \theta$, and $\xi_2 = \dot{\theta}$ and new input $u_1 = \bar{u}_1 - \varepsilon\xi_2^2$ transform (1) into [22]

$$\left\{\begin{array}{rcl} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & -u_1\sin(\xi_1) \\ \dot{w}_1 & = & w_2 \\ \dot{w}_2 & = & u_1\cos(\xi_1)-g \\ \dot{\xi}_1 & = & \xi_2 \\ \dot{\xi}_2 & = & u_2. \end{array}\right. \tag{2}$$

As noted in [5], the main literature on (2) is divided into set point stabilization results (e.g., [22], [27]), and results on tracking or path following (e.g., [5], [6], [7], [15], [17], [18]). The challenge in designing PVTOL stabilizers is that u_1 must be nonnegative valued, and that the system is underactuated. Much of the PVTOL literature uses output feedbacks that only depend on (z_1, w_1, ξ_1). However, one can design globally exponentially stable observers for the velocities; see [7] and Section VII below, and [2], [28] for recent work on state feedback tracking controllers for (2).

Given a reference trajectory for (2), it is natural to ask whether we can design feedback controllers u_1 and u_2 that force all trajectories of (2) to track the reference trajectory, for all initial configurations. This is the problem of rendering the corresponding tracking error dynamics for (2) uniformly globally asymptotically stable to the origin. Several significant papers showed how such controllers can indeed be constructed [2], [7]. The work [2] gave globally stabilizing tracking controllers for a specific class of reference trajectories when u_1 is bounded, and semiglobal stability when both u_1 and u_2 are bounded, while the controllers in [7] are not bounded. However, one would hope to establish uniform global asymptotic stability of the tracking dynamics by *globally bounded* controllers u_1 and u_2, for a more general class of reference trajectories. Also, it is important for the controllers to perform well under uncertainty, so one should design the controllers to give input-to-state stability (ISS) with respect to actuator errors, which are additive uncertainties on the controllers that naturally arise in applications. The present work shows how these additional control objectives can indeed be realized. This is a significant new development.

II. Relationship to the Literature

The fundamental importance of the PVTOL model has led to a vast PVTOL literature involving a variety of techniques. In their pioneering work [13], Hauser *et al.* used approximate input-output linearization to get bounded tracking and asymptotic stability for (2). Later work [26] by Teel developed small gain theory for systems in feedforward form that gave stabilization results for the PVTOL model as a special case, including robustness to uncertainty in the coupling parameter ε. In [18], Martin *et al.* extended [13] by giving output tracking results for a class of slightly or strongly non-minimum phase systems that included the PVTOL. The main idea in [18] was to use the output at the Huygens center of oscillation, which is a fixed point with respect to the aircraft body, and then the controller was defined on a suitable subset of the state space. Also, [23, Section 6.1] designed PVTOL aircraft state feedbacks under the assumption that the coupling parameter is zero and then selected the controller parameters to mitigate the

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effects of nonzero values of ε. Then [15] gave optimal control methods that led to nonlinear state feedback controllers that give hovering control that is robust to uncertainty in the coupling parameter ε. See [4, Section VI.C] for stabilization of equilibrium points under linear dynamic stabilizers.

Subsequent work [22] by Olfati-Saber from 2002 used a change of coordinates from [21] to design a state controller that stabilizes a zero velocity configuration and allows larger values of the parameter ε. Also in 2002, Marconi *et al.* [17] used an internal-based model approach and nested saturations to design an autopilot for the autonomous landing of a PVTOL aircraft on a ship whose deck oscillates under high seas. See also [3] for output tracking along a circle. Later work [8] by Francisco *et al.* used forwarding results [20] for feedforward systems to design distributed delay nested saturation feedbacks that give global asymptotic stability.

Another portion of the PVTOL aircraft literature covers path following. The difference between tracking and path following is that the former leads to controllers that have an a priori parametrization of the curve to be followed, while path following controllers do not involve such a parametrization. See [5] for path following of Jordan curves using continuous feedback (and possibly nonuniqueness of solutions) based on finite time stabilization for initial states near the desired configuration. A possible advantage of path following is that it can sometimes mitigate the adverse effects of moving along a path too quickly [5]. However, the PVTOL tracking error dynamics are amenable to *global* Lyapunov function methods. Lyapunov methods have the advantage that they can quantify the effects of uncertainty, e.g., using ISS [16]. Therefore, tracking and path following are both important.

One natural approach to the PVTOL dynamics involves backstepping [7]. See, e.g., [27], whose feedback leads to a cascade structure that minimizes the norm of the interconnection term between subsystems. When designing PVTOL controllers, it is important to take the maximum amplitude of the feedbacks into account. On the other hand, standard backstepping techniques do not lead to *bounded* feedback stabilizers. There are several generalizations of backstepping that give bounded feedbacks [9], [16], [19]. See, e.g., [16, Chapter 7], where bounded backstepping was used to track sinusoidal PVTOL trajectories, but we believe that none of the existing bounded backstepping results apply to the more general problems we consider in this work. Much of the existing work on global tracking for (2) is based on nonstrict Lyapunov function approaches such as Barbalat's Lemma or LaSalle Invariance, and so does not lend itself to ISS.

Our controllers for (2) are necessarily more complex than those of [2], [7]. However, to the best of our knowledge, the results to follow are original and significant because of (a) the global boundedness of our controllers, (b) their applicability to cases where the velocity measurements may not be available, (c) the uniform global asymptotic stability and uniform local exponential stability of our closed loop tracking dynamics, (d) our allowing a rather general class of reference trajectories, and (e) our use of ISS to quantify the performance under actuator errors of large amplitude.

