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The exponential synchronization of Kuramoto oscillator networks in the presence of combined global and local cues

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***Abstract*— This work addresses the influences of global and local cues on the synchronization of Kuramoto oscillators coupled by nonlinear interactions. We first give a condition under which the network can be globally synchronized. This condition is more general than existing results and it is one of the first analytical results that ensure global synchronization of a general coupling structure when phase differences may be outside of the interval $[-\frac{\pi}{2},\ \frac{\pi}{2}]$. Then we prove that the influence of global cues on the synchronization rate is always favorable, whereas the influence of local cues depends on the oscillators' relative phases (phases defined with reference to the phase of the global cue): if all the relative phases are within the interval $[-\frac{\pi}{2},\ \frac{\pi}{2}]$, the local cue either has no influence or increases the synchronization rate; otherwise the local cue may increase or decrease the synchronization rate. Finally, simulation results are given to illustrate the theoretical results.**

I. Introduction

Synchronization phenomena in large populations of interacting elements are commonplace in science and engineering. In recent years, synchronization phenomena in oscillating dynamical systems have been studied in mobile autonomous agents, biological networks, distributed computing and communication networks.

Synchronization in coupled oscillating systems arises from external alignment adjustment and mutual interaction among constituent oscillators. Therefore, the interplay of global and local cues (or, alternatively, the interplay of external cues and local coupling) is a recurring feature in the achievement of synchronization. For example, in mammalian circadian systems, the circadian pacemaker is located in the suprachiasmatic nucleus (SCN), in which about 20,000 neuronal oscillators produce a 24-h rhythm utilizing intercellular interplays among individual oscillators while at the same time receiving a global driving signal such as the sunlight via the retinal input [1]. In engineering applications, such as the coordination of groups of mobile autonomous agents (e.g., UAV-unmanned aerial vehicles and MANET-mobile ad hoc networks), the signal from central resources (e.g., leader node for UAV and satellite for MANET) acts as the global cue and the interplay between the follower nodes acts as the local cue, both of which are essential to the synchronization of the collective motion [2], [3]. Therefore, studying the influences of global and local cues in the achievement of synchronization is not only important in scientific research, but also of crucial importance in engineering applications.

The Kuramoto model was first proposed in 1975 to model the synchronization of chemical oscillators coupled in an all-to-all manner [4]. Since it is rich enough to display a large variety of synchronization patterns and sufficiently flexible to be adapted to many different contexts, yet simple enough to be mathematically tractable, the Kuramoto model is widely used and is regarded as the most representative model of coupled phase oscillators [5]. Recently, the Kuramoto model has received more attention in the control community, and results have been obtained on the synchronization conditions, both for the original all-to-all coupling topology and for generalized coupling topologies [6], [7], [8], [9], [10].

However, existing studies on the synchronization of the Kuramoto model of nonlinearly coupled oscillators are usually based on a special coupling structure, such as the ring structure [11], or all-to-all structure [6], [10]. Moreover, most of the existing results only consider synchronization conditions or stability of the synchronization manifold, while research on the synchronization rate is very sparse.

This work addresses the global synchronization of Kuramoto oscillator networks. We prove that synchronization can be achieved when the relative phases (phases defined with reference to the phase of the global cue, i.e., the phase deviations from the phase of the global cue) are within $(-\pi,\ \pi)$. This condition is much less conservative than existing synchronization conditions, which require that the phase difference between any two oscillators is within $[-\frac{\pi}{2},\ \frac{\pi}{2}]$. It is one of the first analytical results that prove the achievement of global synchronization for a general coupling structure even when phase differences are outside $[-\frac{\pi}{2},\ \frac{\pi}{2}]$. We also analyze the influences of global and local cues on the synchronization rate, on which sparse results have been obtained in the existing literature.

II. Problem formulation

Suppose the overall network is composed of N oscillators, which will henceforth be referred to as 'nodes'. Each of the

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N oscillator nodes receives alignment/entrainment information from an external global cue.

We denote the phase dynamics of the global cue as

$$\dot{\varphi}_0 = w_0, \quad \varphi_0(0) = \phi_0 \tag{1}$$

and the phase dynamics of the isolated oscillator nodes as

$$\dot{\varphi}_i = w_i, \quad \varphi_i(0) = \phi_i, \quad i = 1, 2, \ldots, N \tag{2}$$

where φ_0 and φ_i ($i = 1, 2, \ldots, N$) are the phases of the global cue and the oscillator nodes, and w_i ($i = 0, 1, 2, \ldots, N$) and ϕ_i ($i = 0, 1, 2, \ldots, N$) are their natural frequencies and initial values, respectively.

