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Dynamic Anti-windup Design in Anticipation of Actuator Saturation

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***Abstract*— In a traditional anti-windup design, the anti-windup mechanism is set to be activated as soon as the control signal saturates the actuator. A recent innovation of delaying the activation of the anti-windup mechanism, both static and dynamic, until the saturation reaches to a certain level of severity has led to a performance improvement of the resulting closed-loop system. More recently, it has been shown that significant further performance improvements can be obtained by activating a static anti-windup mechanism in anticipation of actuator saturation, in comparison with the delayed activation design. This paper demonstrates that anticipatory activation of a dynamic anti-windup mechanism would also lead to significant performance improvements over both the immediate and delayed activation schemes.**

I. INTRODUCTION

Anti-windup, because of its intuitive motivation and its effectiveness in dealing with actuator saturation, has been widely used in industries. Many methods for designing anti-windup compensators have been developed (see, *e.g.*, [1, 3, 5–7, 9–11, 14, 16, 17] for a small sample of the literature). In all these designs, the anti-windup mechanism is set to take effect as soon as actuator saturation occurs.

Motivated by the observation that, leaving the controller to act unassisted in the face of slight or moderate saturation, the robustness properties of the nominal closed-loop system might be more effective in overcoming the adverse effect of the saturation than the anti-windup mechanism could, Sajjadi and Jabbari in a pair of recent papers [12] and [13] propose to design the anti-windup compensators, both static and dynamic ones, for delayed activation (see Fig. 2). Simulation results indicate that the delayed activation design scheme can indeed lead to significantly better transient performance in output tracking than the anti-windup compensator designed for immediate activation.

Inspired by the work [12] and by an intuition that the dynamic nature of the system entails a preventive action to be taken before the actuator saturation actually occurs, we have just proposed to activate the anti-windup mechanism in anticipation of actuator saturation [18] (see Fig. 3). The resulting design leads to significant further performance improvement over the delayed activation design scheme of [12]. The delayed activation anti-windup design was developed and its performance improvement over the immediate activation design has been shown for both static anti-windup mechanism and dynamic anti-windup mechanism, in [12] and [13], respectively. The anticipatory anti-windup design in [18] has however only been developed for static anti-windup.

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The objective of this paper is to develop a dynamic anti-windup design for anticipatory activation and to establish its performance improvement over similar designs for immediate and delayed activation. As in [12], we will develop the anticipatory dynamic anti-windup design and carry out the comparison in the LMI based dynamic anti-windup design framework ([7] and [11]). We note that both the signs for delayed activation and anticipatory activation can be viewed to belong to the general category of nonlinear anti-windup schemes (see, for example, [4, 15, 19]).

The remainder of this paper is organized as follows. In Section II, we recall the formulation of the traditional anti-windup design problem. In Section III, after recalling the traditional dynamic $\mathcal{L}_2$ anti-windup design for immediate activation ([7, 11]) and the delayed activation dynamic $\mathcal{L}_2$ anti-windup design of [13], we develop the anticipatory dynamic anti-windup design. Simulation results that demonstrate the performance improvements of the anticipatory dynamic anti-windup design over the immediate and delayed activation designs are given Section IV. A brief conclusion in Section V ends the paper.

II. PROBLEM FORMULATION: DYNAMIC $\mathcal{L}_2$ ANTI-WINDUP DESIGN

Consider a linear plant subject to actuator saturation

$$\Sigma_\mathrm{p}:\ \begin{cases}\dot{x}_\mathrm{p} = A_\mathrm{p}x_\mathrm{p}+B_1w+B_2\mathrm{sat}_h(u),\\ y=C_2x_\mathrm{p}+D_{21}w,\\ z=C_1x_\mathrm{p}+D_{11}w+D_{12}\mathrm{sat}_h(u),\end{cases} \tag{1}$$

where $x_\mathrm{p}\in\mathbf{R}^{n_\mathrm{p}}$ is the state, $u\in\mathbf{R}$ is the control input, $w\in\mathbf{R}^{n_w}$ is the disturbance, $y\in\mathbf{R}^{n_y}$ is the measurement output, $z\in\mathbf{R}^{n_z}$ is the performance output, and, for a given constant $h>0$, the function $\mathrm{sat}_h:\mathbf{R}\to\mathbf{R}$ is a saturation function defined as $\mathrm{sat}_h(u)=\mathrm{sgn}(u)\min\{|u|,h\}$.

Assume that a linear dynamic controller of the form

$$\Sigma_\mathrm{c}:\ \begin{cases}\dot{x}_\mathrm{c}=A_\mathrm{c}x_\mathrm{c}+B_{\mathrm{c}y}y+B_{\mathrm{c}w}w, \quad x_\mathrm{c}\in\mathbf{R}^{n_\mathrm{c}},\\ u=C_\mathrm{c}x_\mathrm{c}+D_{\mathrm{c}y}y+D_{\mathrm{c}w}w,\end{cases} \tag{2}$$

has been designed that achieves the closed-loop performance specifications in the absence of actuator saturation. A traditional dynamic anti-windup design is to add to the controller a correction signal generated from a linear dynamic anti-windup compensator driven by $q=u-\mathrm{sat}_h(u)$, that is,

$$\begin{cases}\dot{x}_\mathrm{c}=A_\mathrm{c}x_\mathrm{c}+B_{\mathrm{c}y}y+B_{\mathrm{c}w}w+\eta_1, \quad x_\mathrm{c}\in\mathbf{R}^{n_\mathrm{c}},\\ u=C_\mathrm{c}x_\mathrm{c}+D_{\mathrm{c}y}y+D_{\mathrm{c}w}w+\eta_2,\end{cases} \tag{3}$$

