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Stabilization and Stability Connection of Networked Control Systems with Two Quantizers

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Abstract— **Recently, Networked Control Systems (NCSs) and quantized feedback control have received a lot of attention. We consider analysis of quantized estimation and investigation of stability connection between estimation error system (EES) and quantized estimation feedback control system (QEFCS) in this paper. We assume that there are two-quantized signals being passed to the estimator and the controller through network channels in the closed-loop NCS. Using adjustable zoom quantizer parameters, Lyapunov-based quadratically stabilizable conditions of NCSs are presented. We also propose a method to disclose the stability connection of EES and QEFCS by the use of the connected invariant region sequences. Numerical example and simulation results demonstrate the effectiveness of the method.**

I. INTRODUCTION

In recent years, an emerging framework of modern control systems embedded with network medium has received considerable research interest, e.g., [1, 2]. Such networked control systems (NCSs) are much more information-rich than traditional control systems which are based on one of the assumptions that all data in the system can be got with arbitrary precision. Therefore, quantization effect is one of the critical problems in NCSs, e.g., [15]. Due to limited capacities of network channels, command signals or data in NCSs have to be quantized to finite precision before being transmitted, which calls for the development of quantized feedback control theory, see, e.g., references therein.

There are many approaches being developed to study the analysis and design problems for quantized feedback control. The method of [3], [4] was built to analyze the characterizations of the minimum data rate and the quantization level for quantized stabilization. In [5], [6], a type of dynamic quantizer which uses dynamic scaling and hybrid feedback stabilization was presented to achieve global asymptotic stabilization and input-to-state stability of control systems. The ideas are extended in [7]. Results on quantized control with dynamic quantizer can also be found in [9]. During a series of works in [8], [9], [14], Fu and Xie showed a sector bound approach to stabilize a linear system by treating the quantization error as sector bound nonlinearity or uncertainty. Finite level quantized stabilization and quantized H_∞ control problem have also been well studied in the later literature. A quantized-dependent Lyapunov functional approach which could less conservation was given in [10]. The extended Popov-type Lyapunov function focusing on the geometric property of logarithmic quantizer was constructed by [11]. Similar to the classical control theory, quantized estimation is also very important to quantized feedback control as well as a broad range of other applications, see, e.g. [9], [14]. Considering a remote control scheme and shared network channels, a more complicated setup with multi-quantizer and adaptive control framework was employed in [12]. Hybrid stabilization problem for quantized control systems with two quantizers was proposed in [13].

We study two basic problems about quantized feedback stabilization in network based control with multi-quantized measurements in this paper. The first one is stability analysis of estimation error system (EES) and quantized estimation feedback control system (QEFCS) which involves two quantizers, i.e., the case where both output and estimated state are quantized. Figure 1 depicts such a setup. We give the sufficient quadratically stabilizable conditions for both EES and QEFCS based on standard Lyapunov function which can also be used to construct level sets or invariant regions of all state trajectories. The second one is to investigate stability connection or relationship between EES and QEFCS which has also been discussed in some recent works [6], [14]. But differ from the existing methods such as generalization state space model [6], we address two connected invariant region sequences with the same shapes for EES and QEFCS, respectively. Using the dynamic scaling method similar to [5], we show the interplay of quadratically attractive process of states in EES and QEFCS. Finally, numerical simulations are presented to show the effectiveness of the method.

The main contribution of this paper is that we present the stability connection of system parameters of EES and QEFCS. The results provide a unified method to disclose the dynamical relationship of system stabilization. The second one is that we consider the quantized estimation feedback for noise system under multi-quantized measurements. This

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is an important structural extension comparing to existing work, where only one quantizer considered. We argue that this setting is much more reasonable when the distributed quantization is due to physical or practical constraints on the sensors such as network-based control scheme or environment.

Notation: $\mathbb{R}^n$ denotes the set of $n\times 1$ real column vectors ($\mathbb{R}$ stands for $\mathbb{R}^1$), $\mathbb{R}_+^n$ denotes the positive set of $n\times 1$ real column vectors, $\mathbb{R}^{n\times n}$ denotes $n\times n$ real matrix space, $|\cdot|$ denotes the standard Euclidean norm in $\mathbb{R}^n$, $\|\cdot\|$ denotes the corresponding induced matrix norm, $|\cdot|_\infty$ denotes the maximum norm. $\mathbb{Z}_+$ denotes the set of positive integer, $\mathbb{N}$ denotes the set of natural number. We use the following function $\lceil x\rceil := \min\{k\in\mathbb{Z}, k>x\}$. We write $(\cdot)^T$ for the transpose, $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ for the minimum and maximum eigenvalue of a symmetric matrix, respectively.

