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Exhaustive Stability Analysis in a Consensus System with Time Delay and Irregular Topologies

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***Abstract*—A consensus problem and its stability for a group of agents with second order dynamics and communication delays is studied. Communication topologies are taken as irregular but always connected and undirected. The delays are assumed to be quasi-static and they remain the same for all the interagent channels. A decentralized PD-like control structure is proposed to create consensus in the position and velocity of the agents. We deploy a recent factorization technique for the characteristic equation of the system in order to simplify the stability analysis from a prohibitively large dimension to a small scale. Considering all possible topologies we reach a common stability picture utilizing a paradigm named CTCR. The influence of the individual factors on the absolute and relative stability of the system is studied, leading to the introduction of a novel concept of most exigent eigenvalue, for the one that defines the delay margin of the system, and the most critical eigenvalue, which dictates the consensus speed. It is shown that the most exigent eigenvalue is not always the most critical. Case studies and simulations results are presented to verify the analytical derivations.**

I. Introduction

The main objective in the consensus problem is to drive all the agents of the group to reach a common value in some variable of interest that may or may not be related to the motion of the agents. This problem became the center of attraction recently. The work of Olfati-Saber and Murray [1] is one of the earlier studies published, which introduces the consensus problem for multi agent coordination. They focus on agents with first order dynamics and considering different connected communication topologies among the agents. They propose a control that drives the agents to the average value of the group's initial conditions. Under the simplifying features of the first order governing dynamics, they also study the behavior of their protocol when communication delays are present, keeping the topology fixed. Several other researchers [2-4] have performed further extensions on this earlier work, proposing different protocols for *zero consensus* of agents that are driven by second order dynamics. In these studies, the consensus is reached for the target position and the velocity of the agents at zero. They also include variations on the topologies and communication delays. For the analysis of the stability in the delay space, the previous works use approximated methods, based on LMIs. Lin [2], particularly, presents a maximum bound in the time delay for which the system remains stable. None of these earlier works, however, presents an exact and exhaustive determination of the stability boundaries of the system in the time delay space, which is the key contribution of the CTCR paradigm [5-6] used in this work.

Recently [7-8], the authors proposed and analyzed a new control strategy for consensus over a group of agents with second order dynamics which are also operating under a time delayed communication structure. The main contribution of the previous work is the introduction of an interesting technique for the factorization of the characteristic equation in a very distinct form that considerably simplifies the stability analysis. This simplification reduces the problem from a treatment of a system whose order depends on the number of agents to the analysis of some second order sub-systems, each one corresponding to an eigenvalue of a matrix related to the communication topology. When delays are introduced, these subsystems exhibit no commensuracy (integer multiplicity of delays) therefore, they alleviate the complexity. In [8], this factorization technique has been applied to consensus protocols previously reported by different authors.

In this paper, we study the different factors of the characteristic equation and prove that the delay stability margin of the system is dictated by only one of the factors. We call this factor the *most exigent factor*, and the particular eigenvalue of a topology-defined matrix generating it the *most exigent eigenvalue*. We also study the relative stability of the system, introducing the concept of *most critical factor* and *most critical eigenvalue* (of the same matrix), which dictates the consensus speed of the system. For the particular protocol presented here, we show that the dominant eigenvalue is not always the most exigent one and that the consensus speed can be improved to some extent by increasing the communication delay.

II. System Dynamics and Stability Analysis

A. Consensus Protocol and Factorization of the CE

Consider a group of n autonomous agents, which are driven by second order dynamics given by $\ddot{x}_j = u_j$, j=1, 2,…, n, where $x_j \in \Re$ is taken as the scalar position and $u_j \in \Re$ as the control law. Here we treat the motion of the agent as one dimensional, but the entire analysis is still valid for higher dimensions. We declare consensus is achieved when all n agents reach the same position, i.e., $\lim_{t\to\infty} x_j - x_k = 0$ for any $j,k \in [1,n]$. Notice that this consensus definition does not state if the consensus is zero or not.

