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Lyapunov-Based Semistability Analysis for Discrete-Time Switched Network Systems

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Abstract— **In this paper, we address semistability analysis of a class of distributed iterative algorithms for discrete-time switched network systems. Semistability is the property whereby every solution that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. To this end, we use a Lyapunov-based approach to develop a series of sufficient conditions for semistability of discrete-time switched systems. This technique gives us a new perspective to design distributed numerical iterative algorithms for network systems from a dynamical systems viewpoint. Despite the fact that distributed iterative algorithms in general are not dynamical systems, the methods we used for proving convergence of dynamical systems are somehow valid for a large class of distributed iterative algorithms in network systems. The motivation of this paper exactly follows from this general observation. Part of the effort by this paper can be viewed as an attempt to analyze distributed numerical algorithms for network systems from a control perspective.**

I. Introduction

The distributed agreement problem arises in the context of coordination of networks of autonomous agents, and in particular, the consensus or agreement problem among the agents. Distributed agreement problems have been studied extensively in the computer science literature [1]–[4]. Recently it has found a wide range of applications, in areas such as formation control of underwater autonomous vehicles [5], coordination of mobile robots [6], [7], and sensor networks [8], [9].

Many current iterative policies for the agreement problem are assumed to be either stationary structures or switching structures [10]–[17], which can be unified to analyze under a time-invariant differential inclusion framework. However, there is a limitation for many networking problems since time-dependent communication links are widespread in multi-agent coordination, *ad hoc* or peer-to-peer network routing, and distributed algorithm design. Hence, the time-dependent agreement problem was studied under different circumstances [1], [2], [18]–[20]. Even though many time-dependent agreement protocol algorithms have been developed over the last several years in the literature (see [1], [2], [18]–[20] and the numerous references therein), stability properties of these time-varying algorithms have been largely ignored. *Stability here means that small perturbations from the steady state will lead to only small transient excursions from a state of consensus*. It is important to note that this stability is different from the standard notion of convergence and, hence, leading to a more involved analysis.

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In this paper, we address semistability analysis of a class of distributed iterative algorithms for discrete-time switched network systems, which is named for the *semistable dynamic iterative agreement* (SDIA) problem. Semistability is the property whereby every solution that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. Here "dynamic" means that the iterative algorithm is a time-dependent algorithm and "agreement" means that the semistable equilibria are in the subspace of all ones vector. To this end, we use a Lyapunov-based approach to develop sufficient conditions for semistability of the SDIA system. This technique gives us a new perspective to design numerical iterative algorithms from a dynamical systems viewpoint. Despite the fact that iterative algorithms in general are not dynamical systems, the methods we used for proving convergence of dynamical systems are somehow valid for a large class of iterative algorithms. The motivation of this paper exactly follows from this general observation. Part of the effort by this paper can be viewed as an attempt to analyze numerical algorithms from a control perspective [21].

There are several applications in which semistability can be seen to be the appropriate notion of stability. Besides the case of the distributed agreement problem, we can mention the stability of the lateral dynamics of an aircraft in trimmed level flight. For sideslip disturbances affecting the angle between the longitudinal axis and the velocity vector, the vertical tail is designed to influence yaw so as to cause the sideslip angle to converge to zero. However, the heading angle will not generally converge to the pre-disturbance heading angle. The offset in the final equilibrium is a reflection of semistability, which in the literature is referred to as directional or weathercock stability [22]. This form of stability is evident from the form of the rudder-to-heading-angle transfer function which includes an integrator, that is, a simple pole at the origin. Nevertheless, this observation cannot be applied for nonlinear systems like switched systems. Hence, it is worth a thorough investigation on semistability of a class of nonlinear systems like switched systems.