III. Preliminaries

A. Definitions and Notation

A continuous function $\gamma : [0,\infty) \to [0,\infty)$ belongs to class $\mathcal{K}$ (written $\gamma \in \mathcal{K}$) provided it is strictly increasing and $\gamma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if, in addition, $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ is of class $\mathcal{KL}$ (written $\beta \in \mathcal{KL}$) provided for each $s \geq 0$, the function $\beta(\cdot, s)$ belongs to class $\mathcal{K}$; and for each $r \geq 0$, the function $\beta(r, \cdot)$ is non-increasing and $\beta(r, s) \to 0$ as $s \to \infty$. For each subset $\mathcal{D}$ of a Euclidean space, let $\mathcal{M}_\mathcal{D}$ denote the set of all measurable essentially bounded functions $\delta : [0,\infty) \to \mathcal{D}$. For each $\delta \in \mathcal{M}_\mathcal{D}$ and each interval $\mathcal{I} \subseteq (0,\infty)$, let $|\delta|_\mathcal{I}$ denote the essential supremum of δ on $\mathcal{I}$, and $|\delta|_\infty$ its essential supremum on $[0,\infty)$. For any constant $\Delta > 0$, let $\mathcal{B}_\Delta$ denote the closed ball of radius Δ centered at the origin in the appropriate Euclidean space.

Consider a forward complete system $\dot{x} = \mathcal{F}(t, x, \delta)$ evolving on some open subset $\mathcal{X}$ of Euclidean space with measurable essentially bounded disturbances δ valued in some subset $\mathcal{D}$ of a Euclidean space (of possibly different dimension). We assume that $0 \in \mathcal{D}$, that $\mathcal{F} : [0,\infty) \times \mathcal{X} \times \mathcal{D} \to \mathcal{X}$ is such that the standard existence and uniqueness of solutions properties hold for all initial states in $\mathcal{X}$ and all disturbances $\delta \in \mathcal{M}_\mathcal{D}$ [16], and that $\mathcal{F}(t, 0, 0) = 0$ for all $t \geq 0$. The system is *input-to-state stable (ISS)* with respect to disturbances valued in $\mathcal{D}$ [14], [24], [25] provided there are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all solutions $t \mapsto x(t, t_0, x_0, \delta)$ of the system for all initial conditions $x(t_0) = x_0$ and all $\delta \in \mathcal{M}_\mathcal{D}$, we have $|x(t, t_0, x_0, \delta)| \leq \beta(|x_0|, t - t_0) + \gamma(|\delta|_{[t_0,t]})$ for all $t \geq t_0$. The special case of ISS where $\mathcal{F}$ only depends on (t, x) and the γ term in the sum is not present is called *uniform global asymptotic stability (UGAS)*. In this case, we denote the trajectory for each initial condition $x(t_0) = x_0$ by $t \mapsto x(t, t_0, x_0)$, and we say that the system is *uniformly locally exponentially stable (ULES)* provided there are positive constants Δ, c_1, and c_2 such that for all initial conditions $x(t_0) = x_0 \in \mathcal{B}_\Delta$, we have $|x(t, t_0, x_0)| \leq c_1 |x_0| e^{-c_2(t-t_0)}$ for all $t \geq t_0$.

B. Useful Classes of Functions

We use the functions $\sigma_\ell, \varphi_\ell : \mathbb{R} \to \mathbb{R}$ defined by

$$\sigma_\ell(x) = \tfrac{2\ell}{\pi} \arctan\left(\tfrac{\pi x}{2\ell}\right)$$

and

$$\varphi_\ell(x) = 1 - \frac{1}{B_\ell} \int_{4\ell}^{\max\left\{4\ell, \min\{|x|, 6\ell\}\right\}} (q - 4\ell)^4 (q - 6\ell)^4 \mathrm{d}q,$$

where

$$B_\ell = \textstyle\int_{4\ell}^{6\ell} (q - 4\ell)^4 (q - 6\ell)^4 \mathrm{d}q$$

for each constant $\ell > 0$. The constant B_ℓ is chosen such that φ_ℓ is a compactly supported smoothed indicator function for the interval $[-6\ell, 6\ell]$. The key properties of σ_ℓ and φ_ℓ are:

Lemma 1: For each constant $\ell > 0$, we have (a) $\sigma'_\ell(x) \in [0, 1]$ for all $x \in \mathbb{R}$, (b) $\sigma_\ell(x) \geq 0.75x$ for all $x \in [0, \ell/4]$, (c) $|\sigma_\ell(x)| \leq \ell$ for all $x \in \mathbb{R}$, (d) $\varphi_\ell : \mathbb{R} \to [0, 1]$ is C^3 and even, (e) $\varphi_\ell(x) = 1$ on $[-4\ell, 4\ell]$, (f) $\varphi_\ell(x) = 0$ when $|x| \geq 6\ell$, and (g) $\ell \sup_{x \in \mathbb{R}} |\varphi'_\ell(x)| = 315/256$. $\square$

Property (g) holds because $B_\ell = 256\ell^9/315$, so

$$\begin{aligned} \ell \max_{x\in\mathbb{R}} |\varphi_\ell'(x)| &= \frac{\ell}{B_\ell} \max_{q\in[4\ell,6\ell]} (q-4\ell)^4(q-6\ell)^4 \\ &= \frac{315}{256}\,. \end{aligned}$$