After taking into consideration the influences of global cues (alignment/entrainment information from global cues) and local cues (interplay between different nodes), the overall dynamics of the oscillator network can be rewritten as

$$\dot{\varphi}_i = w_i + \sum_{1 \le j \le N, j \ne i} a_{i,j} \sin(\varphi_j - \varphi_i) + g_i \sin(\varphi_0 - \varphi_i) \tag{3}$$

for $1 \le i \le N$, where $a_{i,j} \sin(\varphi_j - \varphi_i)$ is the interplay between node i and node j, $a_{i,j}$ denotes the strength of the interplay. $g_i \sin(\varphi_0 - \varphi_i)$ denotes the influence of the global cue and $g_i > 0$ denotes its strength.

Assumption 1: In the remainder of the paper, $a_{i,j}$ ($1 \le i, j \le N$) are assumed to be non-negative and identical to $a_{j,i}$, i.e., the coupling between pairs of oscillators are symmetric.

Next, based on Assumption 1, we study the influences of global cues, g_i ($1 \le i \le N$), and local cues, $a_{i,j}$ ($1 \le i, j \le N$), on synchronization conditions and synchronization rates. It is noteworthy that $a_{i,j} = 0$ represents the scenario that oscillator i is not influenced by oscillator j, and we do not make any restrictions on $a_{i,j}$ except non-negativity, thus the structure of the coupling is quite general.

For convenience in the analysis, we first transform (3) into a rotating reference frame via the transformations $\varphi_i = w_0 t + \theta_i$ and $\varphi_0 = w_0 t + \phi_0$:

$$\dot{\theta}_i = w_i - w_0 + \sum_{1 \le j \le N, j \ne i} a_{i,j} \sin(\theta_j - \theta_i) + g_i \sin(\phi_0 - \theta_i) \tag{4}$$

where $1 \le i \le N$. The constant ϕ_0 in (4) can be removed by substituting $\theta_i - \phi_0$ with ξ_i:

$$\dot{\xi}_i = w_i - w_0 + \sum_{1 \le j \le N, j \ne i} a_{i,j} \sin(\xi_j - \xi_i) - g_i \sin(\xi_i) \tag{5}$$

Since

$$\xi_i = \theta_i - \phi_0 = w_0 t + \theta_i - (w_0 t + \phi_0) = \varphi_i - \varphi_0, \quad 1 \le i \le N$$

ξ_i is the relative phase of the ith oscillator with respect to the phase of the global cue and, thus, will be referred to as the relative phase.

So far, by studying the properties of system (5), we can obtain the roles of global and local cues in the synchronization of inter-connected Kuramoto oscillator networks:

- Synchronization: If all ξ_i asymptotically converge to 0, then we have $\xi_1 = \xi_2 = \ldots = \xi_N$ when time goes to infinity, i.e., ultimately we have $\theta_1 = \theta_2 = \ldots = \theta_N$, which leads to $\varphi_1 = \varphi_2 = \ldots = \varphi_N$, meaning that all the nodes are synchronized.
- Exponential bound on synchronization rate: The synchronization rate is determined by the rate at which ξ_i decays to 0, namely, it can be measured by the maximal value α ($\alpha > 0$) satisfying
$$\|\xi(t)\| \le C e^{-\alpha t} \|\xi(0)\| \tag{6}$$
where $\| \bullet \|$ is the Euclidean norm and C is a constant. It follows that α measures the exponential bound on the synchronization rate of (5): a larger α leads to a faster synchronization rate.

Assigning arbitrary orientation to each interaction, we can get the $N \times M$ incidence matrix B (M is the number of non-zero $a_{i,j}$ ($1 \le i \le N$, $j < i$), i.e., the number of interaction edges) of the interaction graph [12]: $B_{i,j} = 1$ if edge j enters node i, $B_{i,j} = -1$ if edge j leaves node i, and $B_{i,j} = 0$ otherwise. Then using graph theory, (5) can be written in a more compact matrix form:

$$\dot{\xi} = \Omega - G \sin \xi - BW \sin\left(B^T \xi\right) \tag{7}$$

where

$$\xi^T = \left[\begin{array}{cccc} \xi_1 & \xi_2 & \ldots & \xi_N \end{array}\right]^T \tag{8}$$