II. Mathematical Preliminaries

The notion we use in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\overline{\mathbb{Z}}_+$ (resp., $\mathbb{Z}_+$)

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denotes the set of nonnegative (resp., positive) integers, $\mathbb{R}^n$ denotes the set of $n\times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$ real matrices, $(\cdot)^{\rm T}$ denotes transpose, $(\cdot)^{\#}$ denotes the group generalized inverse, and I_n or I denotes the $n\times n$ identity matrix. Furthermore, we write $\|\cdot\|$ for the Euclidean vector norm, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ for the range space and the null space of a matrix or operator A, and $A\geq 0$ (resp., $A>0$) to denote the fact that the Hermitian matrix A is nonnegative (resp., positive) definite. Finally, we write $\mathcal{B}_{\varepsilon}(x)$, $x\in\mathbb{R}^n$, $\varepsilon>0$, for the open ball with *radius* ε and *center* x.

We consider a network characterized by a directed graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ consisting of the set of nodes $\mathcal{V}=\{1,\ldots,q\}$ and the set of edges $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$, where each edge $(j,i)\in\mathcal{E}$ is an ordered pair of distinct nodes indicating a communication from j to i. The set of neighbors of node i is denoted by $\mathcal{N}_i=\{j\in\mathcal{V}:(j,i)\in\mathcal{E}\}$. Finally, we denote the value of the node $i\in\{1,\ldots,q\}$ at time t by $x_i(t)\in\mathbb{R}$.

Each node i holds an initial value on the network $x_i(0)\in\mathbb{R}$. The network gives the allowed communication between two nodes if and only of they are neighbors. We are interested in computing the average of the initial values, $(1/q)\sum_{i=1}^{q}x_i(0)$, via a *stable* distributed algorithm in which the nodes only communicate with their neighbors.

In this paper, we consider distributed linear iterations given by the form

$$x_i(t+1)=W_{(i,i)}(t)x_i(t)+\sum_{j\in\mathcal{N}_i}W_{(i,j)}(t)x_j(t),\quad i=1,\ldots,q,\quad t\in\overline{\mathbb{Z}}_+,\tag{1}$$

where $W_{(i,j)}(t)$ denotes the weight on x_j at node i and time step t. Letting $W_{(i,j)}(t)=0$ for $j\neq\mathcal{N}_i$, this iteration can be rewritten as a compact form

$$x(t+1)=W(t)x(t),\quad t\in\overline{\mathbb{Z}}_+,\tag{2}$$

where $x(t)=[x_1(t),\ldots,x_q(t)]^{\rm T}\in\mathbb{R}^q$, $W:\overline{\mathbb{Z}}_+\to\mathbb{R}^{q\times q}$ is piecewise continuous with respect to t. Finally, the constraint on the matrix $W(t)$ can be expressed as $W(t)\in\mathcal{W}$, where

$$\mathcal{W}=\{W\in\mathbb{R}^{q\times q}:W_{(i,j)}=0\text{ if }(i,j)\notin\mathcal{E}\text{ and }i\neq j\}.\tag{3}$$

Remark 2.1: The time-dependent discrete-time linear iteration (2) is a *dynamical system*. If we denote the solution to (2) with initial condition $x(t_0)=x_0$ by $s(\cdot,t_0,x_0)$, then the map of the dynamical system given by $s:\overline{\mathbb{Z}}_+\times\overline{\mathbb{Z}}_+\times\mathbb{R}^q$ is continuous and satisfies the consistency property $s(t_0,t_0,x_0)=x_0$ and the semigroup property $s(k,t_0,s(\kappa,t_0,x_0))=s(k+\kappa,t_0,x_0)$ for all $x_0\in\mathbb{R}^q$, $t_0\in\overline{\mathbb{Z}}_+$, and $k,\kappa\in\overline{\mathbb{Z}}_+$. Hence, all the properties of discrete-time dynamical systems in [23] hold for (2).

III. Related Work

Unlike the fixed matrix case where W is a constant matrix [24], [25], the time-varying matrix case is much more involved since it is very difficult to come up with a necessary and sufficient condition like the results in [24], [25] for $W(t)$ being convergent. Here we recall some conditions on the convergence of $W(t)$.