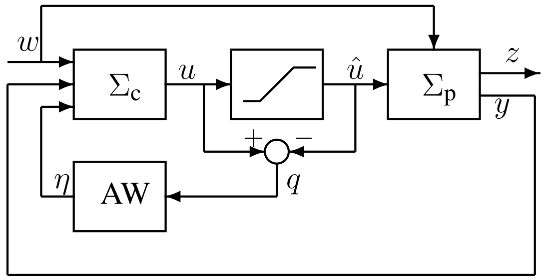

Fig. 1. Traditional anti-windup scheme: immediate activation at the occurrence of saturation.

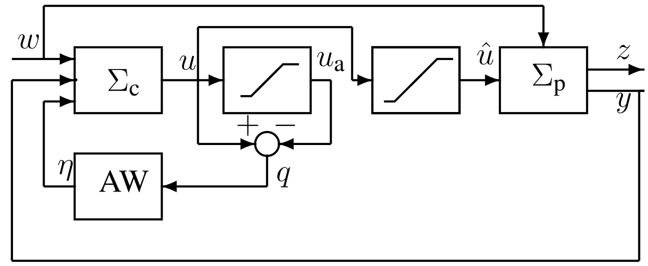

Fig. 3. Anticipatory anti-windup scheme: anticipatory activation of the anti-windup mechanism.

where $\eta = [\eta_1^{\mathrm{T}}\ \eta_2^{\mathrm{T}}]^{\mathrm{T}}$ is the output of the linear dynamic anti-windup compensator

$$\Sigma_{\mathrm{aw}}:\begin{cases}\dot{x}_{\mathrm{aw}} = A_{\mathrm{aw}}x_{\mathrm{aw}} + B_{\mathrm{aw}}q, & q \in \mathbf{R}^{n_{\mathrm{aw}}},\\ \eta = C_{\mathrm{aw}}x_{\mathrm{aw}} + D_{\mathrm{aw}}q.\end{cases} \tag{4}$$

The resulting overall closed-loop system is illustrated in Fig. 1. As the term $u - \mathrm{sat}_h(u)$, which drives the dynamic anti-windup compensator, becomes non-zero as soon as actuator saturation occurs, the anti-windup takes effect immediately after the saturation. Such an anti-windup design can thus be referred to as the anti-windup design for immediate activation.

The closed-loop system with a delayed activation anti-windup compensator, as proposed in [12] and [13], is depicted in Fig. 2. The closed-loop system with a anticipatory activation anti-windup compensator, as proposed in [18], is depicted in Fig. 3.

Under any of these three activation strategies, the problem of dynamic anti-windup design is to compute the anti-windup dynamic compensation coefficient matrices $(A_{\mathrm{aw}}, B_{\mathrm{aw}}, C_{\mathrm{aw}}, D_{\mathrm{aw}})$ to meet various performance indices. In this paper, the design objective is to minimize the $\mathcal{L}_2$ gain from the disturbance w to the controlled output z. The dynamic $\mathcal{L}_2$ anti-windup designs have been developed for immediate activation (see, *e.g.*, [7, 11]) and for delayed activation (see, [13]). Here, we will develop a dynamic $\mathcal{L}_2$ anti-windup designs for anticipatory activation and compares the performance of the resulting closed-loop system with those resulting from the immediate activation and delayed activation designs.

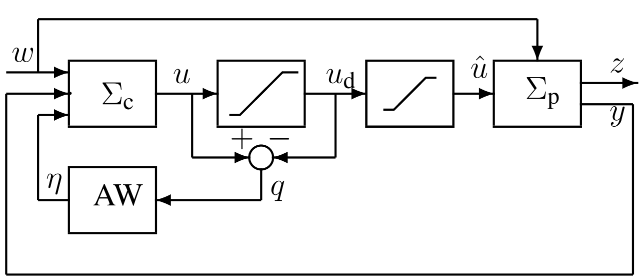

Fig. 2. Dynamic delayed anti-windup scheme: delayed activation of the anti-windup mechanism.

III. $\mathcal{L}_2$ Anti-Windup Design

For the purpose of comparison, we will first recall the designs for immediate activation and for delayed activation, from [7] and [11] and from [13], respectively. In our development of anticipatory dynamic anti-windup design, we will use similar notation as used in [7, 11] and [13], with some minor modification to accommodate the different activation schemes.