$$\Omega^T = \left[\begin{array}{cccc} w_1 - w_0 & w_2 - w_0 & \ldots & w_N - w_0 \end{array}\right]^T \tag{9}$$

$$G = \operatorname{diag}(g_1, g_2, \ldots, g_N), W = \operatorname{diag}(\nu_1, \mu_2, \ldots, \nu_M) \tag{10}$$

and ν_i ($1 \le i \le M$) are the strengths of interaction, i.e., they are a permutation of non-zero $a_{i,j}$ ($1 \le i \le N, j < i$). diag($\bullet$) denotes a diagonal matrix with elements ($\bullet$) on the diagonal.

In the following, based on the derived model (7), we will analyze the influences of global cues, g_i ($1 \le i \le N$), and local cues, $a_{i,j}$ ($1 \le i, j \le N$), on synchronization rates.

III. Main Results

Assumption 2: We assume all the oscillators have identical natural frequencies, i.e., $w_1 = w_2 = \ldots = w_N = w_0$.

Using Assumption 2, (7) reduces to:

$$\dot{\xi} = -G \sin \xi - BW \sin\left(B^T \xi\right) \tag{11}$$

where ξ_i denotes the relative phase of node i with reference to the phase of the global cue, and $B^T \xi$ is an $M \times 1$ vector composed of elements in the form of $\xi_m - \xi_n$ ($1 \le m, n \le N$). In this section, we will analyze the influences of global and local cues on the synchronization rate. To this end, we first give a synchronization condition. In fact, we will prove that the oscillators can be synchronized even when the relative phases are outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$. This condition is less conservative than existing results, which require that all phase differences (including ξ_i representing the phase differences between oscillator nodes and the global cue) must be within the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ [6], [7], [11].

For ease in comparison with existing results, we divide the problem into two subproblems: synchronization rate when all the relative phases $\xi_i\,(1 \le i \le N)$ are within $[-\frac{\pi}{2},\,\frac{\pi}{2}]$ and synchronization rate when the maximal/minimal relative phase is within $(-\pi,\,\pi)$ but outside $[-\frac{\pi}{2},\,\frac{\pi}{2}]$.

A. Synchronization rate when all the relative phases are within the interval $[-\frac{\pi}{2},\,\frac{\pi}{2}]$

To study the synchronization rate, we first consider the condition under which synchronization can be ensured. Following the idea of [7], we have the following theorem:

Theorem 1: For the oscillator network formulated by (11), if all the relative phases $\xi_i\,(1 \le i \le N)$ are within the interval $[-\frac{\pi}{2},\,\frac{\pi}{2}]$, then the oscillator network will always synchronize.

Proof: Construct a Lyapunov function as

$$V = \frac{1}{2}\xi^T\xi \tag{12}$$

where V is a non-negative function and will be zero if and only if all $\xi_i\,(1 \le i \le N)$ are zero, meaning that all the oscillators are synchronized to the global cue.

Differentiating V along trajectories of (11) yields

$$\begin{aligned}\dot{V} &= \xi^T\dot{\xi} = -\xi^T\left(G\sin\xi + BW\sin(B^T\xi)\right)\\ &= -\xi^T G S_1 \xi - \xi^T BWS_2B^T\xi\end{aligned} \tag{13}$$

where S_1 and S_2 are given by

$$S_1 = \text{diag}\left\{\frac{\sin\xi_1}{\xi_1},\,\frac{\sin\xi_2}{\xi_2},\,\ldots,\,\frac{\sin\xi_N}{\xi_N}\right\}, \tag{14}$$

$$S_2 = \text{diag}\left\{\frac{\sin(B^T\xi)_1}{(B^T\xi)_1},\,\frac{\sin(B^T\xi)_2}{(B^T\xi)_2},\,\ldots,\,\frac{\sin(B^T\xi)_M}{(B^T\xi)_M}\right\} \tag{15}$$

with $(B^T\xi)_i\,(1 \le i \le M)$ denoting the ith element of $M\times 1$ dimensional vector $B^T\xi$.