The first result is due to [2], [26]. To state this result, let T_{ij} be the set of times at which node i receives a message from node j, containing the value of $x_j(t)$, which is used by i to update the value of $x_i(t)$. We define $\mathcal{E}(t)$ as the set of ordered pairs (j,i) such that $W_{(i,j)}(t)>0$. Thus, $(\mathcal{V},\mathcal{E}(t))$ is a directed graph indicating the influences between agents at time t. Finally, let $\mathcal{E}$ be the set of (i,j) such that $(i,j)\in\mathcal{E}(t)$ for infinitely many t.

Proposition 3.1 ([26]): Consider (2). Assume $W_{(i,j)}(t)\geq 0$ for all $i,j=1,\ldots,q$ and $t\in\overline{\mathbb{Z}}_+$, $\sum_{j=1}^{q}W_{(i,j)}(t)=1$ for all $i=1,\ldots,q$ and $t\in\overline{\mathbb{Z}}_+$, and $W_{(i,j)}(t)=0$ for all $i,j=1,\ldots,q$ and $t\notin T_{ij}$. Next, assume $D\subseteq\mathcal{V}$ is nonempty and there exists $\alpha>0$ such that if $(j,i)\in\mathcal{E}(t)$, then $W_{(i,j)}(t)\geq\alpha$. Furthermore, assume that there exists $m\in\mathbb{Z}_+$ such that for every t, $\mathcal{E}(t+1)\cup\cdots\cup\mathcal{E}(t+m)=\mathcal{E}$, and the graph $(\mathcal{V},\mathcal{E})$ contains a directed path from every $i\in D$ to every $j\in\mathcal{V}$. Then there exist nonnegative constants $w_1,\ldots,w_q$ such that $\lim_{t\to\infty}x_i(t)=\sum_{j=1}^{n}w_jx_j(0)$ for all $i=1,\ldots,q$.

The above result is a general result for (2) by considering a *directed* graph topology and *joint connected* graphs to guarantee distributed consensus. However, it is hard to use this result to "quantify" the optimal one since the graph constraints are not straightforward in the process of optimization compared to algebraic constraints. The next result, which is a matrix characterization of the convergence of $W(t)$, relies on the assumption that $W(t)$ belongs to a finite set of matrices. We call a square matrix W *paracontracting matrix* if $Wx\neq x$ is equivalent to $\|Wx\|<\|x\|$.

Proposition 3.2 ([27]): Consider (2) where $W(t)$ belongs to a finite set of paracontracting matrices. If $\mathcal{I}$ is the set of matrices W_i that appear infinitely often in the sequence $W(t)$, and for $i\in\mathcal{I}$, $\mathcal{H}(W_i)$ denotes the eigenspace of W_i associated with eigenvalue 1, then the sequence of vectors $x(t)$ has a limit x^* in $\bigcap_{i\in\mathcal{I}}\mathcal{H}(W_i)$.

A direct consequence of this lemma is Wolfowitz's theorem given by [28], which says that if for any matrix sequence $M_{i_1},\ldots,M_{i_j}$ of positive length selecting from a finite set of ergodic matrices $\{M_1,\ldots,M_m\}$, the matrix product $M_{i_j}M_{i_{j-1}}\cdots M_{i_1}$ is ergodic, then there exists a row vector c such that $\lim_{j\to\infty}M_{i_j}M_{i_{j-1}}\cdots M_{i_1}=\mathbf{1}c$. Here any *stochastic matrix* M for which $\lim_{i\to\infty}M^i$ is a matrix of rank 1 is called *ergodic* [29].

Using convex analysis and the set-valued Lyapunov theory, [18] has extended the above result to the general nonlinear system by studying the form of the difference inclusion given by $x(t+1)\in{\rm co}(t,x(t))$, where "co" denotes the convex hull. This famework can be viewed as a generalization of Proposition 3.1. Due to the space limitation, we do not consider this general form of linear iterations.