II. PROBLEM STATEMENT AND PRELIMINARIES

A. Quantizer

According to [6], let $z\in\mathbb{R}^l$ be the variable being quantized. A quantizer is defined as a piecewise constant function $Q:\mathbb{R}^l\to D$, where D is a finite subset of $\mathbb{R}^l$. This leads to a partition of $\mathbb{R}^l$ into a finite number of quantization regions of the form $\left\{z\in\mathbb{R}^l : Q(z)=i\right\}, i\in D$. We assume that there exist positive real numbers M and Δ such that the following two conditions hold:
1. If $|z|\leqslant M$, then $|Q(z)-z|\leqslant\Delta$.
2. If $|z|>M$, then $|Q(z)|>M-\Delta$.

We refer to M and Δ as the quantization range and the quantization error of the quantizer $Q_i, i=1,2$, respectively. In the dynamic scaling strategy to be stated below, we will use quantized measurements of the form $Q_\mu(z)=\mu Q\left(\frac{z}{\mu}\right)$, where $\mu>0$ is the dynamic scaling parameter. The range of this kind of quantizer is changed to be $M\mu$ and the quantization error is $\Delta\mu$. Similar to [6], [13], We can think of μ as a "zoom" variable. The quantization ranges and quantization errors of Q_1 and Q_2 are $M_1\mu_1$, $\Delta_1\mu_1$, $M_2\mu_2$ and $\Delta_2\mu_2$, respectively.

B. Problem statement

Consider the following linear system

$$\begin{cases} x(k+1)=Ax(k)+B_1u(k)+B_2w(k), \\ x(0)=x_0 \\ y(k)=Cx(k)+v(k) \end{cases} \tag{1}$$

where $x(k)\in\mathbb{R}^n$ is the state, $u(k)\in\mathbb{R}^m$ is the control input, $y(k)\in\mathbb{R}$ is the measured output which is scalar-valued for simplicity, $w(k)\in\mathbb{R}^p$ is the process noise, $v(k)\in\mathbb{R}$ is the measurement noise, respectively. $w(k)$ and $v(k)$ are assumed to be bounded with upper bound δ_1 and δ_2, respectively. A, B_1, B_2 and C are known matrices of appropriate dimensions. (A, B_1) is stabilizable and (C, A) is detectable. Initial value x_0 is bounded and the state $x(k)$ can not be obtained directly. The channels are supposed to be free of transmission errors and time delay.

As shown in Figure 1, our quantized estimation feedback control system (QEFCS) consists of four parts: two quantizers (Q_1 and Q_2), the plant as well as the controller, digital network channels and an estimator. Instead of quantizing the measured signal directly. The estimator is chosen to be

$$\begin{cases} \hat{x}(k+1)=A\hat{x}(k)+B_1u(k) \\ +L\mu_1 Q_1\left(\dfrac{y(k)-\hat{\hat{y}}(k)}{\mu_1}\right)-L\left(\hat{y}(k)-\hat{\hat{y}}(k)\right) \\ \hat{y}(k)=C\hat{x}(k) \\ \hat{\hat{y}}(k)=C\hat{\hat{x}}(k) \end{cases} \tag{2}$$

where $\hat{x}(k)\in\mathbb{R}^n$ is the estimate of $x(k)$, $\hat{y}(k)\in\mathbb{R}$ is the estimate of $y(k)$ based on $\hat{x}(k)$, $\hat{\hat{x}}(k)=Q_2(\hat{x}(k))$. L is the estimation gain such that $A-LC$ is Hurwitz. Define the state estimation error as $e(k)=x(k)-\hat{x}(k)$, the prediction error as $\varepsilon_1(k)=y(k)-\hat{y}(k)=Cx(k)+v(k)-C\hat{x}(k)=Ce(k)+v(k)$, and $\varepsilon_2(k)=\hat{y}(k)-\hat{\hat{y}}(k)=C\left(\hat{x}(k)-\hat{\hat{x}}(k)\right)$. So quantized innovation $y(k)-\hat{\hat{y}}(k)=y(k)-\hat{y}(k)+\hat{y}(k)-\hat{\hat{y}}(k)=\varepsilon_1(k)+\varepsilon_2(k)$. In order to get simpler model for analyzing, we assume that the feedback control law is $u(k)=K\hat{\hat{y}}(k)$ instead of $u(k)=K\hat{\hat{x}}(k)$, though the latter is more convenient for control law design, which is not the main purpose of this paper. Where K is the feedback gain such that $A+B_1KC$ is Hurwitz.

The closed-loop system block diagram is shown in Figure 1. There are two quantizers Q_1 and Q_2 in the closed-loop system, The quantized signals are given in Figure 2.