Note that $(B^T\xi)_i\,(1 \le i \le M)$ are in the form of $\xi_m - \xi_n\,(1 \le m,n \le N)$ and thus satisfy $-\pi \le (B^T\xi)_i \le \pi$ if relative phases $\xi_i\,(1 \le i \le N)$ are restricted to the interval $[-\frac{\pi}{2},\,\frac{\pi}{2}]$. Considering the fact that in the interval $[-\pi,\,\pi]$, the sinc function satisfies

$$\text{sinc}(x) \triangleq \frac{\sin x}{x} \ge 0$$

with the equality holding if and only if $x = \pm\pi$, it follows that S_1 is positive definite and S_2 is positive semi-definite, and thus $\dot{V}$ is always negative if ξ is non-zero. Therefore V will decay to zero exponentially, meaning that ξ will converge to zero and all the oscillators are synchronized. ■

Based on a similar derivation, we can get the synchronization rate of the oscillator network:

Theorem 2: For the oscillator network formulated in (11), when the relative phases are within the interval $[-\varepsilon,\,\varepsilon]$ with $0 \le \varepsilon \le \frac{\pi}{2}$, the synchronization rate is no worse than

$$\alpha = g_{\min}\frac{\sin\varepsilon}{\varepsilon} \tag{16}$$

where $g_{\min}$ is determined by

$$g_{\min} = \min\{g_1,\,g_2,\,\ldots,\,g_N\} \tag{17}$$

Proof: Similar to the proof of Theorem 1, we can get

$$\begin{aligned}\dot{V} &\le -\lambda_{\min}(GS_1)\xi^T\xi - \lambda_{\min}(BWS_2B^T)\xi^T\xi\\ &= -\lambda_{\min}(G)\lambda_{\min}(S_1)\xi^T\xi - \lambda_{\min}(BWS_2B^T)\xi^T\xi\end{aligned} \tag{18}$$

where $\lambda_{\min}(\bullet)$ denotes the minimal eigenvalue of matrix $(\bullet)$.

According to the definition of G in (10), we have

$$\lambda_{\min}(G) = g_{\min} \tag{19}$$

with $g_{\min}$ defined in (17).

Since sinc is an odd function and monotonic in the interval $[0,\,\pi]$, we know

$$\min_{x,\,-\varepsilon\le x\le\varepsilon}\frac{\sin x}{x} = \frac{\sin\varepsilon}{\varepsilon} \tag{20}$$

holds for any ε satisfying $0 \le \varepsilon \le \frac{\pi}{2}$, which leads to

$$\lambda_{\min}(S_1) = \frac{\sin\varepsilon}{\varepsilon} \tag{21}$$

From the proof of Theorem 1, BWS_2B^T is a positive semi-definite matrix when all the relative phases $\xi_i\ (1 \le i \le N)$ are in the interval $[-\frac{\pi}{2},\,\frac{\pi}{2}]$. Because W and S_2 are diagonal matrices, it can be easily proven that all row sums of BWS_2B^T are equal to zero. According to consensus theory, it follows that BWS_2B^T always has a zero eigenvalue, i.e.,

$$\lambda_{\min}(BWS_2B^T) = 0 \tag{22}$$

Combining (18), (19), (21) and (22) yields

$$\dot{V} \le -2\alpha\frac{\xi^T\xi}{2} = -2\alpha V \tag{23}$$

with α defined in (16), which further means that

$$V(t) \le C^2e^{-2\alpha t}V(0) \Rightarrow \|\xi(t)\| \le Ce^{-\alpha t}\|\xi(0)\| \tag{24}$$

holds for some positive constant C. Thus the synchronization rate is no worse than α in (16). ■

Remark 1: Although the results are derived using Lyapunov analysis, they are not conservative. As shown in [13], when there is no global cue, i.e., $g_1 = g_2 = \ldots = g_N = g_{\min} = 0$, the network may not synchronize even when the relative phases $\xi_i\,(1 \le i \le N)$ are in the interval $[-\frac{\pi}{2},\,\frac{\pi}{2}]$. So the derived lower bound on the synchronization rate, which is zero if there is no global cue, is attainable.

Remark 2: Since S_2 is positive semi-definite when the relative phases $\xi_i\ (1 \le i \le N)$ are in the interval $[-\frac{\pi}{2},\,\frac{\pi}{2}]$, which leads to $\xi^TBS_2B^T\xi \ge 0$, the local cue will either increase the synchronization rate (when $\xi^TBS_2B^T\xi > 0$) or have no influence on the synchronization rate (when $\xi^TBS_2B^T\xi = 0$).