The rest of Lemma 1 follows from simple calculations, by equating the one-sided derivatives of φ_ℓ at the points $\pm 4\ell$ and $\pm 6\ell$. Lemma 1 implies that $\varphi_\ell(x)x$ and $\varphi_\ell^{(i)}(x)x$ are bounded for each derivative $i = 1, 2, 3$. For each constant $\ell > 0$, we can use property (g) to define the function $U_\ell : \mathbb{R}^2 \to \mathbb{R}$ by

$$U_\ell(Z) = \frac{-\sigma_\ell\left(2Z_2+\sigma_\ell(\ell Z_1)\varphi_\ell(Z_2)\right)-\ell\sigma_\ell'(\ell Z_1)\varphi_\ell(Z_2)Z_2}{2+\sigma_\ell(\ell Z_1)\varphi_\ell'(Z_2)}\,. \tag{3}$$

The following properties are immediate from the compact support of φ_ℓ from Lemma 1:

Lemma 2: For each constant $\ell > 0$, (I) the functions

$$\begin{aligned} &\frac{\partial U_\ell}{\partial Z}(Z),\ \ Z_2\frac{\partial U_\ell}{\partial Z_1}(Z),\ \ Z_2^2\frac{\partial^2 U_\ell}{\partial Z_1^2}(Z),\ \ \frac{\partial^2 U_\ell}{\partial Z_2^2}(Z), \\ &\text{and}\ \ Z_2\frac{\partial^2 U_\ell}{\partial Z_1\partial Z_2}(Z) \end{aligned} \tag{4}$$

are all bounded and (II) $\sup_{Z\in\mathbb{R}^2} |U_\ell(Z)| \le 2(6\ell^2+\ell)$. □

C. Key Lemma

The following lemma is key to our control design for (2):

Lemma 3: Let $\Theta : [0,\infty)\times\mathbb{R}^2 \to \mathbb{R}$ be any C^1 function that admits a constant $\ell > 0$ such that

$$\sup_{t\ge 0} |\Theta(t,X)| \le \frac{\ell}{16}\min\{1, |X_1|\} \tag{5}$$

for all $X = (X_1, X_2) \in \mathbb{R}^2$. Let $\dot\varepsilon = E(t,\varepsilon)$ be any system on some Euclidean state space $\mathbb{R}^p$ that is UGAS and ULES to 0. Assume that $\partial E/\partial\varepsilon$ is bounded on $[0,\infty)\times\mathbb{R}^p$. Let $L : [0,\infty)\times\mathbb{R}^2\times\mathbb{R}^p \to \mathbb{R}$ be any C^1 function that admits a constant $\bar L$ such that $|L(t,X,\varepsilon)| \le \bar L|\varepsilon|$ for all $t \ge 0$, $X \in \mathbb{R}^2$, and $\varepsilon \in \mathbb{R}^p$. Let $\bar\eta \ge 0$ be any constant, and set $\mathcal{Z}_1(X) = 2X_2 + \sigma_\ell(\ell X_1)\varphi_\ell(X_2)$. Then

$$\begin{cases} \dot X_1 &= X_2 + \Theta(t,X) \\ \dot X_2 &= \beta_{\ell,\bar\eta}(t,X) + L(t,X,\varepsilon) + \eta \\ \dot\varepsilon &= E(t,\varepsilon) \end{cases} \tag{6}$$

in closed loop with the bounded C^1 feedback

$$\begin{aligned} &\beta_{\ell,\bar\eta}(t,X) = \\ &\frac{-[1+172\bar\eta/\ell]\sigma_\ell\left(\mathcal{Z}_1(X)\right)-\ell\sigma_\ell'(\ell X_1)\varphi_\ell(X_2)[X_2+\Theta(t,X)]}{2+\sigma_\ell(\ell X_1)\varphi_\ell'(X_2)} \end{aligned} \tag{7}$$

is UGAS and ULES to 0 when $\eta \equiv 0$, and ISS with respect to disturbances $\eta : [0,\infty) \to \mathcal{B}_{\bar\eta}$. □

See [11] for the proof of Lemma 3. The proof constructs a function $\alpha \in \mathcal{K}_\infty \cap C^1$ and a constant $c > 0$ such that

$$\begin{aligned} |Y(t)| \ \le\ & \alpha(|Y(t_0)|)e^{-c(t-t_0)/16} \\ & + \frac{12}{c}\sqrt{\ell^2+1}\left(1+\sqrt{2/\{\ell c\}}\right)|\eta|_\infty \end{aligned} \tag{8}$$

along all trajectories $Y = (X,\varepsilon)$ of (6) for all $t_0 \ge 0$, $t \ge t_0$, and measurable functions $\eta : [0,\infty) \to \mathcal{B}_{\bar\eta}$. Lemma 3 implies that for any constant $\bar\eta > 0$, we can choose the feedback such that (6) is ISS with respect to disturbances that are bounded by $\bar\eta$. Moreover, $\bar\eta$ can be taken as large as desired, so we get ISS with respect to disturbances of arbitrarily large amplitude by choosing $\bar\eta$. Also, $(\partial/\partial x)\beta_{\ell,\bar\eta}$ is bounded if $(\partial/\partial x)\Theta$ is bounded, and $|\beta_{\ell,\bar\eta}(t,X)| \le 2\ell(1+172\bar\eta/\ell)(1+7\ell)$ for all $(t,X) \in [0,\infty)\times\mathbb{R}^2$.