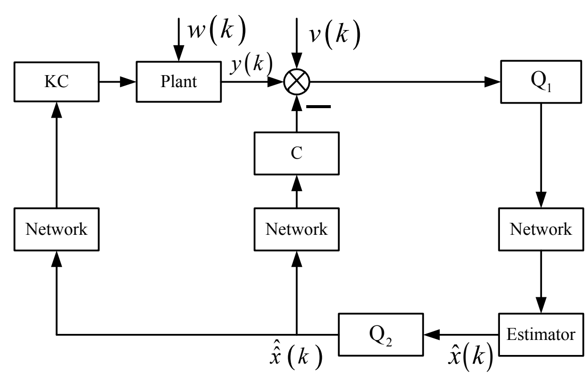

Fig. 1. Quantized estimation

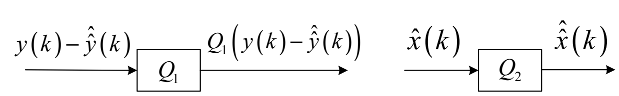

Fig. 2. Quantizers

We can obtain estimation error system (EES) equation as

$$e(k+1) = (A - LC)\, e(k) + B_2 w(k) - L v(k) + L d(k) \tag{3}$$

and QEFCS equation as

$$\begin{gathered} x(k+1) = (A + B_1 KC)\, x(k) - \\ B_1 K(\varepsilon_1(k) + \varepsilon_2(k)) + B_1 K v(k) + B_2 w(k) \end{gathered} \tag{4}$$

where $d(k) = (\varepsilon_1(k) + \varepsilon_2(k)) - \mu_1 Q_1\left(\frac{\varepsilon_1(k)+\varepsilon_2(k)}{\mu_1}\right)$, so $d(k)$ denotes the quantization error of quantizer Q_1.

In the following, we consider dynamic quantizers of Q_1 and Q_2. The objective of this paper is to analyze quadratically stabilizable conditions for both EES and QEFCS, and find out the stability connection between EES and QEFCS under dynamic scaling strategy of quantization.

Definition 1: The discrete-time linear system $x(k+1) = Ax(k) + Bu(k)$ is said to be quadratically stabilizable via feedback if there exist a positive definite function $V(k) = x^T(k) P x(k)$ and such that $V(k+1) - V(k) < -\nu x^T(k) x(k), \nu > 0$ for trajectories of the system.

Consider the dynamic system in (1), if (A, B_1) is stabilizable, (C, A) is detectable, and there are K and L such that $A + B_1 KC$ and $A - LC$ are Hurwitz, then there exist positive definite symmetric matrices P, Q, $\tilde{P}$, $\tilde{Q}$ satisfying

$$(A - LC)^T P (A - LC) - P = -Q \tag{5}$$

$$(A + B_1 KC)^T \tilde{P} (A + B_1 KC) - \tilde{P} = -\tilde{Q} \tag{6}$$

When quantized estimation is used, we have the following results.

Theorem 1: Consider the EES in (3), given an arbitrary scalar $\varepsilon \in (0, 1)$, and $P > 0$, $Q > 0$, satisfying equation (5), assuming that

$$\sqrt{\frac{\lambda_m(P)}{\lambda_M(P)}} \geqslant \frac{R\|C\|}{M_1\mu_1 - \delta_2 - \|C\|\mu_2\Delta_2}, \tag{7}$$

where $R = \frac{b+\sqrt{b^2+ac}}{a}$, $a = (1-\varepsilon)\lambda_m(Q)$, $b = \alpha\delta_1 + \beta\delta_2 + \beta\Delta_1\mu_1$, $c = 2\delta_1\gamma\delta_2 + 2\delta_1\gamma\Delta_1\mu_1 + 2\delta_2\theta\Delta_1\mu_1 + \delta_1^2\tau + \delta_2^2\theta + \Delta_1^2\mu_1^2\theta$, $\alpha = \left\|(A - LC)^T P B_2\right\|$, $\beta = \left\|(A - LC)^T P L\right\|$, $\gamma = \left\|B_2^T P L\right\|$, $\theta = \left\|L^T P L\right\|$, $\tau = \left\|B_2^T P B_2\right\|$. Then, the ellipsoids

$$\begin{gathered} \Re_1(\mu_1, \mu_2) = \\ \left\{ \begin{array}{l} e(k) : e^T(k) P e(k) \\ \leqslant \lambda_m(P) \left(\dfrac{M_1\mu_1 - \delta_2 - \|C\|\mu_2\Delta_2}{\|C\|} \right)^2 \end{array} \right\} \end{gathered} \tag{8}$$

and

$$\Re_2(\mu_1, \mu_2) = \left\{ e(k) : e^T(k) P e(k) \leqslant \lambda_M(P) R^2 \right\} \tag{9}$$

are invariant regions of EES. Moreover, If (7) is obtained, then all solutions of EES that start in the ellipsoid $\Re_1(\mu_1, \mu_2)$ enter the smaller ellipsoid $\Re_2(\mu_1, \mu_2)$ in finite step S.