We assume the j-th agent exchanges its position and velocity information with a set of Δ_j agents, $\Delta_j \leq n-1$, which we call the *informers* of agent j and denote by N_j. This communication topology can be described by an irregular graph of n vertices with degrees Δ_j. It is also assumed that all these communication channels experience the same delay of τ seconds, i.e., agent j knows the position and velocity of its Δ_j informers with τ seconds delay. We consider a PD type control scheme on the agents as:

$$\ddot{x}_j = u_j = P\left(\sum_{k\in N_j}\frac{x_k(t-\tau)}{\Delta_j} - x_j(t)\right) + D\left(\sum_{k\in N_j}\frac{\dot{x}_k(t-\tau)}{\Delta_j} - \dot{x}_j(t)\right) \tag{1}$$

with the control gains P and D positive. The control law (1) tries to bring the position of agent j to the centrode of its informers and consequently the velocity to the velocity of that centrode, using the last known position and velocity of its informers. If the communication topology is connected, and the system is stable, all the agents converge to the same position and velocity. Notice that this protocol differs from all previously proposed schemes in that the communication delay affects the information coming from all other agents, but not the state of the j-th agent.

The dynamics of the whole system can be represented in the augmented state space as $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{x}(t-\tau)$, with the state vector $\mathbf{x} = [x_1 \quad \dot{x}_1 \quad x_2 \quad \dot{x}_2 \quad \cdots \quad x_n \quad \dot{x}_n]^T \in \Re^{2n}$ and:

$$\mathbf{A} = \mathbf{I}_n \otimes \begin{bmatrix} 0 & 1 \\ -P & -D \end{bmatrix},\ \mathbf{B} = \mathbf{C} \otimes \begin{bmatrix} 0 & 0 \\ P & D \end{bmatrix},\ \mathbf{A},\mathbf{B} \in \Re^{2n\times 2n} \tag{2}$$

where $\otimes$ denotes Kronecker multiplication [9], $\mathbf{I}_n$ is the $n\times n$ identity matrix, and $\mathbf{C} = \mathbf{\Delta}^{-1}\mathbf{A}_\Gamma$, is the product of the inverse of the degree matrix, $\mathbf{\Delta}^{-1} = \mathrm{diag}(1/\Delta_1, 1/\Delta_2, ..., 1/\Delta_n) \in \Re^{n\times n}$, of the graph representing the communication topology and its adjacency matrix $\mathbf{A}_\Gamma \in \Re^{n\times n}$.

The characteristic equation of this system,

$$\det\left(s\mathbf{I}_{2n} - \mathbf{A} - \mathbf{B}e^{-\tau s}\right) = 0, \tag{3}$$

required for the stability analysis is a quasi-polynomial of degree $2n$ with exhibit n degree of commensuracy in the delay terms, since the rank of matrix $\mathbf{B}$ is n. However, the particular structure of the matrices $\mathbf{A}$ and $\mathbf{B}$ introduces some unique features to the characteristic equation of the system that allow a reduction in the complexity of the problem. They are stated in several lemmas.

Lemma 1: The characteristic equation (3) can be expressed as a product of n factors of the form:

$$\prod_{j=1}^{n}\left[s^2 + (Ds+P)\left(1-\lambda_j e^{-\tau s}\right)\right] = 0 \tag{4}$$

where λ_j, j=1,2,…,n, are the eigenvalues of the newly defined *weighted adjacency matrix* $\mathbf{C}=\mathbf{\Delta}^{-1}\mathbf{A}_\Gamma$.

Proof: Let $\mathbf{T} \in \Re^{n\times n}$ be the matrix whose columns are the eigenvectors of $\mathbf{C}$. By construction, $\mathbf{C}$ is a *symmetrizable matrix* [10], therefore it is diagonalizable and has n linearly-independent eigenvectors. Thus a similarity transformation can convert it into a diagonal matrix, as $\mathbf{T}^{-1}\mathbf{C}\mathbf{T}=\mathbf{\Lambda}$, where $\mathbf{\Lambda}=\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ has nonzero entries equal to the eigenvalues of $\mathbf{C}$. To achieve this, we perform a state transformation $\mathbf{x} = (\mathbf{T}\otimes\mathbf{I}_2)\xi$, $\xi = [\xi_1\ \dot{\xi}_1\ \xi_2\ \dot{\xi}_2 \cdots \xi_n\ \dot{\xi}_n]^T \in \Re^{2n}$. From (2), the system dynamics in the new state becomes:

$$\begin{aligned}\dot{\xi} &= \left(\mathbf{T}^{-1}\otimes\mathbf{I}_2\right)\left(\mathbf{I}_n \otimes \begin{bmatrix} 0 & 1 \\ -P & -D \end{bmatrix}\right)(\mathbf{T}\otimes\mathbf{I}_2)\xi(t) \\ &\quad + \left(\mathbf{T}^{-1}\otimes\mathbf{I}_2\right)\left(\mathbf{C} \otimes \begin{bmatrix} 0 & 0 \\ P & D \end{bmatrix}\right)(\mathbf{T}\otimes\mathbf{I}_2)\xi(t-\tau)\end{aligned} \tag{5}$$

Using the features of $\otimes$ operation [9], we obtain:

$$\dot{\xi} = \left(\mathbf{I}_n \otimes \begin{bmatrix} 0 & 1 \\ -P & -D \end{bmatrix}\right)\xi(t) + \left(\mathbf{\Lambda} \otimes \begin{bmatrix} 0 & 0 \\ P & D \end{bmatrix}\right)\xi(t-\tau) \tag{6}$$

Since $\mathbf{I}_n$ and $\mathbf{\Lambda}$ are diagonal matrices, equation (6) represents a set of n decoupled second-order blocks of the form:

$$\begin{bmatrix} \dot{\xi}_j \\ \ddot{\xi}_j \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -P & -D \end{bmatrix}\begin{bmatrix} \xi_j(t) \\ \dot{\xi}_j(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ P\lambda_j & D\lambda_j \end{bmatrix}\begin{bmatrix} \xi_j(t-\tau) \\ \dot{\xi}_j(t-\tau) \end{bmatrix} \tag{7}$$

where j=1, 2,…, n. The characteristic equation of each block (7) becomes:

$$s^2 + (Ds+P)\left(1-\lambda_j e^{-\tau s}\right) = 0 \tag{8}$$

and the complete characteristic equation for the system is the product of these n factors of the form (8). That is,

$$\prod_{j=1}^{n}\left[s^2 + Ds\left(1-\lambda_j e^{-\tau s}\right) + P\left(1-\lambda_j e^{-\tau s}\right)\right] = 0 \tag{9}$$

QED.

Lemma 1 simplifies the stability treatment considerably, by transforming it from a $2n$-order system with time delays of commensuracy degree n (3) into n second order systems with single time delay, i.e., the factors in (9) with no commensuracy in the delay terms (i.e., no integer multiples of the delay appears). Although considerable, this simplification is not that helpful unless one finds a factor in (9) which always invites the instability first. We will later prove that there is such a factor and it belongs to the minimum λ.

It can further be proven that λ=1 is always an eigenvalue of $\mathbf{C}$. Then, the corresponding factor

$$s^2 + (Ds+P)\left(1-e^{-\tau s}\right) = 0 \tag{10}$$

is always a factor of the characteristic quasi-polynomial in (9). Without loss of generality, we will assign this eigenvalue to the state ξ_1. It can be shown that the eigenvector corresponding to ξ_1 is always $\mathbf{T}_1 = [1 \quad 1 \quad \cdots \quad 1]^T \in \Re^n$, and it is selected as the first column of the earlier defined matrix $\mathbf{T}$. In fact, it can be shown that this factor governs the dynamics of a weighted average of the positions of the agents with the degree of each agent, Δ_j,

$$\xi_1 = \frac{1}{\Psi}\sum_{j=1}^{n}\Delta_j x_j,\quad \Psi = \sum_{j=1}^{n}\Delta_j \tag{11}$$

We call ξ_1 the *weighted centrode*, which is really a **topology dependent centrode**. The other factors of the characteristic equation in (9) are related to the disagreement dynamics of the system, when they are stable the agents will reach consensus among themselves. This claim is stated in the following lemma.

Lemma 2: If all the agents in the group reach a consensus, the steady state value of ξ_1 will indeed be that consensus, whereas $\xi_j = 0$ for j=2, 3,…,n.