A. Traditional Design for Immediate Activation

Let $x = \left[x_{\mathrm{p}}^{\mathrm{T}}\ x_{\mathrm{c}}^{\mathrm{T}}\ x_{\mathrm{aw}}^{\mathrm{T}}\right]^{\mathrm{T}}$ and $\Lambda = [B_{\mathrm{aw}}^{\mathrm{T}}\ D_{\mathrm{aw}}^{\mathrm{T}}]^{\mathrm{T}}$. Then, the closed-loop system can be written as

$$\Sigma:\begin{cases}\dot{x} = Ax + B_w w + (B_q - B_\eta\Lambda)q,\\ z = C_z x + D_{zw}w + (D_{zq} - D_{z\eta}\Lambda)q,\\ u = C_u x + D_{uw}w + (D_{uq} - D_{u\eta}\Lambda)q,\end{cases} \tag{5}$$

where

$$A = \begin{bmatrix}\hat{A} & B_\eta C_{\mathrm{aw}}\\ 0 & A_{\mathrm{aw}}\end{bmatrix},\ B_w = \begin{bmatrix}\hat{B}_w\\ 0\end{bmatrix},\ B_q = \begin{bmatrix}\hat{B}_q\\ 0\end{bmatrix},$$

$$B_\eta = \begin{bmatrix}0_{(n_{\mathrm{p}}+n_{\mathrm{c}})\times n_{\mathrm{aw}}} & -\hat{B}_{\eta\ (n_{\mathrm{p}}+n_{\mathrm{c}})\times(n_u+n_{\mathrm{c}})}\\ -I_{n_{\mathrm{aw}}} & 0_{n_{\mathrm{aw}}\times(n_u+n_{\mathrm{c}})},\end{bmatrix},$$

$$C_z = \begin{bmatrix}\hat{C}_z & D_{z\eta}C_{\mathrm{aw}}\end{bmatrix},\ D_{zw} = \hat{D}_{zw},\ D_{zq} = \hat{D}_{zq},$$

$$D_{z\eta} = -\hat{D}_{z\eta}\begin{bmatrix}0_{(n_u+n_{\mathrm{c}})\times n_{\mathrm{aw}}} & I_{(n_u+n_{\mathrm{c}})}\end{bmatrix},$$

$$C_u = \begin{bmatrix}\hat{C}_u & D_{u\eta}C_{\mathrm{aw}}\end{bmatrix},\ D_{uw} = \hat{D}_{uw},\ \ D_{uq} = \hat{D}_{uq},$$

$$D_{u\eta} = \begin{bmatrix}0_{n_u\times n_{\mathrm{aw}}} & -\hat{D}_{u\eta\ n_u\times(n_u+n_{\mathrm{c}})}\end{bmatrix},$$

and where

$$\hat{A} = \begin{bmatrix}A_{\mathrm{p}} + B_2 D_{\mathrm{c}y}C_2 & B_2 C_{\mathrm{c}}\\ B_{\mathrm{c}y}C_2 & A_{\mathrm{c}}\end{bmatrix},\ \hat{B}_q = \begin{bmatrix}-B_2\\ 0\end{bmatrix},$$

$$\hat{B}_\eta = \begin{bmatrix}0 & B_2\\ I & 0\end{bmatrix},\ \hat{B}_w = \begin{bmatrix}B_2(D_{\mathrm{c}y}D_{21} + D_{\mathrm{c}w})\\ B_{\mathrm{c}y}D_{21} + B_{\mathrm{c}w}\end{bmatrix},$$

$$\hat{C}_z = \begin{bmatrix}C_1 + D_{12}D_{\mathrm{c}y}C_2 & D_{12}C_{\mathrm{c}}\end{bmatrix},$$

$$\hat{C}_u = \begin{bmatrix}D_{\mathrm{c}y}C_2 & C_{\mathrm{c}}\end{bmatrix},\ \hat{D}_{z\eta} = \begin{bmatrix}0 & D_{12}\end{bmatrix},$$

$$\hat{D}_{zw} = D_{11} + D_{12}D_{\mathrm{c}y}D_{21} + D_{12}D_{\mathrm{c}w},\ \ \hat{D}_{zq} = -D_{12},$$

$$\hat{D}_{uw} = D_{\mathrm{c}y}D_{21} + D_{\mathrm{c}w},\ \hat{D}_{uq} = 0,\ \hat{D}_{u\eta} = \begin{bmatrix}0 & I\end{bmatrix}.$$

The design of the dynamic $\mathcal{L}_2$ anti-windup compensator for immediate activation, as developed in [7, 11], is based on the solution of an LMI problem. We recall from [13] the following theorem, on which the LMI problem can be formulated.

Theorem 1: The closed-loop system (5) is stable and the $\mathcal{L}_2$ gain from w to z is less than γ if there exist a scalar $M > 0$, a symmetric $Q > 0$ and anti-windup matrices A_a, C_a and $\Lambda = [B_{aw}^T\ D_{aw}^T]^T$ such that

$$\begin{bmatrix} AQ+QA^T & * & * & * \\ B_w^T & -\gamma I & * & * \\ C_zQ & D_{zw} & -\gamma I & * \\ \Phi_{41} & D_{uw} & \Phi_{43} & \Phi_{44} \end{bmatrix} < 0, \tag{6}$$

where $\Phi_{41} = MB_q^T - M\Lambda^T B_\eta^T + C_uQ$, $\Phi_{43} = MD_{zq}^T - M\Lambda^T D_{z\eta}^T$, and $\Phi_{44} = -2M + D_{uq}M + MD_{uq}^T - D_{u\eta}\Lambda M - M\Lambda^T D_{u\eta}^T$.