Proof: For EES in (3), let the Lyapunov function $V(k) = e^T(k) P e(k)$.

$$\begin{aligned} & V(k+1) - V(k) \\ & = e^T(k+1) P e(k+1) - e^T(k) P e(k) \\ & \leqslant -\lambda_m(Q) |e(k)|^2 + 2|e(k)| \alpha\delta_1 + 2|e(k)| \beta\delta_2 \\ & +\delta_1^2\tau + \delta_2^2\theta + 2|e(k)| \beta\Delta_1\mu_1 + 2\delta_1\gamma\delta_2 \\ & +2\delta_1\gamma\Delta_1\mu_1 + 2\delta_2\theta\Delta_1\mu_1 + \Delta_1^2\mu_1^2\theta \\ & < -\varepsilon\lambda_m(Q) |e(k)|^2 \end{aligned}$$

The above inequality is obtained if $|d(k)| \leqslant \Delta_1\mu_1$, thus $|\varepsilon_1(k) + \varepsilon_2(k)| \leqslant M_1\mu_1$. Then

$$\begin{aligned} & (1-\varepsilon)\lambda_m(Q) |e(k)|^2 \\ & -2(\alpha\delta_1 + \beta\delta_2 + \beta\Delta_1\mu_1) |e(k)| \\ & - \left(\begin{array}{l} 2\delta_1\gamma\delta_2 + 2\delta_1\gamma\Delta_1\mu_1 + 2\delta_2\theta\Delta_1\mu_1 \\ +\delta_1^2\tau + \delta_2^2\theta + \Delta_1^2\mu_1^2\theta \end{array} \right) > 0 \\ & \Rightarrow a|e(k)|^2 - 2b|e(k)| - c > 0 \end{aligned}$$

where $a = (1-\varepsilon)\lambda_m(Q)$, $b = \alpha\delta_1 + \beta\delta_2 + \beta\Delta_1\mu_1$, $c = 2\delta_1\gamma\delta_2 + 2\delta_1\gamma\Delta_1\mu_1 + 2\delta_2\theta\Delta_1\mu_1 + \delta_1^2\tau + \delta_2^2\theta + \Delta_1^2\mu_1^2\theta$. It is easy to confirm that

$$|e(k)| > \frac{b + \sqrt{b^2 + ac}}{a} \triangleq R, 0 < \varepsilon < 1. \tag{10}$$

Define two balls $B_1(\mu_1, \mu_2)$ and $B_2(\mu_1, \mu_2)$ as

$$\begin{gathered} B_1(\mu_1, \mu_2) \triangleq \\ \left\{ e(k) : |e(k)| \leqslant \frac{M_1\mu_1 - \delta_2 - \|C\|\mu_2\Delta_2}{\|C\|} \right\} \end{gathered} \tag{11}$$

$$B_2(\mu_1, \mu_2) \triangleq \{ e(k) : |e(k)| \leqslant R \}. \tag{12}$$

We get the corresponding level sets of EES in (3).

$$\begin{gathered} \Re_1(\mu_1, \mu_2) = \\ \left\{ \begin{array}{l} e(k) : e^T(k) P e(k) \\ \leqslant \lambda_m(P) \left(\dfrac{M_1\mu_1 - \delta_2 - \|C\|\mu_2\Delta_2}{\|C\|} \right)^2 \end{array} \right\} \end{gathered} \tag{13}$$

$$\Re_2(\mu_1, \mu_2) = \left\{ e(k) : e^T(k) P e(k) \leqslant \lambda_M(P) R^2 \right\}. \tag{14}$$

When (7) holds, we have the relationship

$$\begin{gathered} B_2(\mu_1, \mu_2) \subset \Re_2(\mu_1, \mu_2) \\ \subset \Re_1(\mu_1, \mu_2) \subset B_1(\mu_1, \mu_2) \end{gathered}$$

Using the fact that $V(k)$ decreases for any $e(k)$ not in $B_2(\mu_1, \mu_2)$, we see immediately that the ellipsoids $\Re_1(\mu_1, \mu_2)$ and $\Re_2(\mu_1, \mu_2)$ are both invariant regions of EES. Now we use the inequality concerning the difference of $e^T(k) P e(k)$, to show that all trajectories starting in $\Re_1(\mu_1, \mu_2)$ will reach $\Re_2(\mu_1, \mu_2)$ in finite step S. When the estimation error vector $e(k)$ enters into $\Re_1(\mu_1,\mu_2)$, we obtain that

$$\begin{aligned} V(k+1)-V(k) &< -\varepsilon\frac{\lambda_m(Q)}{\lambda_M(P)}V(k) \\ \Rightarrow V(k+S) &\leqslant \left(1-\varepsilon\frac{\lambda_m(Q)}{\lambda_M(P)}\right)^S V(k) \\ &\leqslant \left(1-\varepsilon\frac{\lambda_m(Q)}{\lambda_M(P)}\right)^S \lambda_m(P) \\ &\cdot\left(\frac{M_1\mu_1-\delta_2-\|C\|\mu_2\Delta_2}{\|C\|}\right)^2 \end{aligned}$$