Proof: From the definition of the state **ξ**, we have col(ξ_1,ξ_2…,ξ_n)=$\mathbf{T}^{-1}$col(x_1,x_2…,x_n). If consensus is reached, that means $\lim_{t\to\infty} x_j = \bar{x}$, j=1, 2, 3,…,n. Then:

$$\lim_{t\to\infty}\{\,\mathrm{col}\,(\xi_1,\,\xi_2,\cdots,\xi_n)\} = \bar{x}\;\mathbf{T}^{-1}\begin{bmatrix}1 & 1 & \cdots & 1\end{bmatrix}^T \tag{12}$$

Since $\mathbf{T}^{-1}\mathbf{T}_1$=[1 0 … 0]$^{\mathbf{T}}$, (12) leads to $\lim_{t\to\infty}\xi_1 = \bar{x}$ and $\lim_{t\to\infty}\xi_j = 0$ for j=2, 3,…,n.

QED.

B. Stability Analysis

If the communication topology of the agents is fixed, the stability of each factor in (9) can be assessed separately, using the facilitating feature introduced by Lemma 1. Then the stability regions in the parametric space (P, D, τ) are intersected to find those compositions that produce stable operation for the whole system. We can also claim that this common zone in (P, D, τ) space renders stability robustness bounds of the system against variations of one (or all) of the three parameters.

The stability analysis for each factor of (9) is performed following the Cluster Treatment of Characteristic Roots (CTCR) methodology [5-6]. The first step of CTCR requires an exhaustive determination of the possible imaginary roots of the characteristic equation. For this, a procedure is used, which is similar to the one presented in the analysis of the Delayed Resonator active vibration absorption system [11].

Before engaging the stability analysis, it is important to pay attention to the eigenvalues of **C**. From the way in which this matrix is created, its main diagonal has only zeroes, and the summation of all the elements in any row is always 1. Then, based on the Gershgorin disk theorem [12], we can claim that all the eigenvalues of **C** are in the interval [–1, 1], with λ_1=1 being always present. This boundedness of the eigenvalues of **C** will be exploited later.

The generic form of each individual factor is:

$$s^2 + Ds + P = \lambda(Ds + P)e^{-\tau s} \tag{13}$$

For a possible imaginary root $s=\omega i$, the magnitude and phase of both sides of (13) should be equal. From the magnitude equality, we obtain:

$$\gamma^2 + \left(\mu D^2 - 2P\right)\gamma + \mu P^2 = 0 \tag{14}$$

where $\gamma=\omega^2$ and $\mu = 1-\lambda^2 \in [0,1]$. All the real positive roots of (14) represent imaginary roots of (13). Equation (14) is quadratic of the form $\gamma^2+b\gamma+c=0$ and it will have real roots if its discriminant b^2–$4c$ is positive. Since λ^2_j<1, μ>0, those roots will both be positive if b is negative. Furthermore b>0 yields no positive real root. It is trivial to show that, the positive-real root generating two conditions of b<0 and b^2–$4c$>0 reduce in the following single relation between P, D and μ:

$$D^2 < \left(2P\left(1-\sqrt{\mu}\right)\right)/\mu \tag{15}$$

Within the range defined in (15), equation (13) has two crossings on the imaginary axis, at the frequencies given by the solutions of (14):

$$\gamma_{1,2} = P - \frac{\mu}{2}D^2 \pm \frac{\sqrt{\Delta}}{2} \tag{16}$$

where $\Delta = \mu^4 D^4 - 4\mu^2 D^2 P + 4P^2(1-\mu)$ is the discriminant of (14). Outside this range, no real roots exist thus no stability change occurs.

It is not hard to see that the interval of D defined in (15) increases monotonically as μ decreases, and it becomes infinite for μ=0, which corresponds to λ=±1. In this case, the solutions to (14) are ω=0 and $\omega=\sqrt{2P}$. It can be proven that the first solution is actually a stationary root at s=0, whereas the second one is a stability switching point.