B. Synchronization rate when the maximal/minimal relative phase is within the interval $(-\pi,\,\pi)$ but outside $[-\frac{\pi}{2},\,\frac{\pi}{2}]$

In this section, we analyze the synchronization condition and synchronization rate of oscillator networks when the maximal/minimal relative phase is outside $[-\frac{\pi}{2},\,\frac{\pi}{2}]$. We first derive synchronization conditions under which the whole oscillator network can be synchronized. It is one of the first analytical results that prove the achievement of global synchronization for a general coupling structure even when phase differences are outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$. We also analyze influences of global and local cues on the synchronization rate. In fact, we will prove that a stronger global cue always leads to a faster synchronization rate, whereas a stronger local cue may increase or decrease the synchronization rate, depending on the topology of local cues. Theorem 3 gives a synchronization condition:

Theorem 3: For the oscillator network formulated in (11), if we denote $\lambda_{\max}(\bullet)$ as the maximal eigenvalue, then

1) when the relative phases $\xi_i\,(1 \le i \le N)$ are within the interval $[-\varepsilon, \varepsilon]$ with $\frac{\pi}{2} \le \varepsilon \le \tilde{\varepsilon}$, where $\tilde{\varepsilon}$ is the solution to
$$2\tilde{\varepsilon}\cos(2\tilde{\varepsilon}) = \sin(2\tilde{\varepsilon}), \quad \frac{\pi}{2} < \tilde{\varepsilon} < \pi \tag{25}$$
and can be solved numerically, then the network will be synchronized if the following inequality is satisfied:
$$g_{\min} > (-\cos\varepsilon)\lambda_{\max}(BWB^T) \tag{26}$$
2) when the relative phases $\xi_i\,(1 \le i \le N)$ are within the interval $[-\varepsilon, \varepsilon]$ with $\tilde{\varepsilon} < \varepsilon < \pi$, the network will be synchronized if the following inequality is satisfied:
$$g_{\min} > \left(-\frac{\varepsilon\sin(2\tilde{\varepsilon})}{2\tilde{\varepsilon}\sin\varepsilon}\right)\lambda_{\max}(BWB^T) \tag{27}$$

Proof: Using the same Lyapunov function as in Theorem 1, we have
$$\begin{aligned}\dot{V} &= -\xi^T GS_1\xi - \xi^T BWS_2B^T\xi \\ &\le -\lambda_{\min}(GS_1)\xi^T\xi - \xi^T BWS_2B^T\xi \\ &= -\lambda_{\min}(G)\lambda_{\min}(S_1)\xi^T\xi - \xi^T BWS_2B^T\xi\end{aligned} \tag{28}$$
where S_1 and S_2 are given in (14) and (15), respectively.

When the relative phases $\xi_i\,(1 \le i \le N)$ are within the interval $[-\varepsilon, \varepsilon]$ with $\frac{\pi}{2} \le \varepsilon \le \tilde{\varepsilon}$, $(B^T\xi)_i\,(1 \le i \le M)$, which are in the form of $\xi_m - \xi_n\,(1 \le m, n \le N)$, are within the interval $[-2\varepsilon, 2\varepsilon]$. Making use of the fact that $\frac{\sin x}{x}$ and $\frac{\sin(2x)}{2x}$ are monotonically decreasing when $x \in [0, \varepsilon]$ for $\frac{\pi}{2} \le \varepsilon \le \tilde{\varepsilon}$ (which can be proven using the first derivative test [14]), we have
$$\dot{V} \quad \le -\left(g_{\min}\tfrac{\sin\varepsilon}{\varepsilon} + \tfrac{\sin(2\varepsilon)}{2\varepsilon}\lambda_{\max}(BWB^T)\right)\xi^T\xi \tag{29}$$
According to the Lyapunov theory, a sufficient condition to guarantee convergence of ξ is
$$g_{\min}\frac{\sin\varepsilon}{\varepsilon} + \frac{\sin(2\varepsilon)}{2\varepsilon}\lambda_{\max}(BWB^T) > 0$$
which equates to (26) using trigonometric identities.