IV. Tracking Objective

We begin by choosing any C^2 reference trajectory $(z_{1r}, z_{2r}, w_{1r}, w_{2r}, \xi_{1r}, \xi_{2r}) : [0,\infty) \to \mathbb{R}^6$ for (2) that admits a constant $c_1 \in (0,\pi/2)$ such that $\xi_{1r}(t) \in [-\pi/2 + c_1, \pi/2 - c_1]$ for all $t \ge 0$. We also assume that $\dot\xi_{1r}$ and $\ddot\xi_{1r}$ are bounded, and that there is a corresponding C^2 reference input $u_r = (u_{1r}, u_{2r})$ such that for all $t \ge 0$, we have

$$\begin{cases} \dot z_{1r}(t) &= z_{2r}(t) \\ \dot z_{2r}(t) &= -u_{1r}(t)\sin(\xi_{1r}(t)) \\ \dot w_{1r}(t) &= w_{2r}(t) \\ \dot w_{2r}(t) &= u_{1r}(t)\cos(\xi_{1r}(t)) - g \\ \dot\xi_{1r}(t) &= \xi_{2r}(t) \\ \dot\xi_{2r}(t) &= u_{2r}(t)\,. \end{cases} \tag{9}$$

Finally, we assume that u_r, $\dot u_r$, and $\ddot u_r$ are all bounded, and that there is a constant $c_2 > 0$ such that $\inf_{t\ge 0} u_{1r}(t) \ge c_2$.

We wish to track reference trajectories that satisfy the requirements from the previous paragraph, using bounded C^1 feedbacks. Equivalently, we must design bounded C^1 state feedbacks u_i to drive the error variables $\tilde z_i = z_i - z_{ir}(t)$, $\tilde w_i = w_i - w_{ir}(t)$, and $\tilde\xi_i = \xi_i - \xi_{ir}(t)$ to 0 for $i = 1, 2$. This means that the u_i's must render the tracking dynamics

$$\begin{cases} \dot{\tilde z}_1 &= \tilde z_2 \\ \dot{\tilde z}_2 &= -u_1\sin(\xi_1) + u_{1r}(t)\sin(\xi_{1r}(t)) \\ \dot{\tilde w}_1 &= \tilde w_2 \\ \dot{\tilde w}_2 &= u_1\cos(\xi_1) - u_{1r}(t)\cos(\xi_{1r}(t)) \\ \dot{\tilde\xi}_1 &= \tilde\xi_2 \\ \dot{\tilde\xi}_2 &= u_2 - u_{2r}(t) \end{cases} \tag{10}$$

globally asymptotically stable to 0.

V. Choice of u_1 and New Coordinates

Using part (II) of Lemma 2 and the constants $c_i > 0$ defined in Section IV, we have $\inf_{t\ge 0}\{u_{1r}(t)\cos(\xi_{1r}(t))\} > 0$, and we can fix a small enough constant $\lambda > 0$ such that

$$v = \arctan\left(\tan(\xi_{1r}(t)) - \frac{U_\lambda(\tilde z)}{u_{1r}(t)\cos(\xi_{1r}(t))}\right) \tag{11}$$

admits a constant $c_3 \in (0, c_1)$ such that $v \in [-\pi/2 + c_3, \pi/2 - c_3]$ for all $t \ge 0$ and $\tilde z \in \mathbb{R}^2$. We choose

$$u_1 = \frac{1}{\cos(v)}\left[u_{1r}(t)\cos(\xi_{1r}(t)) + U_\lambda(\tilde w)\right]. \tag{12}$$

By reducing $\lambda > 0$ to a sufficiently small value without relabeling and again using part (II) of Lemma 2, we can assume that u_1 is positive valued. We then set

$$K(t,\tilde z) = \frac{1}{1+\tan^2(v)}\frac{1}{u_{1r}(t)\cos(\xi_{1r}(t))}\,, \tag{13}$$

where v depends on t and $\tilde z$, and we define the functions

$$\begin{aligned} &\mathcal{S}_\lambda(t,\tilde z,\tilde w) = \\ &-\dot\xi_{1r}(t) + \frac{\{1+\tan^2(\xi_{1r}(t))\}\dot\xi_{1r}(t)}{1+\tan^2(v)} \\ &+ K(t,\tilde z)\left[\frac{U_\lambda(\tilde z)\frac{d}{dt}\{u_{1r}(t)\cos(\xi_{1r}(t))\}}{u_{1r}(t)\cos(\xi_{1r}(t))} - \frac{\partial U_\lambda(\tilde z)}{\partial\tilde z_1}\tilde z_2\right] \\ &- K(t,\tilde z)\frac{\partial U_\lambda(\tilde z)}{\partial\tilde z_2}\left[\sin(\xi_{1r}(t))u_{1r} - \sin(v)u_1\right] \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\mathcal{T}_\lambda(t,\varpi_1,\tilde z,\tilde w) = \\ &u_1 K(t,\tilde z)\frac{\partial U_\lambda(\tilde z)}{\partial\tilde z_2}\left[\sin(\varpi_1+v) - \sin(v)\right], \end{aligned} \tag{15}$$

where $\varpi_1 = \xi_1 - v$. Since $\dot{\tilde z}_2$, $\dot{\tilde w}_2$, $\dot\xi_{1r}$, and $\ddot\xi_{1r}$ are all bounded, simple calculations based on Lemma 2 show that