The phase equivalency of (13) yields:

$$\angle e^{-i\omega\tau} = \angle\left(P - \omega^2 + D\omega i\right) - \angle\left(P + D\omega i\right) - \angle\lambda \tag{17}$$

where the angle of λ is 0 if λ>0 or π if λ<0. Further developing (16), we arrive at:

$$\tau_\ell = \frac{1}{\omega}\left(\arctan\left(\frac{-D\omega^3}{P^2 + \omega^2\left(D^2 - P\right)}\right) + 2\ell\pi\right) \quad \text{for } \lambda>0 \tag{18a}$$

$$\tau_\ell = \frac{1}{\omega}\left(\arctan\left(\frac{-D\omega^3}{P^2 + \omega^2\left(D^2 - P\right)}\right) + (2\ell+1)\pi\right) \quad \text{for } \lambda<0 \tag{18b}$$

Using equations (14) and (18), the imaginary roots of each factor and the respective delays can be determined. When τ increases from the value given in (18), these imaginary roots could cross to the left or to the right half of the complex plane. The direction of this crossing is found by evaluating the root tendency [5], defined as:

$$RT = \mathrm{sgn}\left[\mathrm{Re}\left(\frac{ds}{d\tau}\right)\right]\Bigg|_{P,D,\tau=\tau_l,s=\omega i} \tag{19}$$

If RT is –1, the root moves to the left, stabilizing the system, whereas if it is 1, the root moves to the right, introducing instability. The second proposition of CTCR [5], states that RT is invariant for a given imaginary root which is created by the periodically distributed set of delays determined by (17). This property is crucial in handling this infinite dimensional systems described by (9) (a.k.a. quasi-polynomials).

From (13), the root sensitivity is:

$$\frac{ds}{d\tau} = \frac{s\,\lambda_j(Ds+P)\,e^{-\tau s}}{2s + D + \lambda_j\left[\tau(Ds+P) - D\right]e^{-\tau s}} \tag{20}$$

and the root tendencies for the crossings are found using the definition (18) .

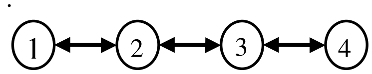

Fig. 1: Communication topology used in the examples.

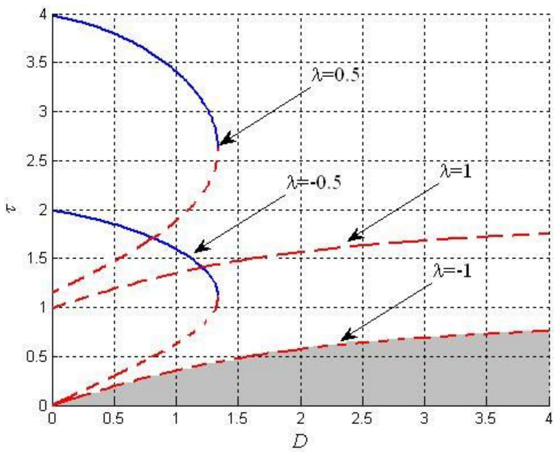

Fig. 2: Stability boundaries in (D, τ) domain for the communication topology of Fig. 1 with P=5. Shaded region depicts the stable zone.

Using equations (14), (15), (18), and (20), the stability of the whole system can be determined in 3-D parametric domain for any combination of parameters P, D and time delay τ, by intersecting the stability regions of each factor.

As an example of this construction, consider four agents interacting under a simple communication topology of Fig. 1. The **C** matrix corresponding to this topology has the eigenvalues: ±0.5 and ±1. The factors generated by these eigenvalues produce the stability outlook presented in Fig. 2 for constant P=5 and changing D and τ. The shaded region represents the absolute stability zone: parametric selections inside this zone make the agents reach consensus at a constant position and zero velocity. The red dashed lines indicate destabilizing crossings as τ increases, and the blue solid lines mark stabilizing crossings.

From Fig. 2, it is clear that the factor corresponding to λ=–1 has the most restrictive stability boundary for this topology. The shaded region declares the stability region for the entire system. Since the stability boundary is determined by –1 eigenvalue, one may be tempted to think that this factor also introduces the dominant pole of the system for the entire parametric settings inside the shaded region of stability. That is, the behavior of the factor with λ=–1 determines the settling time for the combined system consensus. It turns out that this observation is not true.

In the following section, we present two critical findings:

a) The concept of *most exigent* λ is introduced, and we show that it is always the **smallest eigenvalue** of the weighted adjacency matrix **C**.

b) We analyze the particular factors which define the consensus speed for a particular setting of the parameters (P, D, and τ), leading to the definition of the *most critical* λ.