To express (6) into an LMI, let

$$Q = \begin{bmatrix} Y & S \\ S & S \end{bmatrix}, \tag{7}$$

and define $F_1 = A_{aw}S$, $F_2 = C_{aw}S$, $F_3 = B_{aw}M$ and $F_4 = D_{aw}M$. Then, (6) reduces to the following LMI,

$$\begin{bmatrix} \Omega_{11} & * & * & * & * \\ \Omega_{12}^T & F_1+F_1^T & * & * & * \\ \hat{B}_w^T & 0 & -\gamma I & * & * \\ \Omega_{41} & \Omega_{42} & \hat{D}_{zw} & -\gamma I & * \\ \Omega_{51} & \Omega_{52} & \hat{D}_{uw} & \Omega_{54} & \Omega_{55} \end{bmatrix} < 0, \tag{8}$$

where $\Omega_{11} = \hat{A}Y + Y\hat{A}^T + \hat{B}_\eta F_2 + F_2^T\hat{B}_\eta^T$, $\Omega_{12} = \hat{A}S + \hat{B}_\eta F_2 + F_1^T$, $\Omega_{41} = \hat{C}_zY + \hat{D}_{z\eta}F_2$, $\Omega_{42} = \hat{C}_zS + \hat{D}_{z\eta}F_2$, $\Omega_{51} = M\hat{B}_q^T + F_4^T\hat{B}_\eta^T + \hat{C}_uY + \hat{D}_{z\eta}F_2$, $\Omega_{52} = F_3^T + \hat{C}_uS + \hat{D}_{u\eta}F_2$, $\Omega_{54} = M\hat{D}_{zq}^T + F_4^T\hat{D}_{z\eta}^T$, and $\Omega_{55} = -2M + \hat{D}_{uq}M + M\hat{D}_{uq}^T + \hat{D}_{u\eta}F_4 + F_4^T\hat{D}_{u\eta}^T$.

Clearly, if the LMI (8) is feasible, then the dynamic anti-windup matrices can be obtained as $A_{aw} = F_1S^{-1}$, $B_{aw} = F_3M^{-1}$, $C_{aw} = F_2S^{-1}$ and $D_{aw} = F_4M^{-1}$.

B. Design for Delayed Activation

Under a delayed activation anti-windup compensator, the closed-loop system, as depicted in Fig. 2, can be described as

$$\Sigma_d: \begin{cases} \dot{x} = A(g)x + B_w(g)w + (B_q(g) - B_\eta(g)\Lambda)q, \\ z = C_z(g)x + D_{zw}(g)w + (D_{zq}(g) - D_{z\eta}(g)\Lambda)q, \\ u = C_u(g)x + D_{uw}(g)w + (D_{uq}(g) - D_{u\eta}(g)\Lambda)q, \end{cases} \tag{9}$$

where all the matrices are the same as in Section III-A, except with B_2 and D_{12} respectively replaced with gB_2 and gD_{12} and the dependency on g of the resulting matrices are explicitly marked. Let $h > 0$ be the level of the actuator saturation and h/g_d, $g_d \in (0,1)$, be the level of the additional saturation introduced to delay the actuation of the anti-windup mechanism. Letting $\hat{u}(t) = g(t)u(t)$, $g(t) \in [g_d, 1]$, to result in a pseudo LPV closed-loop system when $|u(t)| \le h/g_d$, and noting that $\hat{u}(t) = g_d(t)u_d(t)$ when $|u(t)| > h/g_d$, the authors of [13] arrive at the following characterization of the $\mathcal{L}_2$ gain of the resulting closed-loop system.

Theorem 2: The closed-loop system (9) is stable with an $\mathcal{L}_2$ gain from w to z less than γ if there exist a scalar $M > 0$ and matrices $Y > S > 0$, F_1, F_2, F_3 and F_4 such that

$$\begin{bmatrix} \Omega_{11}(g) & * & * & * \\ \Omega_{12}^T(g) & F_1+F_1^T & * & * \\ \hat{B}_w^T(g) & 0 & -\gamma I & * \\ \Omega_{41}(g) & \Omega_{42}(g) & \hat{D}_{zw}(g) & -\gamma I \end{bmatrix} < 0, g = 1,$$

and

$$\begin{bmatrix} \Omega_{11}(g) & * & * & * & * \\ \Omega_{12}^T(g) & F_1+F_1^T & * & * & * \\ \hat{B}_w^T(g) & 0 & -\gamma I & * & * \\ \Omega_{41}(g) & \Omega_{42}(g) & \hat{D}_{zw} & -\gamma I & * \\ \Omega_{51}(g) & \Omega_{52}(g) & \hat{D}_{uw} & \Omega_{54}(g) & \Omega_{55}(g) \end{bmatrix} < 0, g = g_d,$$

where $\Omega_{ij}(g)$ are defined in terms of M, S, Y, F_1, F_2, F_3 and F_4 in the same way as Ω_{ij} in Section III-A. If the above LMIs are feasible, then the anti-windup matrices can be obtained as $A_{aw} = F_1S^{-1}$, $B_{aw} = F_3M^{-1}$, $C_{aw} = F_2S^{-1}$ and $D_{aw} = F_4M^{-1}$.

C. Design for Anticipatory Activation

The closed-loop system under an anticipatory activation anti-windup compensator is depicted in Fig. 3). Let $h > 0$ be the level of the actuator saturation and $h/g_a, g_a > 1$, be the level of the additional saturation introduced to implement the anticipatory activation of the anti-windup compensator.

Based on the magnitude of the control input $u(t)$, the closed-loop system operates in one of the following three modes:

Mode 1: $|u(t)| \le h/g_a$.