$\dot{\mathcal{S}}_\lambda$ is bounded. Fix any constant $a>0$ such that

$$\max\left\{\left|\frac{\partial}{\partial\varpi_1}\mathcal{T}_\lambda(t,\varpi_1,\tilde{z},\tilde{w})\right|,\left|\mathcal{T}_\lambda(t,\varpi_1,\tilde{z},\tilde{w})\right|\right\}\ \le\ \tfrac{a}{16} \tag{16}$$

on $[0,\infty)\times\mathbb{R}^5$. Notice that (10) with the choice (12) of u_1 and the new variable $\varpi_2=\tilde{\xi}_2-\mathcal{S}_\lambda$ can be rewritten as

$$\begin{cases} \dot{\tilde{z}}_1 \ = \ \tilde{z}_2 \\ \dot{\tilde{z}}_2 \ = \ -\frac{\sin(\varpi_1+v)}{\cos(v)}\big[u_{1r}(t)\cos(\xi_{1r}(t))+U_\lambda(\tilde{w})\big] \\ \qquad\quad +\,u_{1r}(t)\sin(\xi_{1r}(t)) \\ \dot{\tilde{w}}_1 \ = \ \tilde{w}_2 \\ \dot{\tilde{w}}_2 \ = \ \Big[\frac{\cos(\varpi_1+v)}{\cos(v)}-1\Big]u_{1r}(t)\cos(\xi_{1r}(t)) \\ \qquad\quad +\,U_\lambda(\tilde{w})\frac{\cos(\varpi_1+v)}{\cos(v)} \\ \dot{\varpi}_1 \ = \ \varpi_2-\mathcal{T}_\lambda(t,\varpi_1,\tilde{z},\tilde{w}) \\ \dot{\varpi}_2 \ = \ u_3 \end{cases} \tag{17}$$

where $u_3=u_2-u_{2r}(t)-\dot{\mathcal{S}}_\lambda$ and v depends on $(t,\tilde{z})$. Then the UGAS and ULES properties for (10) are equivalent to those of (17), so we have reduced the stabilization problems for (10) to those for (17).

VI. Main Result

A. Statement of Theorem

Since $\dot{\mathcal{S}}_\lambda$, u_{2r}, and our thrust controller (12) are C^1 and bounded, we will have our bounded feedbacks for (10) once we design a C^1 bounded feedback u_3 that renders (17) UGAS and ULES to 0. Our construction of u_3 is:

Theorem 1: Let the constants $a>0$ and $\lambda>0$ satisfy the requirements from Section V. Set $\mathcal{Z}_{1,a}(\varpi)=2\varpi_2+\sigma_a(a\varpi_1)\varphi_a(\varpi_2)$. Then the bounded C^1 function

$$u_3=\frac{-\sigma_a\Big(\mathcal{Z}_{1,a}(\varpi)\Big)-a\sigma_a'(a\varpi_1)\varphi_a(\varpi_2)\Big[\varpi_2-\mathcal{T}_\lambda(t,\varpi_1,\tilde{z},\tilde{w})\Big]}{2+\sigma_a(a\varpi_1)\varphi_a'(\varpi_2)} \tag{18}$$

renders (17) UGAS and ULES to the origin. □

See Section VIII for our extension to models with actuator errors. We cannot eliminate $\mathcal{T}_\lambda$ the way we eliminated $\mathcal{S}_\lambda$, because the unboundedness of $\dot{\varpi}_1$ makes $\dot{\mathcal{T}}_\lambda$ unbounded.

B. Proof of Theorem 1

The dynamics (17) in closed loop with (18) are forward complete because its right side grows affinely in the state uniformly in t. Therefore, the fact that the ϖ subdynamics of (17) satisfies a UGAS and ULES estimate of the form

$$|\varpi(t)|\ \le\ \alpha_1(|\varpi(t_0)|)e^{-c(t-t_0)} \tag{19}$$

for some $\alpha_1\in C^1\cap\mathcal{K}_\infty$ and some constant $c>0$ follows from the proof of Lemma 3 with the choices $\varepsilon(t)\equiv 0$, (16), and the fact that (18) agrees with the controller (7) when we take $L\equiv 0$, $\ell=a$, $X=\varpi$, $\bar{\eta}=0$, and $\Theta(t,\varpi)=-\mathcal{T}_\lambda(t,\varpi_1,\tilde{z}(t),\tilde{w}(t))$.

Next note that the $\tilde{w}$ dynamics in (17) can be written as

$$\begin{cases} \dot{\tilde{w}}_1 \ = \ \tilde{w}_2 \\ \dot{\tilde{w}}_2 \ = \ U_\lambda(\tilde{w})+L(t,\tilde{w},\varpi) \end{cases} \tag{20}$$

for an appropriate function L that admits a constant $\bar{L}>0$ so that $|L(t,\tilde{w},\varpi)|\le\bar{L}|\varpi_1|$ for all $t\ge 0$. Hence, the UGAS and ULES estimate

$$|(\tilde{w}(t),\varpi(t))|\ \le\ \alpha_2\left(|(\tilde{w}(t_0),\varpi(t_0))|\right)e^{-c(t-t_0)/16} \tag{21}$$

for the $(\tilde{w},\varpi)$ subsystem of (17) for a suitable function $\alpha_2\in C^1\cap\mathcal{K}_\infty$ also follows from Lemma 3, this time applied with $X=\tilde{w}$, $\Theta\equiv 0$, $\bar{\eta}=0$, and $\varepsilon=\varpi$.