In general, finding a necessary and sufficient condition for semistability of (2) is a widely open problem. In this paper, we focus on a subclass of this problem labeled as the *semistable dynamic iterative agreement* (SDIA) problem. We follow the idea of [25] to develop sufficient conditions for *semistability* of (2) using Lyapunov functions. The merit of using Lyapunov equations to characterize the semistability of (2) is that numerical and optimization techniques such as sums of squared polynomials (SOS) can be developed to solve the SDIA problem since it turns the SDIA problem into a semidefinite programming (SDP) problem [30], [31]. Hence, this method looks more promising in the sense of computation than Propositions 3.1 and 3.2 to solve the SDIA problem, particularly for a large-scale network systems. In fact, it is not quite clear on how to use Propositions 3.1 and 3.2 to solve the proposed SDIA problem since these results only address the convergence of (2) and the semistability of these results has been largely ignored.

IV. Lyapunov-Based Analysis

In this section, we present a Lyapunov-based analysis framework for semistability of discrete-time switched network systems given by the form of (2). First, we give the definition of semistability for switched systems.

A. Semistability

The following definition introduces the new notion of semistability for (2). This new concept is motivated from semistability theory of continuous-time nonlinear dynamical systems [14], [17] and stability theory of time-varying discrete-time dynamical systems [23]. To state this new concept, we define the *equilibrium point* of (2) as a point $z \in \mathbb{R}^q$ satisfying $W(t)z = z$ for all $t \geq 0$. The set of all the equilibrium points of (2) is denoted by E, that is, $E \triangleq \{z \in \mathbb{R}^q : W(t)z = z \; \forall \; t \geq 0\}$. The following assumption is a standing assumption in the paper.

Assumption 4.1: E is a connected set.

This assumption implies that (2) possesses a continuum of equilibria instead of isolated equilibria. It creates a major difference between our semistability theory and classic (asymptotic) stability theory in [23]. The classic (asymptotic) stability theory is not an appropriate notion for the stability of dynamical systems having a continuum of equilibria as pointed out in [14]. Hence, we need the following notion of semistability.

Definition 4.1: i) The linear time-varying iteration (2) is *Lyapunov stable* if, for every $\varepsilon > 0$, every $x_{\rm e} \in E$, and $t_0 \in \overline{\mathbb{Z}}_+$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|x(t_0) - x_{\rm e}\| < \delta$ implies that $\|x(t) - x_{\rm e}\| < \varepsilon$ for all $t \geq t_0$.

ii) The linear time-varying iteration (2) is *semistable* if it is Lyapunov stable and, for every $x_{\rm e} \in E$ and every $t_0 \in \overline{\mathbb{Z}}_+$, there exists $\delta = \delta(t_0) > 0$ such that $\|x(t_0) - x_{\rm e}\| < \delta$ implies that the sequence $\{x(t)\}_{t=t_0}^{\infty}$ has a limit. The linear time-varying iteration (2) is *globally semistable* if it is Lyapunov stable and the sequence $\{x(t)\}_{t=t_0}^{\infty}$ has a limit for all $x(t_0) \in \mathbb{R}^q$ and $t_0 \in \overline{\mathbb{Z}}_+$.

It is important to note that semistability is not merely equivalent to asymptotic stability of the set of equilibria. Indeed, it is possible for a trajectory to converge to the set of equilibria without converging to any one equilibrium point as examples in [32] show. Conversely, semistability does not imply that the equilibrium set is asymptotically stable in any accepted sense [33]. This is because stability of sets is defined in terms of distance (especially in case of noncompact sets), and it is possible to construct examples in which the dynamical system is semistable, but the domain of semistability contains no ε-neighborhood (defined in terms of the distance) of the (noncompact) equilibrium set, thus ruling out asymptotic stability of the equilibrium set. Hence, semistability and set stability of the equilibrium set are independent notions.