Suppose that estimation error $e(k)$ starts from $\Re_1(\mu_1,\mu_2)$ and enters into $\Re_2(\mu_1,\mu_2)$ by finite step S, so we have

$$\left(1-\varepsilon\frac{\lambda_m(Q)}{\lambda_M(P)}\right)^s \leqslant \frac{\lambda_M(P)R^2}{\lambda_m(P)\left(\frac{M_1\mu_1-\delta_2-\|C\|\mu_2\Delta_2}{\|C\|}\right)^2},$$

so that $S \geqslant \frac{E_1-E_2}{F}$. Let $S = \left\lceil\frac{E_1-E_2}{F}\right\rceil$, where $E_1 = \lg\left(\lambda_M(P)R^2\right)$, $E_2 = \lg\left(\lambda_m(P)\left(\frac{M_1\mu_1-\delta_2-\|C\|\mu_2\Delta_2}{\|C\|}\right)^2\right)$, $F = \lg\left(1-\varepsilon\frac{\lambda_m(Q)}{\lambda_M(P)}\right)$. ■

The similar result also can be obtained for QEFCS in (4).

Corollary 1: Consider the QEFCS in (4), given an arbitrary scalar $\xi \in (0,1)$, and $\tilde{P} > 0$, $\tilde{Q} > 0$ satisfying equation (6), then the state trajectories of QEFCS are quadratically attracted by the ball $\left\{x(k) : |x(k)| \leqslant \tilde{R}\right\}$, where $\tilde{R} = \frac{\tilde{b}+\sqrt{\tilde{b}^2+\tilde{a}\tilde{c}}}{\tilde{a}}$, $\tilde{a} = (1-\xi)\lambda_m\left(\tilde{Q}\right)$, $\tilde{b} = \tilde{\alpha}M_1\mu_1 + \tilde{\alpha}\delta_2 + \tilde{\beta}\delta_1$, $\tilde{c} = 2M_1\mu_1\tilde{\gamma}\delta_2 + 2M_1\mu_1\tilde{\theta}\delta_1 + 2\delta_2\tilde{\theta}\delta_1 + M_1^2\mu_1^2\tilde{\gamma} + \delta_2^2\tilde{\gamma} + \delta_1^2\tilde{\tau}$, $\|\tilde{\alpha}\| = \left\|(A+B_1KC)^T\tilde{P}B_1K\right\|$, $\tilde{\beta} = \left\|(A+B_1KC)^T\tilde{P}B_2\right\|$, $\tilde{\gamma} = \left\|K^TB_1^T\tilde{P}B_1K\right\|$, $\tilde{\theta} = \left\|K^TB_1^T\tilde{P}B_2\right\|$, $\tilde{\tau} = \left\|B_2^T\tilde{P}B_2\right\|$.

Proof: Let the Lyapunov function is $\tilde{V}(k) = x^T(k)\tilde{P}x(k)$, the rest proof is similar to Theorem 1. ■

Remark 1: Since the process noise $w(k)$ and the measurement noise $v(k)$ are both bounded, in view of R and $\tilde{R}$ in Theorem 1 and Corollary 1, there are minimum invariant regions $\Re_{\min}$ and $\tilde{\Re}_{\min}$ near the equilibrium point of EES and QEFCS, respectively. When $\mu_1 = \mu_2 = 0$, we obtain that

$$\Re_{\min} = \left\{e(k) : e^T(k)Pe(k) \leqslant \lambda_M(P)R_{\min}^2\right\} \quad (15)$$

and

$$\tilde{\Re}_{\min} = \left\{x(k) : x^T(k)\tilde{P}x(k) \leqslant \lambda_M\left(\tilde{P}\right)\tilde{R}_{\min}^2\right\}, \quad (16)$$

where $R_{\min} = \frac{\alpha\delta_1+\beta\delta_2+\sqrt{\Delta_{R_{\min}}}}{(1-\varepsilon)\lambda_m(Q)}$, $\Delta_{R_{\min}} = (\alpha\delta_1+\beta\delta_2)^2 + (1-\varepsilon)\lambda_m(Q)\left(2\delta_1\gamma\delta_2+\delta_1^2\tau+\delta_2^2\theta\right)$, $\tilde{R}_{\min} = \frac{\tilde{\alpha}\delta_2+\tilde{\beta}\delta_1+\sqrt{\Delta_{\tilde{R}_{\min}}}}{(1-\xi)\lambda_m(\tilde{Q})}$, $\Delta_{\tilde{R}_{\min}} = \left(\tilde{\alpha}\delta_2+\tilde{\beta}\delta_1\right)^2 + (1-\xi)\lambda_m\left(\tilde{Q}\right)\left(2\delta_2\tilde{\theta}\delta_1+\delta_1^2\tilde{\tau}+\delta_2^2\tilde{\gamma}\right)$.

