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Analysis of Chattering in Sliding Mode Control Systems with Continuous Boundary Layer Approximation of Discontinuous Control

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Abstract — **It has been a widely accepted notion that approximation of discontinuous control by certain continuous function in a boundary layer results in chattering elimination in sliding mode (SM) control systems. It is shown through three different types of analysis that in the presence of parasitic dynamics, this approach to chattering elimination would work only if the slope of the continuous nonlinear function within the boundary layer is low enough, which may result in the deterioration of performance of the system. A few examples are provided.**

I. INTRODUCTION

SLIDING mode (SM) control features such valuable properties as low sensitivity to external disturbances (theoretically insensitive) and robustness with respect to plant parameter uncertainties and variations. However, the price being paid for that is an undesirable phenomenon called chattering, which is revealed as high-frequency vibrations (oscillations) of the control, which in turns results in the oscillations of the state trajectory around the sliding surface [1]. This drawback of the SM control is well known, and many efforts of the researched were aimed at elimination of chattering or mitigation of its effects. One of the proposed remedies is the replacement of the discontinuous control with its continuous approximation in the vicinity of the sliding surface, which is also known as boundary layer (around the sliding surface) introduction [2]-[4].

However, it is well known that chattering happens in a SM control system when parasitic dynamics (that were not accounted for in the design of the sliding surface) are present. It should be noted that the presence of some kind of parasitic dynamics is inevitable [5], [6]. This is why chattering is inevitable when the control is discontinuous. Yet, despite the popularity of the boundary layer approximation, the conditions of the occurrence of chattering were not analyzed if parasitic dynamics are present. The idea of chattering suppression basically stems from the logical conclusion that chattering is a feature of discontinuous control, and replacement of discontinuous control with a continuous one would result in chattering elimination. However, it was shown in [7] that chattering (as high-frequency vibrations of continuous nature) may exist in SM systems with continuous control too. This fact motivates the present research aimed at finding out if and when chattering may exist in SM systems with continuous approximation of discontinuous control in the boundary layer.

The paper is organized as follows. At first the models of the continuous approximations are considered. Then analysis with the used of the describing function (DF) method is undertaken. After that the sufficient conditions are derived through the use of the Popov's criterion of absolute stability [8]. After that, exact analysis of chattering is proposed with application of the Poincare map to the problem. Finally, an example of analysis is given.

II. CONTINUOUS APPROXIMATION OF DISCONTINUOUS CONTROL IN BOUNDARY LAYER

At first, we consider a linear plant and a discontinuous SM controller. Let the linear part of the system, which includes the plant, the sliding surface, and some kind of parasitic dynamics (for example, the dynamics of sensors and actuators not accounted for in the model that is used for determination of the sliding surface), be described by the following equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}u , \qquad (1)$$

$$\sigma = \mathbf{C}\mathbf{x} , \qquad (2)$$

where $\mathbf{x} \in \mathbf{R}^n$ is the state vector (that includes the states of both the principal and the parasitic dynamics), σ is the sliding variable, $\mathbf{A} \in \mathbf{R}^{n\times n}$, $\mathbf{B} \in \mathbf{R}^{n\times 1}$, $\mathbf{C} \in \mathbf{R}^{1\times n}$ are matrices, $\mathbf{A}$ is assumed nonsingular (for singular matrix $\mathbf{A}$, the approach of [9] can be used). The sign "-" before $\mathbf{B}u$ is attributed to the negative feedback, so that conventional shapes of nonlinearities with location in 1st and 3rd quadrants can be considered. It is worth noting that the model (1), (2) includes parasitic dynamics in the form of the actuator dynamics but not of the sensor dynamics because it is assumed below that the sliding variable σ is the direct input to the controller. In real life applications, there would be some additional dynamics (of sensors) between the sliding variable that combines the plants states (and, therefore, involves measurements of these states) and the controller. However, parasitic dynamics of an actuator and of a sensor possess the *duality* property because both of them are connected in series with the plant, which results in the same effect regardless of where they are connected: at the plant input or output [6]. Therefore, the effect of parasitic

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dynamics in the model (1), (2) is contained in the relative degree of the linear part from the control u to the output σ: the relative degree is equal to *one* if there are no parasitic dynamics, and it is greater than *two* if there are parasitic dynamics [6]. Also, we do not use the popular term *unmodeled dynamics* because we are including these dynamics in the model; however, we realize that there still a mismatch between the actual actuator (and sensor) dynamics and their models, so that there still may be unaccounted for or remaining unmodeled dynamics in this case too [10].