III. Most Exigent and Most Critical λ

A. *Most Exigent* λ

Definition 1: For a group of agents interacting under the consensus protocol defined in (1), the *most exigent* λ, is the eigenvalue of the weighted adjacency matrix **C** that generates the particular factor in the characteristic equation which introduces the first destabilizing crossing as τ increases for fixed P and D.

Lemma 3: For the swarm dynamics and consensus protocol described by (1) and a given set of control parameters (P, D), the first destabilization always appears as τ increases, due to that factor of the characteristic equation (9) corresponding to the smallest eigenvalue of the weighted adjacency matrix **C**.

Proof: From (14), it is clear that factors corresponding to $\pm\lambda$ generate the same crossing frequencies. Furthermore from (18), it is easy to see that, the factor with $-\lambda$ has a smaller τ, so negative eigenvalues generate the dominant roots which always cross before those generated by the positive eigenvalue λ. We wish to study the sensitivity of this crossing delay τ with respect to λ for those critical factors with λ<0. From (18b), we can see that τ is a function of only ω. Then, we need to see first how ω changes with λ, and then how ω affects τ.

Using implicit differentiation with respect to μ in (14):

$$\frac{d\gamma}{d\mu} = \frac{-\left(D^2\gamma + P^2\right)}{2\gamma + \mu D^2 - 2P} \tag{21}$$

and replacing (16) in (21), it is trivial to see that $d\gamma/d\mu$ is negative for the larger of the two solutions, obtained using the + sign in (16), and positive for the smaller one. Since $\gamma=\omega^2$, the sign of $d\omega/d\mu$ is the same of $d\gamma/d\mu$. Then, the minimum μ, μ=0 (λ=±1), produces both the smallest, ω=0, and the largest, $\omega=\sqrt{2P}$, possible crossing frequencies.

The first crossing delay for factors with λ<0 is given by (18b), which can be expressed as:

$$\tau_0\omega = \arctan\left(\frac{-D\omega^3}{P^2 + \omega^2\left(D^2 - P\right)}\right) + \pi \tag{22}$$

By using implicit differentiation w.r.t. ω in (22) we obtain:

$$\frac{d\tau_0}{d\omega} = \frac{1}{\omega}\left(-\frac{\left(3P^2 + \left(D^2 - P\right)\omega^2\right)\omega^2 D}{\left(P^2 + D^2\omega^2\right)\left(\left(P - \omega^2\right)^2 + D^2\omega^2\right)} - \tau_0\right) \tag{23}$$

Since P, D, ω, τ_0 are all positive, this derivative is always negative if $\left(D^2 - P\right)\omega^2 + 3P^2 > 0$, and this condition is always satisfied when D^2>P. If D^2<P, $\omega^2 < 3P^2/\left(P - D^2\right)$ will guarantee the objective. But the latter condition is always satisfied, given the maximum crossing frequency $\omega=\sqrt{2P}$. We conclude that $d\tau_0/d\omega$ is always a negative quantity, implying that increasing crossing frequency ω, always decreases the corresponding delay τ. Thus the smallest τ corresponds to the largest frequency.

With the aid of the chain rule, we can state that $d\tau_0/d\mu = (d\tau_0/d\omega)(d\omega/d\mu)$. It was already established that $d\omega/d\mu$<0 for the larger solution of (14), which, according to (23), introduces the first crossing. Then, $d\tau_0/d\mu$>0: the crossing delay increases as μ does, but, since μ=1–λ^2, τ_0 smaller corresponds to larger λ^2. Since we are considering only factors with $\lambda<0$, larger λ^2 in fact implies smaller λ. Thus the smallest τ_0, which corresponds to the first stability change, is introduced by the smallest λ, and it is, by definition, the **most exigent λ** from the stability perspective.

QED.

The concept of the most exigent λ is extremely important. By knowing this λ, we simply reduce the complexity of stability analysis of the consensus system, given by the characteristic equation (3), to the stability assessment of a single factor in this equation given in (9). The reduction in complexity is phenomenal, a quasi-polynomial of 2n degree with n degree of commensuracy in the delay terms in (3), is reduced to analyzing a single 2nd order quasi-polynomial with no commensuracy, which is the most exigent factor in (9). As the number of agents, n, increases, this feature becomes extremely helpful.