In this mode, no saturation occurs and the closed-loop system reduces to a linear system, for which,

$$\begin{bmatrix} QA^T(g)+A(g)Q & * & * \\ B_w^T(g) & -\gamma I & * \\ C_z(g)Q & D_{zw}(g) & -\gamma I \end{bmatrix} < 0,\ g = 1, \tag{10}$$

implies [7] that

$$\frac{d}{dt}\left(x^TQ^{-1}x\right) + \gamma^{-1}z^Tz - \gamma w^Tw < 0. \tag{11}$$

Let Q be in the form of (7). Then (10) reduces to the following LMI:

$$\begin{bmatrix} \Omega_{11}(g) & * & * & * \\ \Omega_{12}^T(g) & F_1+F_1^T & * & * \\ \hat{B}_w^T(g) & 0 & -\gamma I & * \\ \Omega_{41}(g) & \Omega_{42}(g) & \hat{D}_{zw}(g) & -\gamma I \end{bmatrix} < 0,\ g = 1, \tag{12}$$

where Ω_{ij} and other matrices are the same as in Theorem 2.

Mode 2: $|u| > h$.

In this mode, both saturation elements are in effect, $u_a(t) = \text{sgn}(u(t))h/g_a$, and $\hat{u}(t) = h = g_a u_a(t)$. In view of $\hat{u}(t) = g_a u_a(t) = g_a(u(t) - q(t))$, the closed-loop system can be described as

$$\Sigma_a: \begin{cases} \dot{x} = A(g_a)x + B_w(g_a)w + (B_q(g_a) - B_\eta(g_a)\Lambda)q, \\ z = C_z(g_a)x + D_{zw}(g_a)w + (D_{zq}(g_a) - D_{z\eta}(g_a)\Lambda)q, \\ u = C_ux + D_{uw}w + (D_{uq} - D_{u\eta}\Lambda)q, \end{cases} \tag{13}$$

where $x = [x_p^T\ x_c^T\ x_{aw}^T]^T$ and all matrices, functions of g, are as defined in Theorem 2.

Let us consider the quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Q^{-1}x$. We will analysis the derivative of V along the trajectories of the closed-loop system operating in this mode. We first note that, as u and u_{a} have the same sign and $|u_{\mathrm{a}}| \le |u|$, for any $W > 0$, $qW(q-u) = -(u-u_{\mathrm{a}})Wu_{\mathrm{a}} \le 0$. We next invoke the S-procedure on the desired inequality (11) to obtain

$$\frac{d}{dt}\left(x^{\mathrm{T}}Q^{-1}x\right) + \gamma^{-1}z^{\mathrm{T}}z - \gamma w^{\mathrm{T}}w - 2\tau qW(q-u) < 0,\ \tau > 0,$$

which, in view of the closed-loop system equation (13), can be expanded into,

$$\begin{bmatrix} x \\ w \\ q \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Psi_{11} & * & * \\ \Psi_{12}^{\mathrm{T}} & \Psi_{22} & * \\ \Psi_{13}^{\mathrm{T}} & \Psi_{23}^{\mathrm{T}} & \Psi_{33} \end{bmatrix} \begin{bmatrix} x \\ w \\ q \end{bmatrix} < 0, \tag{14}$$

where $\Psi_{11} = Q^{-1}A + A^{\mathrm{T}}Q^{-1} + \gamma^{-1}C_z^{\mathrm{T}}C_z$, $\Psi_{12} = Q^{-1}B_w + \gamma^{-1}C_z^{\mathrm{T}}D_{zw}$, $\Psi_{13} = \tau C_u^{\mathrm{T}}W^{\mathrm{T}} + \gamma^{-1}C_z^{\mathrm{T}}(D_{zq} - D_{z\eta}\Lambda) + Q^{-1}(B_q - B_\eta\Lambda)$, $\Psi_{22} = \gamma^{-1}D_{zw}^{\mathrm{T}}D_{zw} - \gamma I$, $\Psi_{23} = \tau WD_{uw} + \gamma^{-1}D_{zw}^{\mathrm{T}}(D_{zq} - D_{z\eta}\Lambda)$, and $\Psi_{33} = \gamma^{-1}(D_{zq} - D_{z\eta}\Lambda)^{\mathrm{T}}(D_{zq} - D_{z\eta}\Lambda) - 2\tau W(I - D_{uq} + D_{u\eta}\Lambda)$.

Note that (14) is guaranteed by

$$\begin{bmatrix} \Psi_{11} & * & * \\ \Psi_{12}^{\mathrm{T}} & \Psi_{22} & * \\ \Psi_{13}^{\mathrm{T}} & \Psi_{23}^{\mathrm{T}} & \Psi_{33} \end{bmatrix} < 0, \tag{15}$$

which, by Schur complement and a congruent transformation blkdiag$\{Q, I, M, I\}$, $M = \frac{1}{\tau}W^{-1}$, is equivalent to the following matrix inequality,

$$\begin{bmatrix} A(g)Q + QA^{\mathrm{T}}(g) & * & * & * \\ B_w^{\mathrm{T}}(g) & -\gamma I & * & * \\ C_z(g)Q & D_{zw}(g) & -\gamma I & * \\ \Phi_{41}(g) & D_{uw}(g) & \Phi_{43}(g) & \Phi_{44}(g) \end{bmatrix} < 0, \quad g = g_{\mathrm{a}}, \tag{16}$$