Later on we will also use a slightly extended Definition 4.1 in which the ambient space is a positive invariant set instead of $\mathbb{R}^q$. In this case, Definition 4.1 still holds with respect to this positive invariant set by replacing $\mathbb{R}^q$ with this set.

B. Semistability Analysis Using Common Lyapunov Functions

In this part of the paper, we establish sufficient conditions for semistability of (2) using common *time-varying* Lyapunov functions. The following lemma is a standard result for Lyapunov stability of (2) using time-dependent common Lyapunov functions. For completeness, we include our proof here.

Lemma 4.1: Consider the linear time-varying iteration (2). Assume that there exists a continuous, positive-definite matrix function $P(t)$ (that is, $P(t)$ is positive definite for every $t \geq 0$) such that $P(t) \geq \alpha I_q > 0$, $\alpha > 0$, for all $t \geq 0$, and

$$P(t) = W^{\rm T}(t) P(t+1) W(t) + R(t), \tag{4}$$

where $R(t) \geq 0$ for all $t \geq 0$. Then (2) is Lyapunov stable.

The next lemma represents a convergence result for (2) based on common Lyapunov functions.

Lemma 4.2: Consider the linear time-varying iteration (2). Assume that there exists a continuous, positive-definite matrix function $P(t)$ such that $0 < \alpha I_q \leq P(t)$, $\alpha > 0$, for all $t \geq 0$, and (4) holds, where $R(t) \geq Q \geq 0$ for all $t \geq 0$. Then $x(t) \to \mathcal{N}(Q)$ as $t \to \infty$.

Next, we have a general result on the relationship between convergence and semistability for (2).

Lemma 4.3: Consider the linear time-varying iteration (2). If (2) is Lyapunov stable and $x(t) \to E$ as $t \to \infty$, then (2) is semistable, and hence, $x(t) \to x^* \in E$ as $t \to \infty$.

Proposition 4.1: Consider the linear time-varying iteration (2). Assume that there exists a continuous, positive-definite matrix function $P(t)$ such that $0 < \alpha I_q \leq P(t)$, $\alpha > 0$, for all $t \geq 0$, and (4) holds, where $R(t) \geq Q \geq 0$ for all $t \geq 0$. Furthermore, assume that $\mathcal{N}(Q) = E$. Then (2) is semistable.

Next, we extend the notion of semiobservability for matrices [25], [34] to that for operators. First, define the high order difference operator Δ^n as $\Delta^0 x(t) = x(t)$, $\Delta x(t) = x(t+1) - x(t)$, and $\Delta^n x(t) = \Delta(\Delta^{n-1} x(t))$, where $n = 1, 2, \ldots$. In this case, $\mathcal{N}(\Delta) = \{x \in \mathbb{R}^q : W(t)x = x, \forall t \in \overline{\mathbb{Z}}_+\} = E$.

Definition 4.2: Let $C \in \mathbb{R}^{m \times q}$. The pair (Δ, C) is *semiobservable* if

$$\bigcap_{k=1}^{q} \mathcal{N}(C \Delta^{k-1}) = \mathcal{N}(\Delta). \tag{5}$$

Proposition 4.2: Consider the linear time-varying iteration (2). Assume that there exists a continuous, positive-definite matrix function $P(t)$ such that $0 < \alpha I_q \le P(t)$, $\alpha > 0$, for all $t \ge 0$, and (4) holds, where $R(t) \ge Q \ge 0$ for all $t \ge 0$. Furthermore, assume that (Δ, Q) is semiobservable. Then (2) is semistable.