In order to analyze quadratically stabilizable conditions and to find out qualitative connection between stability of EES and QEFCS. We constructed an ellipsoid set as follows:

$$\tilde{\Re}_0(\mu_1,\mu_2) = \left\{\begin{array}{l} x(k) : x^T(k)Px(k) \\ \leqslant \Gamma^2\lambda_m(P)\left(\frac{M_1\mu_1-\delta_2-\|C\|\mu_2\Delta_2}{\|C\|}\right)^2 \end{array}\right\} \quad (17)$$

where $\Gamma = \tilde{R}\big/R$, $\tilde{R} = \frac{\tilde{b}+\sqrt{\tilde{b}^2+\tilde{a}\tilde{c}}}{\tilde{a}}$, $R = \frac{b+\sqrt{b^2+ac}}{a}$.

III. QUADRATICALLY STABILIZABLE CONDITIONS AND CONNECTION

In this section, we use dynamic scaling strategy by adjusting quantization parameter $\mu_1(k)$ and $\mu_2(k)$ to change the size of invariant regions. Since only quantized estimation measurements $\hat{x}(k)$ and $Q_1\left(y(k)-\hat{y}(k)\right)$ are available, we have to find out the approximate location of $e(k)$ and $x(k)$ at each instant before presenting the quadratically stabilizable conditions of EES and QEFCS.

Like [12], we make several assumptions, one is that a bound on the initial state is known, that is $|x(0)|_\infty \leqslant \zeta$ for some $\zeta > 0$, the other one is made for simplicity, that is $A+B_1KC$ and $A-LC$ have only real eigenvalues. Let the Jordan canonical forms of the matrices $A+B_1KC$ and $A-LC$ be obtained as $J_1 = H_1(A+B_1KC)H_1^{-1}$, $J_2 = H_2(A-LC)H_2^{-1}$, respectively, where $H_1, H_2 \in \mathbb{R}^{n\times n}$. Let the new state variables be $z_1(k) = H_1x(k)$ and $z_2(k) = H_2e(k)$. For each $i = 1,2$, let $r_i(k) \in \mathbb{R}_+^n$, and define a rectangle $\Im_i(k) \subset \mathbb{R}^n$ by $\Im_i(k) := \left\{z\in\mathbb{R}^n : |z_j| \leqslant [r_i(k)]_j, j = 1,2,\cdots,n\right\}$, $k \in \mathbb{Z}_+$. Define an operator on matrices and vectors that takes absolute values of the entries by $\overline{H} : \mathbb{R}^{n_1\times n_2} \to \mathbb{R}^{n_1\times n_2}$, $\overline{H} = [|h_{ij}|]$ where $H = [h_{ij}]$. For logarithmic quantizer, the quantization error is proportional to the input componentwise [12] i.e., for $x \in \mathbb{R}^n$

$$\begin{array}{l} \left|[x-Q_i(x)]_j\right| \leqslant \dfrac{\Delta_i-1}{2}|x_j|, \\ j = 1,2,\cdots,n, i = 1,2. \end{array}$$

Using the operator $\overline{H}$, we get some matrices below

$$\begin{array}{c} A_{r11} = \overline{J_1} + \frac{\Delta_2-1}{2}\overline{H_1B_1KC}\overline{H_1^{-1}} \\ A_{r12} = \overline{H_1B_1KCH_2^{-1}} + \frac{\Delta_2-1}{2}\overline{H_1B_1KC}\overline{H_2^{-1}} \\ A_{r13} = \overline{H_1B_2}, A_{r14} = 0 \\ A_{r21} = \frac{\Delta_1-1}{2}\frac{\Delta_2-1}{2}\overline{H_2LC}\overline{H_1^{-1}} \\ A_{r22} = \overline{J_2} + \frac{\Delta_1-1}{2}\overline{H_2LCH_2^{-1}} \\ +\frac{\Delta_1-1}{2}\frac{\Delta_2-1}{2}\overline{H_2LC}\overline{H_2^{-1}} \\ A_{r23} = \overline{H_2B_2}, A_{r24} = \frac{\Delta_1+1}{2}\overline{H_2L} \end{array}$$

Let $r_{1j}(0) = \|H_1\|\zeta, r_{2j}(0) = \|H_2\|\zeta,\ j = 1, 2, \cdots, n,$ we adopt the following linear system of $r_i(k), i = 1, 2.$

$$\begin{bmatrix} r_1(k+1) \\ r_2(k+1) \end{bmatrix} = \begin{bmatrix} A_{r_{11}} & A_{r_{12}} \\ A_{r_{21}} & A_{r_{22}} \end{bmatrix} \begin{bmatrix} r_1(k) \\ r_2(k) \end{bmatrix} + \begin{bmatrix} A_{r_{13}} & A_{r_{14}} \\ A_{r_{23}} & A_{r_{24}} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$

Now we find out the approximate location of $e(k)$ and $x(k)$ for dynamic system in (3) and (4).