where $\Phi_{41}(g)$, $\Phi_{43}(g)$ and $\Phi_{44}(g)$ are defined the same way as in Theorem 1. By letting Q to be in the form of (7), we can reduce inequality (16) to the following LMI,

$$\begin{bmatrix} \Omega_{11}(g) & * & * & * & * \\ \Omega_{12}^{\mathrm{T}}(g) & F_1 + F_1^{\mathrm{T}} & * & * & * \\ \hat{B}_w^{\mathrm{T}}(g) & 0 & -\gamma I & * & * \\ \Omega_{41}(g) & \Omega_{42}(g) & \hat{D}_{zw}(g) & -\gamma I & * \\ \Omega_{51}(g) & \Omega_{52}(g) & \hat{D}_{uw}(g) & \Omega_{54}(g) & \Omega_{55}(g) \end{bmatrix} < 0, \quad g = g_{\mathrm{a}}. \tag{17}$$

Mode 3: $h/g_{\mathrm{a}} < |u| < h$.

In this mode, we have $\hat{u} = u$ and $|u_{\mathrm{a}}(t)| = h/g_{\mathrm{a}}$. Define $g(t) = \frac{\hat{u}(t)}{u_{\mathrm{a}}}$, then $g(t) \in [1, g_{\mathrm{a}}]$ and the closed-loop system takes the form of (13) with g_{a} replaced by $g(t) \in [1, g_{\mathrm{a}}]$. Also, it is easy to see that $|u|\, g_{\mathrm{a}} - h \in (0, hg_{\mathrm{a}})$. Thus, as in Mode 2, the S-procedure can be invoked to show that (11) is implied by

$$\begin{bmatrix} \Omega_{11}(g) & * & * & * & * \\ \Omega_{12}^{\mathrm{T}}(g) & F_1 + F_1^{\mathrm{T}} & * & * & * \\ \hat{B}_w^{\mathrm{T}}(g) & 0 & -\gamma I & * & * \\ \Omega_{41}(g) & \Omega_{42}(g) & \hat{D}_{zw}(g) & -\gamma I & * \\ \Omega_{51}(g) & \Omega_{52}(g) & \hat{D}_{uw}(g) & \Omega_{54}(g) & \Omega_{55}(g) \end{bmatrix} < 0, \quad g \in \{1, g_{\mathrm{a}}\}. \tag{18}$$

Combining the derivation in all the three modes above, we arrive at following theorem that characterizes the $\mathcal{L}_2$ gain of the closed-loop system as depicted in Fig. 3.

Theorem 3: The closed-loop system with the anticipatory activation of the anti-windup compensator, as depicted in Fig. 3, is stable and the $\mathcal{L}_2$ gain from w to z is less than γ if there exist a scalar $M > 0$ and matrices $Y > S > 0$, F_1, F_2, F_3 and F_4 such that inequalities (12) and (18) hold. If these LMIs are feasible, then the anti-windup matrices can be obtained as $A_{\mathrm{aw}} = F_1S^{-1}$, $B_{\mathrm{aw}} = F_3M^{-1}$, $C_{\mathrm{aw}} = F_2S^{-1}$ and $D_{\mathrm{aw}} = F_4M^{-1}$.

IV. Simulation Results

Consider system (1) and controller (2) with $h = 1$ and

$$\left[\begin{array}{c|c|c} A_{\mathrm{p}} & B_2 & B_1 \\ \hline C_2 & D_{22} & D_{21} \\ \hline C_1 & D_{11} & D_{12} \end{array}\right] = \left[\begin{array}{ccc|c|c} -10.6 & -6.09 & -0.9 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline -1 & -11 & -30 & 0 & 0 \\ \hline -1 & -11 & -30 & -1 & 0 \end{array}\right],$$

$$\left[\begin{array}{c|c|c} A_{\mathrm{c}} & B_{\mathrm{c}y} & B_{\mathrm{c}w} \\ \hline C_{\mathrm{c}} & D_{\mathrm{c}y} & D_{\mathrm{c}w} \end{array}\right] = \left[\begin{array}{cc|c|c} -80 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ \hline 20.25 & 1600 & 80 & -80 \end{array}\right],$$

which were originally considered in [8] and [12].

We carried out the anti-windup designs for immediate, delayed and anticipatory activation, respectively. For the immediate activation scheme, we obtained a $\gamma = 58.5760$. For the delayed activation scheme, we selected $g_{\mathrm{d}} = 0.1700$ and obtained a $\gamma = 59.5600$. For the anticipatory activation scheme, we selected $g_{\mathrm{a}} = 1.0017$ and obtained a $\gamma = 58.9520$. The resulting dynamic anti-windup compensators are given respectively by