Finally, $U_\lambda(\tilde{z})=[\tan(\xi_{1r}(t))-\tan(v)]u_{1r}(t)\cos(\xi_{1r}(t))$, so the $\tilde{z}$ subdynamics in (17) can be rewritten as

$$\begin{cases} \dot{\tilde{z}}_1 \ = \ \tilde{z}_2 \\ \dot{\tilde{z}}_2 \ = \ U_\lambda(\tilde{z})+\mathcal{L}(t,\tilde{z},\tilde{w},\varpi)\ , \end{cases} \tag{22}$$

where

$$\mathcal{L}(t,\tilde{z},\tilde{w},\varpi)=\frac{\sin(v)-\sin(\xi_1)}{\cos(v)}\big[u_{1r}(t)\cos(\xi_{1r}(t))+U_\lambda(\tilde{w})\big]-\tan(v)U_\lambda(\tilde{w}).$$

Using the properties of v and U_λ, we can find a constant $\bar{\mathcal{L}}$ such that $|\mathcal{L}(t,\tilde{z},\tilde{w},\varpi)|\le\bar{\mathcal{L}}|(\tilde{w},\varpi)|$ on $[0,\infty)\times\mathbb{R}^6$. Then the assumptions of Lemma 3 are satisfied with $X=\tilde{z}$, $\varepsilon=(\tilde{w},\varpi)$, $L=\mathcal{L}$, $\bar{L}=\bar{\mathcal{L}}$, $\Theta\equiv 0$, and $\bar{\eta}=0$, so the proof of Lemma 3 gives a function $\alpha_3\in C^1\cap\mathcal{K}_\infty$ such that $|(\tilde{z}(t),\tilde{w}(t),\varpi(t))|\le\alpha_2(|(\tilde{z}(t_0),\tilde{w}(t_0),\varpi(t_0))|)e^{-c(t-t_0)/256}$ along all trajectories of (17), which gives the desired conclusions.

VII. Tracking Without Velocity Measurements

If only z_1, w_1, and ξ_1 are measured, then we can apply the observer approach to output feedback from [7]. Due to space constraints, we only sketch the approach; see [11] for details. First, the proofs of Lemma 3 with $\bar{\eta}=0$ and Theorem 1 provide a positive definite proper function $\mathcal{V}_0(\tilde{z},\tilde{w},\varpi)$ and a C^1 function $\Gamma_0:[0,\infty)\to(0,\infty)$ such that $\dot{\mathcal{V}}_0$ is negative definite along all trajectories of (17) in closed loop with (12) and (18) for all $t\ge t_0+\Gamma_0(|(\tilde{z},\tilde{w},\varpi)(t_0)|)$, and such that $\mathcal{V}_1=\ln(1+\mathcal{V}_0)$ has a bounded gradient. The changes of coordinates used to transform (10) into (17) give state feedbacks $u_{1,s}(t,\tilde{z},\tilde{w},\tilde{\xi})$ and $u_{2,s}(t,\tilde{z},\tilde{w},\tilde{\xi})$ that are globally Lipschitz in the state uniformly in t, a proper positive function $\mathcal{V}_2(t,\tilde{z},\tilde{w},\tilde{\xi})$, and a C^1 function $\Gamma_2:[0,\infty)\to(0,\infty)$ such that $|(\partial\mathcal{V}_2/\partial\tilde{z})(t,\tilde{z},\tilde{w},\tilde{\xi})|$, $|(\partial\mathcal{V}_2/\partial\tilde{w})(t,\tilde{z},\tilde{w},\tilde{\xi})|$, and $|(\partial\mathcal{V}_2/\partial\tilde{\xi})(t,\tilde{z},\tilde{w},\tilde{\xi})|$ are all bounded, and such that $\dot{\mathcal{V}}_2$ is negative definite along all of the closed loop trajectories of (10) for all $t\ge t_0+\Gamma_2(|(\tilde{z},\tilde{w},\tilde{\xi})(t_0)|)$, namely, $\mathcal{V}_2(t,\tilde{z},\tilde{w},\tilde{\xi})=\mathcal{V}_1(\tilde{z},\tilde{w},\tilde{\xi}_1+\xi_{1r}(t)-v(t,\tilde{z}),\tilde{\xi}_2-S_\lambda(t,\tilde{z},\tilde{w}))$. It follows that

$$\begin{cases} \dot{\tilde{z}}_1 \ = \ \tilde{z}_2 \\ \dot{\tilde{z}}_2 \ = \ -u_{1,s}(t,\hat{z},\hat{w},\hat{\xi})\sin(\xi_1) \\ \qquad\quad +u_{1r}(t)\sin(\xi_{1r}(t)) \\ \dot{\tilde{w}}_1 \ = \ \tilde{w}_2 \\ \dot{\tilde{w}}_2 \ = \ u_{1,s}(t,\hat{z},\hat{w},\hat{\xi})\cos(\xi_1) \\ \qquad\quad -u_{1r}(t)\cos(\xi_{1r}(t)) \\ \dot{\tilde{\xi}}_1 \ = \ \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 \ = \ u_{2,s}(t,\hat{z},\hat{w},\hat{\xi})-u_{2r}(t) \\ \dot{\hat{z}}_1 \ = \ \hat{z}_2+k_1(\tilde{z}_1-\hat{z}_1) \\ \dot{\hat{z}}_2 \ = \ -u_{1,s}(t,\hat{z},\hat{w},\hat{\xi})\sin(\xi_1) \\ \qquad\quad +u_{1r}(t)\sin(\xi_{1r}(t))+k_2(\tilde{z}_1-\hat{z}_1) \\ \dot{\hat{w}}_1 \ = \ \hat{w}_2+k_3(\tilde{w}_1-\hat{w}_1) \\ \dot{\hat{w}}_2 \ = \ u_{1,s}(t,\hat{z},\hat{w},\hat{\xi})\cos(\xi_1) \\ \qquad\quad -u_{1r}(t)\cos(\xi_{1r}(t))+k_4(\tilde{w}_1-\hat{w}_1) \\ \dot{\hat{\xi}}_1 \ = \ \hat{\xi}_2+k_5(\tilde{\xi}_1-\hat{\xi}_1) \\ \dot{\hat{\xi}}_2 \ = \ u_{2,s}(t,\hat{z},\hat{w},\hat{\xi})-u_{2r}(t)+k_6(\tilde{\xi}_1-\hat{\xi}_1) \end{cases} \tag{23}$$