Lemma 1: If there exists matrices K and L satisfying (5) and (6), then the dynamic system of $r_1(k)$ and $r_2(k)$ determines the rectangles $\Im_i(k), i = 1, 2,$ that $z_i(k) \in \Im_i(k), i = 1, 2$ for $k \in \mathbb{Z}_+$.

Proof: The proof is the extension of [12], due to the limit of the space, it is omitted here. ■

Remark 2: When systems in (3) and (4) are stable, noting that the Jordan matrices J_1 and J_2 are Hurwitz and are upper triangular since they have only real eigenvalues by assumption. Hence, $\overline{J_1}$ and $\overline{J_2}$ are Hurwitz as well, so is the state space model of $r_1(k)$ and $r_2(k)$.

Remark 3: From $z_i(k) \in \Im_i(k)$, we can obtain the approximate location of $e(k)$ and $x(k)$ in EES and QEFCS; when $r_1(k) \in \tilde{\Re}_0(\mu_1, \mu_2)$ and $r_2(k) \in \Re_1(\mu_1, \mu_2)$, we conclude that $x(k) \in \tilde{\Re}_0(\mu_1, \mu_2)$ and $e(k) \in \Re_1(\mu_1, \mu_2)$.

Theorem 2: Assume that M_1 is large enough compared to Δ_1 so that we have

$$\sqrt{\frac{\lambda_m(P)}{\lambda_M(P)}} \geqslant \frac{\alpha\delta_1 + \beta\delta_2 + \beta\Delta_1 + \rho}{\lambda_m(Q)} \cdot \frac{\|C\|}{M_1 - \delta_2 - \|C\|\Delta_2} \tag{18}$$

where$\rho = \sqrt{(\alpha\delta_1 + \beta\delta_2 + \beta\Delta_1)^2 + \lambda_m(Q)c_\rho}$, $c_\rho = 2\delta_1\gamma\delta_2 + 2\delta_1\gamma\Delta_1 + 2\delta_2\theta\Delta_1 + \delta_1^2\tau + \delta_2^2\theta + \Delta_1^2\theta$, then for the quantized estimation feedback control law $u(k) = K\hat{y}(k)$, QEFCS and EES are quadratically stabilizable.

Proof: Due to the limit of the space, it is omitted here. ■

Remark 4: We conclude that EES and QEFCS satisfying bounded-input-bounded-output(BIBO) stable, since state $x(k)$ and error $e(k)$ will run into their minimum invariant regions around the equilibrium point.

Given dynamic scaling quantization approach in Theorem 2, it is possible to reveal quadratic stability connection between EES and QEFCS, as we now show.

Theorem 3: Assume the parameters $\varepsilon, \xi, P, Q, \tilde{P}, \tilde{Q}$ satisfying

$$\frac{\xi}{\varepsilon} \geqslant \frac{\lambda_M(\tilde{P})}{\lambda_M(P)} \Big/ \frac{\lambda_m(\tilde{Q})}{\lambda_m(Q)}, \tag{19}$$

then $x(k)$ in QEFCS starting in $\tilde{\Re}_0(\mu_1, \mu_2)$ will move along the invariant region sequence $\left\{\tilde{\Re}_t(\mu_1(U), \mu_2(U)), U \in \mathbb{N}\right\}$, and enter into the minimum invariant region $\tilde{\Re}_{\min}$ in finite step S. where

$$\begin{aligned} &\tilde{\Re}_t(\mu_1(U), \mu_2(U)) \\ &= \left\{ \begin{aligned} &x(k) : x^T(k)\tilde{P}x(k) \\ &\leqslant \left(\Gamma^2 \frac{\lambda_M(\tilde{P})}{\lambda_m(P)}\right)\eta^{tS}\cdot \\ &\lambda_m(P)\left(\frac{M_1\mu_1(U) - \delta_2 - \|C\|\mu_2(U)\Delta_2}{\|C\|}\right)^2 \end{aligned} \right\} \\ &\eta = \frac{1 - \xi\lambda_m(\tilde{Q})/\lambda_M(\tilde{P})}{1 - \varepsilon\lambda_m(Q)/\lambda_M(P)}, \Gamma \quad is \quad in \quad (17), \\ &U = k_0 + tS, t = 1, 2, \cdots \end{aligned}$$

Likewise, $e(k)$ in EES starting in $\Re_1(\mu_1, \mu_2)$ will move along the invariant region sequence $\{\Re_t(\mu_1(U), \mu_2(U)), U \in \mathbb{N}\}$ which with the same shape of $\tilde{\Re}_t(\mu_1(U), \mu_2(U))$, and enter into the minimum invariant region $\Re_{\min}$ in finite step S, where

$$\begin{aligned} &\Re_t(\mu_1(U), \mu_2(U)) = \\ &\left\{ \begin{aligned} &e(k) : e^T(k)Pe(k) \\ &\leqslant \left(1 - \varepsilon\frac{\lambda_m(Q)}{\lambda_M(P)}\right)^{tS}\lambda_m(P)\cdot \\ &\left(\frac{M_1\Sigma^t\mu_1(k_0) - \delta_2 - \|C\|\Sigma^t\mu_2(k_0)\Delta_2}{\|C\|}\right)^2 \end{aligned} \right\} \\ &\Sigma \quad is \quad in \quad (22), U = k_0 + tS, t = 1, 2, \cdots \end{aligned}$$