When the relative phases $\xi_i\,(1 \le i \le N)$ are within the interval $[-\varepsilon, \varepsilon]$ with $\tilde{\varepsilon} < \varepsilon < \pi$, $\frac{\sin 2x}{2x}$ is monotonically decreasing in the interval $x \in [0, \tilde{\varepsilon}]$ and monotonically increasing in the interval $x \in [\tilde{\varepsilon}, \pi]$. $\frac{\sin x}{x}$ is still monotonically decreasing in the interval $x \in [0, \varepsilon]$ for $\tilde{\varepsilon} < \varepsilon < \pi$. Thus, based on a similar derivation to the $\frac{\pi}{2} \le \varepsilon \le \tilde{\varepsilon}$ case, we can get
$$\dot{V} \le -\left(g_{\min}\frac{\sin\varepsilon}{\varepsilon} + \frac{\sin 2\tilde{\varepsilon}}{2\tilde{\varepsilon}}\lambda_{\max}(BWB^T)\right)\xi^T\xi \tag{30}$$
A sufficient condition to guarantee convergence of ξ is
$$g_{\min}\frac{\sin\varepsilon}{\varepsilon} + \frac{\sin 2\tilde{\varepsilon}}{2\tilde{\varepsilon}}\lambda_{\max}(BWB^T) > 0$$
which is equivalent to (27) after some trigonometric manipulations. Thus the theorem is proven. ■

Remark 3: A numerical solution to (25) yields $\tilde{\varepsilon} = 2.2467$. Moreover, since the parameters on the right hand sides of (26) and (27) are all known, the conditions in (26) and (27) can easily be verified.

Remark 4: Theorem 3 shows that the oscillator network can be exponentially synchronized even when relative phases are greater than $\frac{\pi}{2}$. Considering the fact that 'relative phases' defined with reference to the phase of the global cue in this paper is only a subset of 'phase differences' defined as the phase discrepancies between any two oscillators in the literature, our result is less conservative than existing synchronization conditions which require that all phase differences must be within the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Remark 5: It is worthwhile to mention that all the eigenvalues of BWB^T are non-negative [12], which means that $\lambda_{\max}(BWB^T) > 0$.

When the conditions in Theorem 3 can be satisfied, i.e., the oscillator network can be synchronized, we can also analyze the synchronization rate of the oscillator network. The results are detailed in Theorem 4.

Theorem 4: For the oscillator network formulated in (11), if the synchronization conditions in (26) or (27) are satisfied, then the rate of synchronization can be bounded as follows:

1) when the relative phases $\xi_i\,(1 \le i \le N)$ are within the interval $[-\varepsilon, \varepsilon]$ with $\frac{\pi}{2} \le \varepsilon \le \tilde{\varepsilon}$, the synchronization rate is no worse than
$$\alpha_1 = g_{\min}\frac{\sin\varepsilon}{\varepsilon} + \frac{\sin(2\varepsilon)}{2\varepsilon}\lambda_{\max}(BWB^T) \tag{31}$$
2) when the relative phases $\xi_i\,(1 \le i \le N)$ are within the interval $[-\varepsilon, \varepsilon]$ with $\tilde{\varepsilon} < \varepsilon < \pi$, the synchronization rate is no worse than
$$\alpha_2 = g_{\min}\frac{\sin\varepsilon}{\varepsilon} + \frac{\sin(2\tilde{\varepsilon})}{2\tilde{\varepsilon}}\lambda_{\max}(BWB^T) \tag{32}$$

Proof: Following the direction of the proof in Theorem 3, we have
$$\dot{V} \le -2\alpha_1 V \tag{33}$$
when the relative phases $\xi_i\,(1 \le i \le N)$ are within the interval $[-\varepsilon, \varepsilon]$ for $\frac{\pi}{2} \le \varepsilon \le \tilde{\varepsilon}$, and
$$\dot{V} \le -2\alpha_2 V \tag{34}$$
when the relative phases $\xi_i\,(1 \le i \le N)$ are within the interval $[-\varepsilon, \varepsilon]$ with $\tilde{\varepsilon} < \varepsilon < \pi$. Thus the relative phases will converge to zero (meaning that the phases of all oscillators will become identical to the phase of the global cue) with a rate no worse than α_1 (in the $\frac{\pi}{2} \le \varepsilon \le \tilde{\varepsilon}$ case) or α_2 (in the $\tilde{\varepsilon} < \varepsilon < \pi$ case), which completes the proof. ■

Remark 6: From (31) and (32), it is clear that a stronger global cue leads to a faster synchronization rate. However, since S_2 in (28) can be positive semi-definite, negative semi-definite or indefinite when the relative phases $\xi_i\,(1 \le$

$i \leq N$) are within the interval $[-\varepsilon, \varepsilon]$ for $\frac{\pi}{2} \leq \varepsilon < \pi$, $\xi^T BWS_2B^T\xi$ can be positive, negative or zero, thus the local cue may increase, decrease or have no influence on the synchronization rate. This conclusion is confirmed by numerical simulations in Sec. IV.