The above discussion is on the stability boundary of (1) being dictated by the most exigent λ of **C**, the weighted adjacency matrix. In the following Section, we study the behavior of the system inside the stable region. The objective is to determine the critical factor of (9) which introduces the dominant eigenvalue, and thus the consensus speed, of the system. We show that, unexpectedly, this dominant pole is not necessarily introduced by the factor corresponding to the most exigent λ, responsible for the stability boundary.

B. *Most Critical λ*

Definition 2: For a group of agents interacting under the consensus protocol defined in (1), the *most critical* λ is the eigenvalue of the matrix **C** that generates the factor of the characteristic equation which carries the supremum of the rightmost poles for a given P, D and τ composition. This dominant eigenvalue defines the consensus speed of the system

In order to find the dominant eigenvalue, we study the dominant root of each factor as in (9) using the QPmR routine of [13]. Without loss of generality, we use again the example communication topology of Fig. 1, which has a complete eigenvalue set of –1, –0.5, 0.5 and 1. We exclude from this analysis the factor (10), generated by λ=1, as it is associated with the centrode dynamics (see Section III) and does not affect the consensus speed.

In Fig. 3, we associate the rightmost root of the system for varying D and τ and constant P=5. The thick white line is the absolute stability boundary, created by λ=–1, the smallest, which is therefore the most exigent λ for this communication topology. For a given D value, delays beyond that line render the system unstable. For zero delay, the dominant eigenvalue is introduced by the factor created by λ=0.5, so it is the most critical λ for small delays. This dominant root moves to the left when τ increases. However, the roots of the factor corresponding to λ=–1 must be moving to the right at the same time so that it becomes the most critical λ for larger delays (e.g., τ=0.4, and D=1.5).

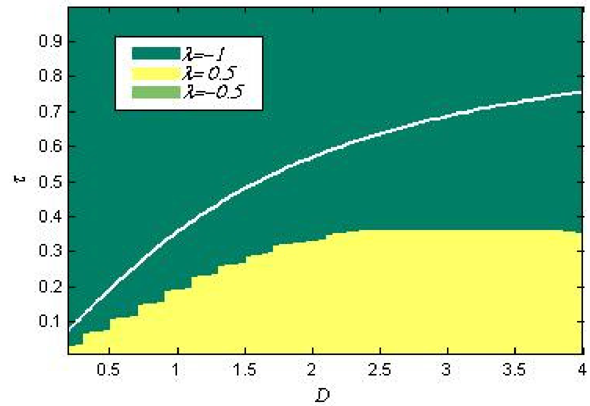

Fig. 3: Eigenvalue generating the dominant root of the system as a function of D and τ for different constant P=5. The white solid line indicates the absolute stability boundary.

Figure 3 clearly shows what was stated earlier: the most exigent λ does not create the dominant pole of the system within the entire stability region. We have observed in several different examples that the dominant eigenvalue for small delays is always generated by the largest λ, until some boundary at which the smallest λ (i.e., the most exigent λ) brings the dominant eigenvalue. This concept is tortuous to prove analytically. We are reporting this phenomenon here leaving the complete proof to a future publication.

Figure 4 shows the variation of Re(s_{dom}) as the delay increases, with P=5 and D=1.6. It is clear that for $0<\tau<0.3$ seconds, the real part of the dominant root of the system is reduced. This shows that, inside the stable region, larger delays can increase the consensus speed. The following example cases verify these observations.

C. *Example Cases*

This subsection presents three different cases of the behavior of four agents using the control law (1) and interacting under the communication topology of Fig. 2.

In the first case, the control parameters are set as P=5, D=1.6 and τ=0.1, corresponding to point (a) in Fig. 4. For the second case the parametric values are P=5, D=1.6 and τ=0.28, point (b) in Fig. 4. Finally, P=5, D=1.6 and τ=0.35 are used in the third case and they correspond to point (c) in Fig. 4.

Figure 5 shows the evolution in time of the norm of the disagreement vector $\xi' = \mathbf{col}(\xi_2, \xi_3, \xi_4)$. It is clear that the agents reach consensus faster in the second case, despite the higher delay. This parametric selection actually corresponds to the optimum delay for the given values of P and D.