$$\left\{\begin{array}{l} A_{\mathrm{aw}} = \begin{bmatrix} -1.0874 \times 10^{10} & -1.1492 \times 10^{11} & -6.4631 \times 10^{10} & 3.8083 \times 10^{3} & 2.1109 \times 10^{5} \\ 1.9051 \times 10^{4} & 2.0133 \times 10^{5} & 1.1323 \times 10^{5} & -7.0336 \times 10^{-3} & -3.6408 \times 10^{-1} \\ -1.5403 \times 10^{3} & -1.6278 \times 10^{4} & -9.1551 \times 10^{3} & 5.6837 \times 10^{-4} & 2.9465 \times 10^{-2} \\ 3.6226 \times 10^{8} & 3.8286 \times 10^{9} & 2.1532 \times 10^{9} & -1.9314 \times 10^{2} & -6.6148 \times 10^{3} \\ 1.4950 \times 10^{9} & 1.5800 \times 10^{10} & 8.8857 \times 10^{9} & -5.2795 \times 10^{2} & -2.9571 \times 10^{4} \end{bmatrix}, \\ B_{\mathrm{aw}} = \begin{bmatrix} -5.1409 \times 10^{-1} \\ -\ 1.7534 \times 10^{-6} \\ -\ 1.0654 \times 10^{-7} \\ 1.7130 \times 10^{-2} \\ 7.0678 \times 10^{-2} \end{bmatrix}, \\ C_{\mathrm{aw}} = \begin{bmatrix} 4.4387 \times 10^{8} & 4.6911 \times 10^{9} & 2.6382 \times 10^{9} & -1.4321 \times 10^{2} & -8.1762 \times 10^{3} \\ 1.4939 \times 10^{9} & 1.5789 \times 10^{10} & 8.8796 \times 10^{9} & -5.2857 \times 10^{2} & -2.9551 \times 10^{4} \\ -1.0874 \times 10^{10} & -1.1492 \times 10^{11} & -6.4630 \times 10^{10} & 3.7880 \times 10^{3} & 2.0949 \times 10^{5} \end{bmatrix}, \\ D_{\mathrm{aw}} = \begin{bmatrix} 2.0959 \times 10^{-2} \\ 7.0630 \times 10^{-2} \\ 4.8565 \times 10^{-1} \end{bmatrix} \text{ (immediate activation),} \end{array}\right.$$

$$
\left\{
\begin{aligned}
A_{\mathrm{aw}} &= \begin{bmatrix}
-1.2883\times10^{6} & -1.3612\times10^{7} & -7.6452\times10^{6} & -8.0582\times10^{-4} & 4.7805\times10^{-5}\\
2.7944\times10^{4} & 2.9526\times10^{5} & 1.6583\times10^{5} & 1.7191\times10^{-5} & -1.0118\times10^{-6}\\
1.9236\times10^{2} & 2.0335\times10^{3} & 1.1415\times10^{3} & 1.4990\times10^{-7} & -9.3817\times10^{-9}\\
-4.0605\times10^{5} & -4.2905\times10^{6} & -2.4097\times10^{6} & -5.2909\times10^{1} & -8.4145\times10^{0}\\
5.1013\times10^{5} & 5.3903\times10^{6} & 3.0281\times10^{6} & -1.2432\times10^{1} & -1.0436\times10^{3}
\end{bmatrix},\\
B_{\mathrm{aw}} &= \begin{bmatrix}
-1.7316\times10^{-4}\\ -7.5069\times10^{-6}\\ -1.3068\times10^{-7}\\ -1.0818\times10^{-4}\\ 1.2999\times10^{-4}
\end{bmatrix},\\
C_{\mathrm{aw}} &= \begin{bmatrix}
-1.2104\times10^{6} & -1.2789\times10^{7} & -7.1830\times10^{6} & 2.7089\times10^{1} & -8.4144\times10^{0}\\
5.2080\times10^{5} & 5.5030\times10^{6} & 3.0913\times10^{6} & -1.3432\times10^{1} & -1.0436\times10^{3}\\
-9.9063\times10^{5} & -1.0467\times10^{7} & -5.8770\times10^{6} & -2.0251\times10^{1} & -1.6000\times10^{3}
\end{bmatrix},\\
D_{\mathrm{aw}} &= \begin{bmatrix}
-4.9997\times10^{-4}\\ 1.3487\times10^{-4}\\ 9.9853\times10^{-1}
\end{bmatrix} \text{ (delayed activation),}
\end{aligned}
\right.
$$

and

$$
\left\{
\begin{aligned}
A_{\mathrm{aw}} &= \begin{bmatrix}
-8.9300\times10^{7} & -9.4372\times10^{8} & -5.2966\times10^{8} & 6.9725\times10^{0} & 1.7226\times10^{2}\\
2.1252\times10^{4} & 2.2458\times10^{5} & 1.2604\times10^{5} & -2.0453\times10^{-3} & -3.1784\times10^{-2}\\
1.2953\times10^{2} & 1.3699\times10^{3} & 7.6827\times10^{2} & 2.0221\times10^{-5} & -9.7502\times10^{-4}\\
1.1142\times10^{7} & 1.1775\times10^{8} & 6.6086\times10^{7} & -6.4107\times10^{1} & 5.4501\times10^{2}\\
4.4961\times10^{7} & 4.7515\times10^{8} & 2.6668\times10^{8} & -2.1919\times10^{1} & -1.5963\times10^{3}
\end{bmatrix},\\
B_{\mathrm{aw}} &= \begin{bmatrix}
-1.5668\times10^{-2}\\ -1.0810\times10^{-7}\\ -2.5062\times10^{-8}\\ 1.9661\times10^{-3}\\ 7.9086\times10^{-3}
\end{bmatrix},\\
C_{\mathrm{aw}} &= \begin{bmatrix}
1.1345\times10^{7} & 1.1989\times10^{8} & 6.7289\times10^{7} & 1.4330\times10^{1} & 5.8151\times10^{2}\\
4.4959\times10^{7} & 4.7512\times10^{8} & 2.6667\times10^{8} & -2.2899\times10^{1} & -1.5968\times10^{3}\\
-8.7085\times10^{7} & -9.2031\times10^{8} & -5.1653\times10^{8} & -1.3452\times10^{1} & -1.4320\times10^{3}
\end{bmatrix},\\
D_{\mathrm{aw}} &= \begin{bmatrix}
1.9584\times10^{-3}\\ 7.9087\times10^{-3}\\ 9.8434\times10^{-1}
\end{bmatrix} \text{ (anticipatory activation).}
\end{aligned}
\right.
$$