IV. Simulation results

In this section, simulation results are given to illustrate the analytical results. We consider an oscillator network composed of $N = 9$ inter-connected oscillators, and each oscillator receives alignment/entrainment information from the global cue. The strengths of the global cues, $g_1, g_2, \ldots, g_N$, are set to an identical positive value and the strengths of the local cues, $a_{i,j}(1 \leq i, j \leq N)$, are randomly chosen from the interval $[0, 1]$. We use the synchronization index to measure the degree of synchrony [15], [16]:

$$SI = \left| \frac{1}{N} \sum_{i=0}^{N} e^{j\varphi_i} \right|$$

$SI \in [0, 1]$ reflects the degree of synchrony and will approach 1 as the network is perfectly synchronized, and 0 if the phases are randomly distributed [15].

In order to show the influences of global and local cues on the rate of synchronization, we simulated the oscillator network under different strengths of global and local cues. For convenience in notation, we use $m \times g\,(m = 1, 2, \ldots)$ to represent the situation that the strength of the global cue is made m times larger than the original one. Similarly, we use $n \times a_{i,j}\,(n = 1, 2, \ldots)$ to represent the situation that the strengths of local cues are made n times larger than the original ones. Synchronization is defined to be achieved when the SI exceeds 0.99.

We first simulated the oscillator network when all the initial relative phases were set uniformly distributed in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The evolution of the 9 oscillators' relative phases is illustrated in Fig 1. It can be seen that all the relative phases converge to zero. Hence the oscillator network is synchronized. With a fixed strength of the local cue, the times to synchronization under different strengths of global cues are given in Fig. 2. It is clear that the synchronization rate increases with an increase in the global cue. To show the influences of local cues on the synchronization rate, we also simulated the network under a fixed global cue and different strengths of local cues. The times to synchronization are given in Fig. 3. It can be seen that when all the relative phases are within the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, a stronger local cue is favorable to synchronization.

When the maximal/minimal relative phase is outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$ but is in $(-\pi, \pi)$, the evolution of the 9 oscillators' relative phases is illustrated in Fig. 4. All relative phases converge to zero or an integral multiple of 2π, which is equivalent to zero in the rotating reference framework. Hence the oscillator network is synchronized. The times to synchronization under a fixed local cue and different strengths of global cues are given in Fig. 5, which illustrates that the synchronization rate increases with an increase in the

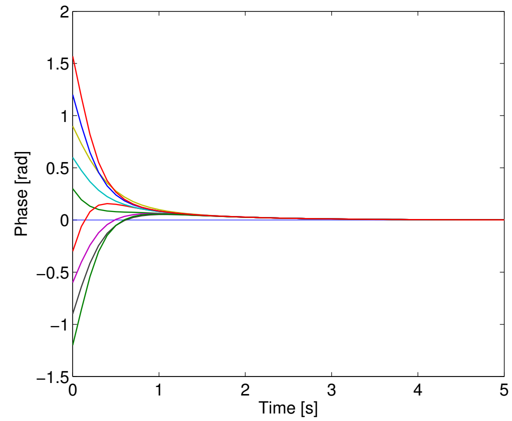

Fig. 1. Evolution of the relative phases $\xi_i(1 \leq i \leq N)$ (with all the relative phases within $[-\frac{\pi}{2}, \frac{\pi}{2}]$).

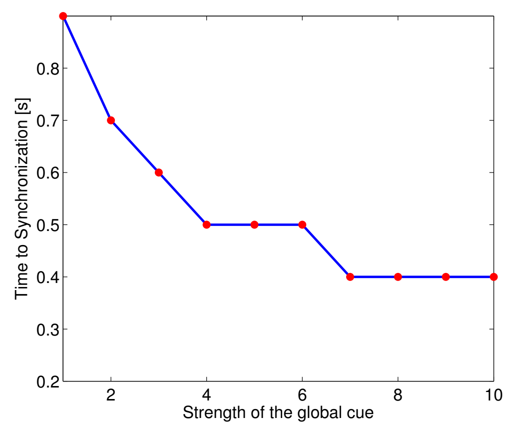

Fig. 2. Times to synchronization under different strengths of global cues and a fixed local cue (with all the relative phases within $[-\frac{\pi}{2}, \frac{\pi}{2}]$).