The traces of the transformed position states ξ_2, ξ_3 and ξ_4, in the three cases are shown in Fig. 6. In this plot it is possible to appreciate that the last factor to settle, is different in each case, and corresponds to the dominant eigenvalue for each case. For τ=0.1, for example, the red dashed line, corresponding to λ=0.5, has the largest time constant, whereas for τ=0.35, the green solid line oscillates longer than the other two. Furthermore, the time constants of the dominant roots in each case agree with the values in Fig. 4.

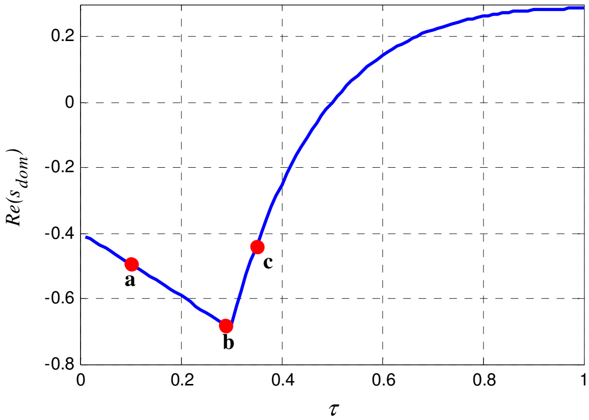

Fig. 4: Real part of the dominant root of the system for P=5, D=1.6, as a function of τ.

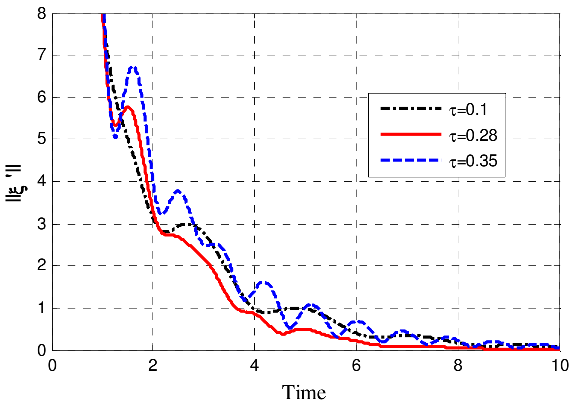

Fig. 5: Time evolution of the disagreement vector for different τ values.

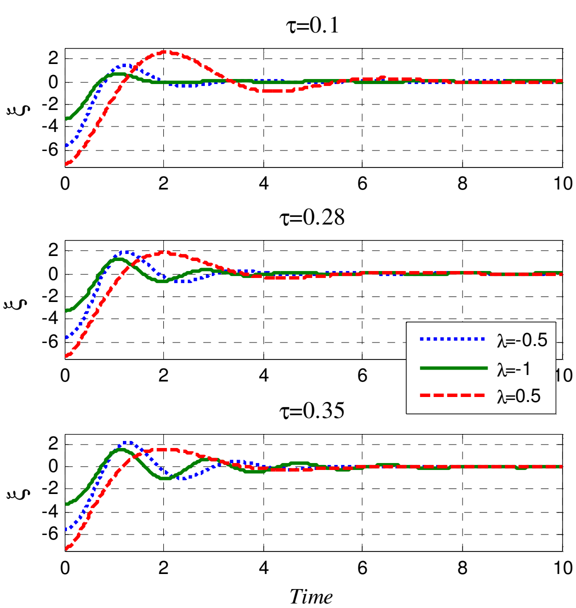

Fig. 6: Evolution of the individual components of the disagreement vector for different τ values.

IV. Conclusions

This paper proposes a consensus protocol for a system of n agents with second order dynamics, under the assumption that there is a constant communication delay in all the communication channels. Procedures and novel concepts are all developed for irregular and undirected topologies.

The stability of the multi agent system, for fixed topologies is analyzed taking advantage of a particular construction of the characteristic equation of the system introduced by the proposed control logic. This control law simplifies the stability analysis tremendously, to the study of a small number of second order delayed dynamics.

The concept of *most exigent* λ, the eigenvalue of the weighted adjacency matrix of the communication topology that defines the absolute stability boundary of the consensus system is introduced to further simplify the stability analysis. We prove that the most exigent λ is always the smallest one.

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