C. Semistability Analysis Using Multiple Lyapunov Functions

The common Lyapunov function approach is restricted in many cases. For example, for multi-agent coordination under switching topology [10], it was shown that there does not exist a common quadratic Lyapunov function [35]. In this case, it is more reasonable to use *multiple Lyapunov functions* [36] to characterize semistability of (2) in which the graph topology may not be fixed. Hence, we begin by addressing semistability analysis of (2) where $W(t) = W_{\sigma(t)}(t) \in \mathcal{P} = \{1, \ldots, m\}$ becomes a sequence of matrices, that is, (2) is given by

$$\mathcal{G}_\sigma: \quad x(t+1) = W_{\sigma(t)}(t)x(t), \quad t \in \overline{\mathbb{Z}}_+, \tag{6}$$

where $\sigma(t)$ is a piecewise constant switching signal. Here $W(t)$ belongs to a finite set $\mathcal{P}$ due to the fact that the total number of possible connected graph topologies for (2) is finite. Let $t_k \in \overline{\mathbb{Z}}_+$ denotes the time instant when the graph topology of (6) changes, $k = 0, 1, 2, \ldots$. For $1 \le p < \infty$, we define ℓ_p to be the collection of all real sequences $x = (x_n)$ for which $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

Lemma 4.4: Consider the switched time-varying iteration $\mathcal{G}_\sigma$ given by (6). Assume that there exist positive-definite continuous functions $V_i : \mathbb{R}^q \to \mathbb{R}$ with $V_i(0) = 0$, $i = 1, \ldots, m$, and nonnegative functions $\gamma_{ij}(\cdot) \in \ell_1$, $i, j = 1, \ldots, m$, $i \ne j$, such that for $i, j = 1, \ldots, m$,

$$\begin{aligned} &V_{i_k}(x(t)) - V_{i_k}(x(s)) \le 0, \quad k = 0, 1, 2, \ldots, \\ &\quad s, t \in \overline{\mathbb{Z}}_+, \quad t_k \le s \le t < t_{k+1}, \end{aligned} \tag{7}$$

$$\begin{aligned} &V_j(x(t)) - V_j(x(s)) \le \sum_{\tau=s}^{t-1} \gamma_{i_k j}(\tau), \quad j \ne i_k, \\ &\quad k = 0, 1, 2, \ldots, \quad s, t \in \overline{\mathbb{Z}}_+, \quad t_k \le s \le t < t_{k+1}. \end{aligned} \tag{8}$$

Then $\mathcal{G}_\sigma$ is Lyapunov stable.

Remark 4.1: The condition (8) is weaker than the conventional non-positive condition (4). Hence, Lemma 4.4 is stronger than Lemma 4.1.

Next, we have a convergence result for (6).

Proposition 4.3: Consider the switched time-varying iteration (6). Assume that there exist positive-definite continuous functions $V_i : \mathbb{R}^q \to \mathbb{R}$ with $V_i(0) = 0$, $i = 1, \ldots, m$, and continuous nonnegative functions $\beta_{ij}(\cdot)$, $i, j = 1, \ldots, m$, $i \ne j$, such that $\beta_{ij} \circ x \in \ell_1$, $i, j = 1, \ldots, m$, $i \ne j$, where $\circ$ denotes the composition operator, (7) holds, and for $i, j = 1, \ldots, m$,

$$\begin{aligned} &V_j(x(t)) - V_j(x(s)) \le \sum_{\tau=s}^{t-1} \beta_{i_k j}(x(\tau)), \quad j \ne i_k, \\ &s, t \in \overline{\mathbb{Z}}_+, \quad k = 0, 1, 2, \ldots, \quad t_k \le s \le t < t_{k+1}. \end{aligned} \tag{9}$$

If $\beta_{ij}^{-1}(0) = \mathcal{M}$ for all $i, j = 1, \ldots, m$, $i \ne j$, then $x(t) \to \mathcal{M}$ as $t \to \infty$.

V. Conclusions

In the present paper we extend the notion of semistability to a class of switched network systems. To do this, our goal is to capture the notion that the solution of the switched system converges on some set without specifying the equilibrium to which it converges. We give several Lyapunov type sufficient conditions for convergence and semistability.

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