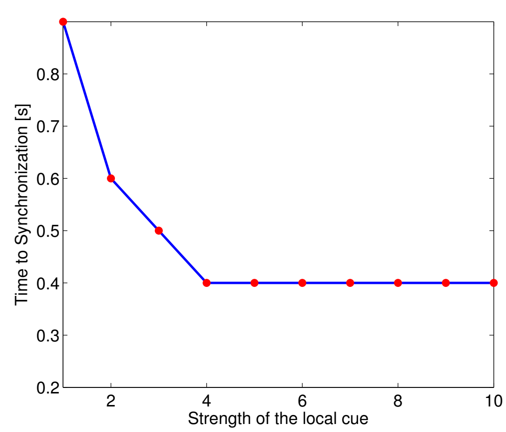

Fig. 3. Times to synchronization under different strengths of local cues and a fixed global cue (with all the relative phases within $[-\frac{\pi}{2}, \frac{\pi}{2}]$).

global cue. The times to synchronization under a fixed global cue and different strengths of local cues are given in Fig. 6. We can see that with an increase in the local cue, the synchronization rate may be increased or decreased, which confirms the theoretical results in Sec. III-B.

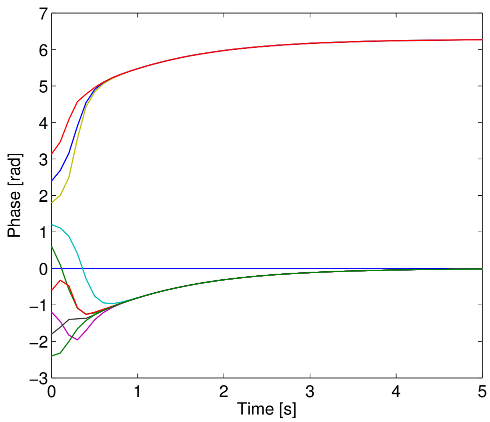

Fig. 4. Evolution of the relative phases $\xi_i (1 \leq i \leq N)$ (with the maximal/minimal relative phase outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$).

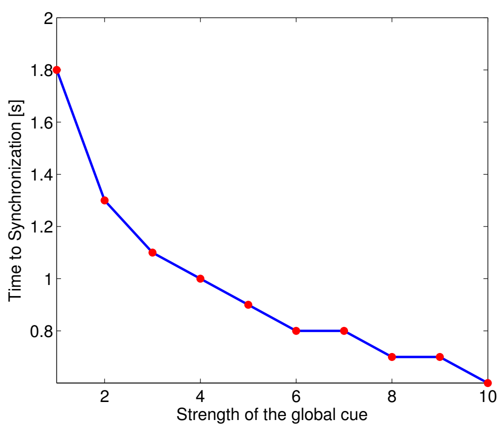

Fig. 5. Times to synchronization under different strengths of global cues and a fixed local cue (with the maximal/minimal relative phase outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$).

V. CONCLUSIONS

The influences of global and local cues on the synchronization rate of the Kuramoto model of sinusoidally interconnected oscillators are analyzed. First a global synchronization condition is proposed. It is one of the first conditions that ensure global synchronization of a general coupling structure when phase differences may be outside of the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Then, it is proven analytically that a stronger global cue always leads to a faster synchronization rate whereas a stronger local cue has no influence or increases the synchronization rate when the relative phases are within the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, and it may increase or decrease the synchronization rate when the relative phases cannot be bounded by that interval. Simulations results are given to illustrate the analytical results.

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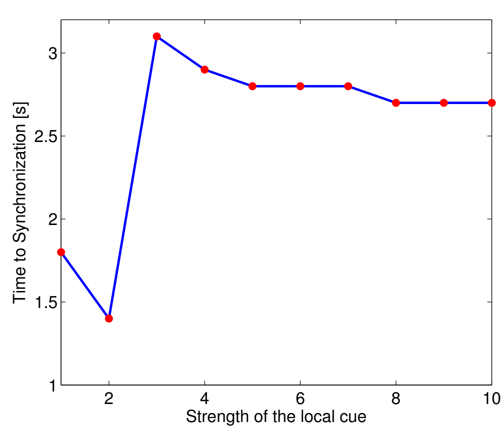

Fig. 6. Times to synchronization under different strengths of local cues and a fixed global cue (with the maximal/minimal relative phase outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$).