We note that the $\mathcal{L}_2$ gain resulting from the anticipatory activation design is slightly lower than the $\mathcal{L}_2$ gain resulting from the delayed activation design, but it is slightly higher than the $\mathcal{L}_2$ gain resulting from the immediately activation design. The real performance difference appears in the transience response in tracking a reference signal (see Figs. 4-6). The anticipatory anti-windup design achieves a much better transience performance than both the delayed activation and immediate activation design, especially when the magnitude of the reference signal is high.

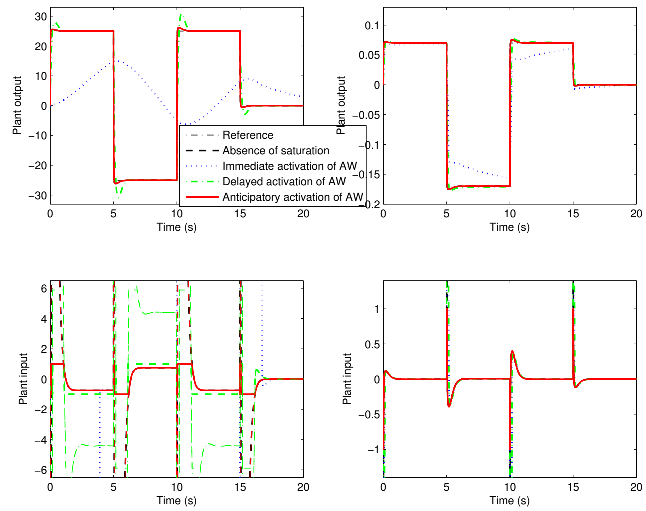

Fig. 4. Transience performance for two different reference inputs under three different anti-windup schemes: immediate activation, delayed activation and anticipatory activation. Left plots: larger reference input. Right plots: smaller reference input.

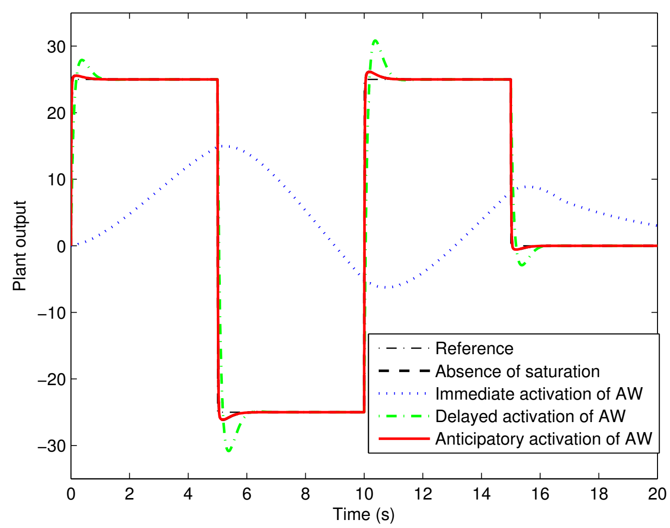

Fig. 5. A zoom-in view of top left plot in Fig. 4.

V. Conclusions

This paper considered the problem of dynamic anti-windup design. We proposed to activate the anti-windup compensation in anticipation of actuator saturation. Conditions are established in terms of LMIs under which the resulting closed-loop system is stable with a prescribed level of $\mathcal{L}_2$ gain from the disturbance to the controlled output. Based on these conditions, the problem of designing a dynamic anti-

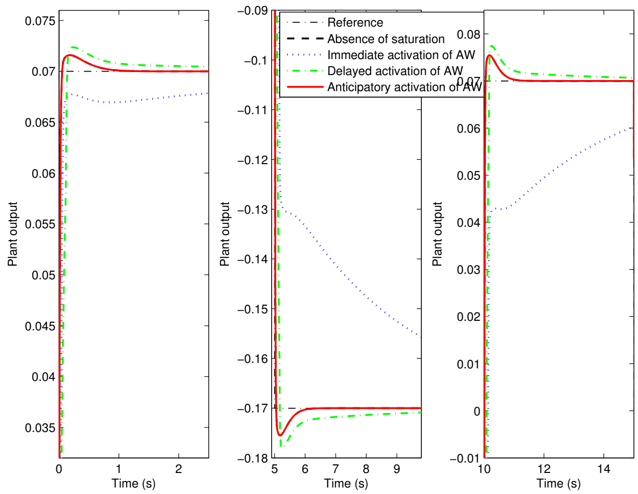

Fig. 6. A zoom-in view of top right plot in Fig. 4.

windup compensator for anticipatory activation is formulated and solved as an LMI optimization problem. Simulation results show that such a "precautionary" approach has the potential of resulting in better closed-loop performances in tracking reference signals, in comparison with both the design for immediate activation and the design for delayed activation.

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