is UGAS to 0 where the constants k_i are positive. Indeed, it is clear that (23) is forward complete, and that $Y_e = (\tilde{z} - \hat{z}, \tilde{w} - \hat{w}, \tilde{\xi} - \hat{\xi})$ converges exponentially to zero. This and the boundedness of the gradient of $\mathcal{V}_2$ in the state imply that $(\tilde{z}, \tilde{w}, \tilde{\xi})$ converges to zero; see [16, Section 7.8] for analogous arguments. Similar arguments show that the $(\hat{z}, \hat{w}, \hat{\xi})$ subsystem converges to zero. Finally, analyzing the local properties shows that (23) is UGAS and ULES to 0.

VIII. Input-to-State Stability

We can also use Lemma 3 to show that the perturbed PVTOL error dynamics

$$\left\{\begin{array}{rcl} \dot{\tilde{z}}_1 & = & \tilde{z}_2 \\ \dot{\tilde{z}}_2 & = & -[u_1 + \delta_1]\sin(\xi_1) + u_{1r}(t)\sin(\xi_{1r}(t)) \\ \dot{\tilde{w}}_1 & = & \tilde{w}_2 \\ \dot{\tilde{w}}_2 & = & [u_1 + \delta_1]\cos(\xi_1) - u_{1r}(t)\cos(\xi_{1r}(t)) \\ \dot{\tilde{\xi}}_1 & = & \tilde{\xi}_2 \\ \dot{\tilde{\xi}}_2 & = & u_2 - u_{2r}(t) + \delta_2 \end{array}\right. \tag{24}$$

with actuator errors δ_i, in closed loop with the feedbacks we designed above, is ISS with respect to measurable essentially bounded actuator errors $\delta : [0, \infty) \to \bar{\eta}\mathcal{B}_2$, where the feedback formulas must now depend on the bound $\bar{\eta}$ on the disturbance. The argument is similar to the proof of Theorem 1 except with actuator errors added in the control channels. We can allow any bound $\bar{\eta}$, through a proper choice of the feedbacks. We illustrate this property in Section X.

IX. Trackable Reference Trajectories

Our assumptions are satisfied by a broad class of reference trajectories and corresponding reference inputs. For example, assume that $(z_{1r}, w_{1r}) : [0, \infty) \to \mathbb{R}^2$ is a C^4 time-periodic function such that $\ddot{w}_{1r}(t) + g$ is positive valued. Then the PVTOL dynamics (9) are satisfied with the controls

$$u_{1r} = \sqrt{(\ddot{z}_{1r})^2 + (\ddot{w}_{1r} + g)^2} \;\; \text{and} \;\; u_{2r} = \ddot{\xi}_{1r} \;, \tag{25}$$

and with $\xi_{2r} = \dot{\xi}_{1r}$, $z_{2r} = \dot{z}_{1r}$, $w_{2r} = \dot{w}_{1r}$, and

$$\xi_{1r} = \arcsin\left(\frac{-\ddot{z}_{1r}}{\sqrt{(\ddot{z}_{1r})^2 + (\ddot{w}_{1r} + g)^2}}\right) . \tag{26}$$

Also, $u_r \in C^2$ because $(z_{1r}, w_{1r}) \in C^4$. Therefore, our assumptions are satisfied by the corresponding reference trajectory $(z_{1r}, z_{2r}, w_{1r}, w_{2r}, \xi_{1r}, \xi_{2r}) : [0, \infty) \to \mathbb{R}^6$. Positivity of $\ddot{w}_{1r}(t) + g$ holds for circular trajectories $(z_{1r}(t), w_{1r}(t)) = g_0(\bar{K} + \cos(t), \bar{K} + \sin(t))$ for any constants $\bar{K} \geq 1$ and $g_0 \in (0, g)$, so we can track trajectories along these circles. In the next section, we illustrate this tracking in a simulation.

X. Simulation

To validate our method, we took the reference profile

$$(z_{1r}(t), w_{1r}(t)) = 5\,(1.5 + \cos(t), 1.5 + \sin(t)) \tag{27}$$

for the center of mass. As we saw in the preceding section, the corresponding reference trajectory is obtained by taking $z_{2r} = \dot{z}_{1r}$, $w_{2r} = \dot{w}_{1r}$, ξ_{1r} as defined in (26), and $\xi_{2r} = \dot{\xi}_{1r}$. The reference inputs are $u_{1r} = \sqrt{(\ddot{z}_{1r})^2 + (\ddot{w}_{1r} + 9.81)^2}$ and $u_{2r} = \ddot{\xi}_{1r}$. Simple calculations show that the requirements from Section V are satisfied with $\lambda = 2$ and $a = 10.14$, so Theorem 1 gives UGAS and ULES of the corresponding (transformed) PVTOL tracking error dynamics (17) in closed loop with the feedbacks (12) and (18).