Proof: Due to the limit of the space, it is omitted here. ■

IV. ILLUSTRATIVE EXAMPLE

In this section, we give a numerical example to illustrate quadratically stabilizable conditions and stability connection of EES and QEFCS in quantized estimation feedback control problem. Consider the unstable system as follows:

$$\begin{cases} x(k+1) = \begin{bmatrix} 0 & 1 \\ 0.9 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.8 \\ 0.5 \end{bmatrix} u(k) \\ + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} w(k) \\ y(k) = \begin{bmatrix} 1.2 & 2.0 \end{bmatrix} x(k) + v(k) \end{cases}$$

Let the positive definite symmetric matrices $Q = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \tilde{Q} = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$, the Lyapunov matrices $P = \begin{bmatrix} 6.9079 & 1.0257 \\ 1.0257 & 11.8051 \end{bmatrix}, \tilde{P} = \begin{bmatrix} 6.7570 & -6.5943 \\ -6.5943 & 3.4917 \end{bmatrix}$, feedback gain $K = -0.5000$, estimation gain $L = \begin{bmatrix} 0.0334 & 0.3999 \end{bmatrix}^T$.

The upper bound values of $w(k)$ and $v(k)$ are $\delta_1 = 0.3353$ and $\delta_2 = 0.3246$, respectively. The parameters of logarithmic quantizer are taken as $M_1 = 7.0880$, $M_2 =$

10.2067, Δ_1 = 0.5907, Δ_2 = 0.8506. In view of Theorem 1, 2 and 3, the simulation results are depicted in Figure 3-8.

The closed-loop system trajectories x_1 and e_1 are shown in Figure 3-4. From the above simulation results, we can see that in order to force the states of EES and QEFCS to enter into the initial ellipsoid, there exists a moving direction of state trajectory of x_1 leaving away from equilibrium point in the zooming-out stage in Figure 5. The reason is that the quantized estimation system is open-loop at those instants and QEFCS is not stable. While in zooming-in stage, the state trajectories of EES and QEFCS enter into the invariant ellipsoids, respectively. With step increasing, the two state trajectories enter into the minimum invariant regions near the equilibrium points ultimately, and will not jump out of the minimum invariant regions since then, see Figure 6 and Figure 8. Stability connection of EES and QEFCS is also depicted in Figure 5 and Figure 7, which present the same shapes of two invariant region sequences and indicate quadratically attractive trajectories of x and e when (23) holds.

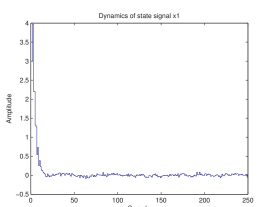

Fig. 3. Response of state x_1

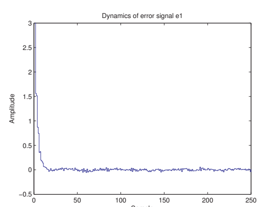

Fig. 4. Response of error e_1

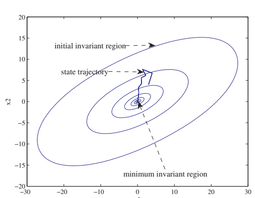

Fig. 5. Invariant region sequences of state x

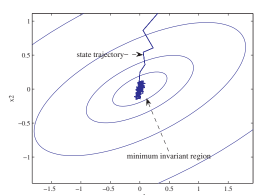

Fig. 6. Magnified image of Figure 5

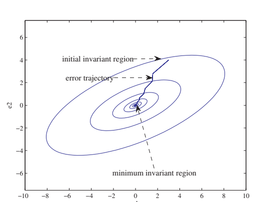

Fig. 7. Invariant region sequences of error e

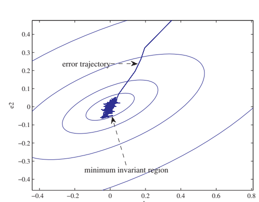

Fig. 8. Magnified image of Figure 7

V. CONCLUSIONS

This paper analyzes stability of quantized estimation and investigates stability connection between EES and QEFCS. We propose that there are multi-quantized signals being passed to the estimator and the controller through network channels in the closed-loop networked control system. Using adjustable zoom quantizer parameters, Lyapunov-based quadratically stabilizable conditions of networked control systems are presented. We also introduce a method to disclose the stability relationship between EES and QEFCS by the use of the connected invariant region sequences. Numerical example and simulation results illustrate the effectiveness of the method.

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