Using the preceding data, we performed two simulations. First, we simulated (17) with the initial state $(0.31, 0.31, 0.31, 0.21, 0.21, 0.41)$, (12), and (18). In Figures 1-3, we report the resulting tracking of the center of mass, the trajectory for the roll angle $\theta = \varpi_1 + v$, and the closed loop thrust input u_1. In Figure 1, the reference trajectory (27) for the center of mass is blue and dashed, the simulated trajectory $(z_1(t), w_1(t))$ for the center of mass is red and solid, and the plot covers times $t = 10$ to $t = 20$.

In our second simulation, we added the constant actuator error $\delta_2 = 0.15$ in the u_3 channel in (17) such that instead of $\dot{\varpi}_2 = u_3$, we now have $\dot{\varpi}_2 = u_3 + 0.15$, and we changed the control component u_3 to

$$u_3 = \frac{-[1+172\bar{\eta}/a]\sigma_a\left(\mathcal{Z}_{1,a}(\varpi)\right) - a\sigma_a'(a\varpi_1)\varphi_a(\varpi_2)\left[\varpi_2 - \mathcal{T}_\lambda(t, \varpi_1, \tilde{z}, \tilde{w})\right]}{2 + \sigma_a(a\varpi_1)\varphi_a'(\varpi_2)}$$

with the disturbance bound $\bar{\eta} = 0.25$, in accordance with Section VIII. Otherwise, everything was the same as the first simulation. In Figure 4, we plot the resulting tracking of the center of mass again over times $t = 10$ to $t = 20$.

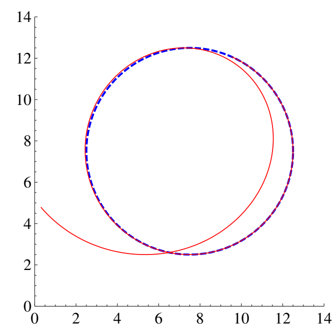

Fig. 1. Center of Mass (z_1, w_1) Tracking (27) Without Disturbances.

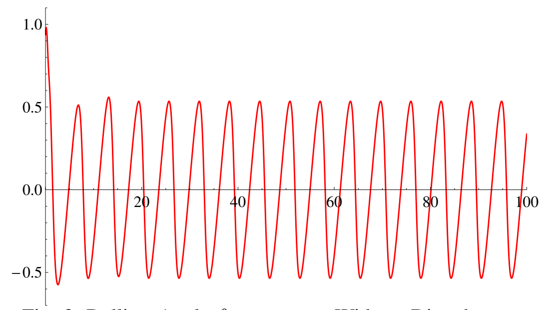

Fig. 2. Rolling Angle $\theta = \varpi_1 + v$ Without Disturbances.

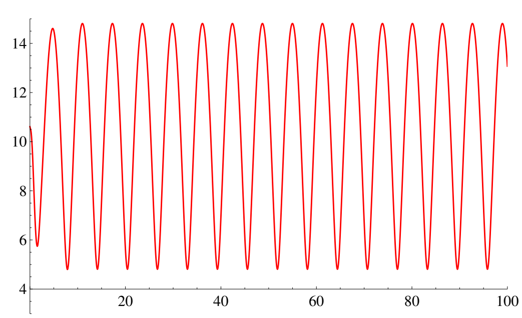

Fig. 3. Thrust Input u_1 from (12) Without Disturbances.

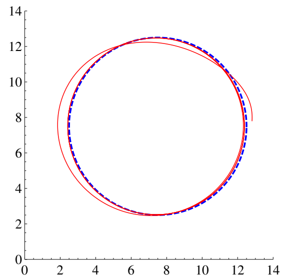

Fig. 4. Center of Mass (z_1, w_1) Tracking (27) With Disturbance $\delta_2 = 0.15$.

Remark 1: We can also track along Cassini's Oval [5]

$$\begin{aligned} \big(z_{1r}(t), w_{1r}(t)\big) \;&=\; R(t)\big(\cos(t), \sin(t)\big), \;\text{ where} \\ R(t) \;&=\; \sqrt{a^2\cos(2t) + \sqrt{b^4 - (a^2\sin(2t))^2}}\;, \end{aligned} \tag{28}$$

for certain choices of the constants $a > 0$ and $b > a$ when we take the gravitational constant $g = 9.81$; see Figure 5. For example, with the choices $a = 2.65$ and $b = 2.9$,

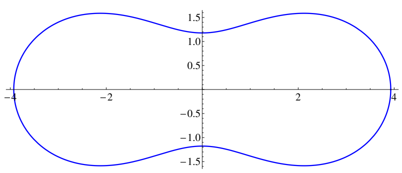

Fig. 5. Cassini's Oval (28) with $a = 2.65$ and $b = 2.9$.

Mathematica gives $\ddot{w}_{1r}(t) + g \geq 0.552321$ for all $t \geq 0$. It follows from our discussion from Section IX that we can track reference trajectories with the center of mass profile (28) using the parameter values $a = 2.65$ and $b = 2.9$.

XI. Conclusions

The PVTOL aircraft dynamics is a benchmark model that is of continuing ongoing research interest. Much of the PVTOL literature deals with tracking. However, we believe that our bounded global tracking controllers are significant because they guarantee UGAS and ULES of the closed loop tracking dynamics under general conditions on the reference trajectories and can be adjusted to give ISS performance to actuator error disturbances for any a priori bound on the admissible disturbances. Our feedback design is based on a new bounded backstepping method which we anticipate being useful for other models in feedforward form.

XII. Acknowledgements

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