The discontinuous control is given as follows:

$$u = \begin{cases} h & \text{if } \sigma > 0 \\ -h & \text{if } \sigma < 0 \end{cases} \tag{3}$$

It is well known that if some kind of parasitic dynamics are present (which always happens and results in the relative degree from the control u to the output y higher than *two* [6]) then chattering occurs in the SM control system. It was proposed that chattering could be eliminated through a continuous approximation of the discontinuous control (3) in a boundary layer around the sliding surface. In particular, it was proposed in [2] that a certain boundary layer should be created in the vicinity of the sliding surface:

$$B = \{x, |\sigma(x)| \le \Phi\}, \tag{4}$$

where $\Phi \ge 0$ is the boundary layer thickness. According to [2], outside B, the control is given by (3) as before (which would guarantee that the boundary layer is attractive and, therefore, invariant: all trajectories stay inside B once get there). It is supposed to provide a guaranteed precision rather then ideal precision. Yet inside B, it was proposed that the linear function should be used as the following linear control:

$$u = h\sigma / \Phi\,,\ \mathbf{x} \in B \tag{5}$$

And, therefore, in overall the discontinuous control (3) is replaced with the saturation function:

$$u = h\,\mathrm{sat}(\sigma / \Phi)\,, \tag{6}$$

Other types of continuous approximation of discontinuous control in the boundary layer are possible too. The following example gives another continuous control function [3], [4]:

$$u = h\frac{\sigma}{|\sigma| + \varepsilon}\,, \tag{7}$$

where $\varepsilon > 0$ is the boundary layer width (a small number).

The idea of the boundary layer, from the point of view of its authors and subsequent contributors, was supposed to totally eliminate chattering. However, the aspect of growth of the relative degree due to the introduction of parasitic dynamics was not considered and the conclusion about chattering elimination was made for the system containing only principal dynamics. We aim to analyze performance of SM systems with boundary layer continuous control, in which parasitic dynamics are present. The analysis is presented in the following three sections, with the use of three different methods: the describing function, the Popov's method, and the Poincare map method.

III. Describing Function Analysis of SM Systems with Boundary Layer Continuous Control

We apply the describing function (DF) method [11] to the analysis of motions in the system (1), (2) having the control (6), (7) or another single-valued monotone odd-symmetric continuous function $f(\sigma)$ fully located in the 1st and 3rd quadrants and satisfying the following conditions:

$$|f(\sigma)| \le h\,, \tag{8}$$

$$\left.\frac{\mathrm{d}\,f(\sigma)}{\mathrm{d}\,\sigma}\right|_{\sigma=0} = \frac{h}{\Phi} \tag{9}$$

$$\frac{\mathrm{d}^2\,f(\sigma)}{\mathrm{d}\,\sigma^2} < 0 \tag{10}$$

The first condition specifies that the function must be saturating, and the second condition sets the limit on the derivative of this function, matching it with the derivative of the linear approximation (5). The nonlinearity satisfying conditions (8)-(10) is fully located in the area defined by the saturation function (6) (Fig. 1).

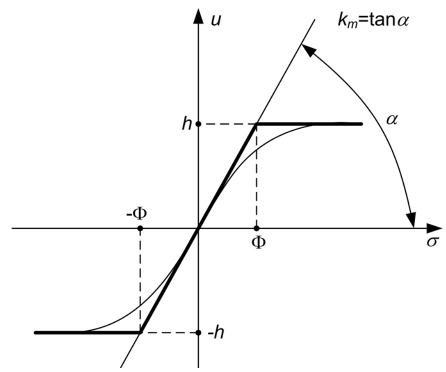


Fig. 1. Boundary layer nonlinearities

We find the DF of the nonlinearity $f(\sigma)$ as follows [11]:

$$N(a) = \frac{2}{\pi a}\int_0^{\pi} f(a\sin\psi)d\psi\,, \tag{11}$$

Because the function is odd-symmetric, we rewrite (11):

$$N(a) = \frac{4}{\pi a}\int_0^{\pi/2} f(a\sin\psi)d\psi\,, \tag{12}$$

It follows from (8)-(10) that the DF of the nonlinearity $f(\sigma)$ is a monotone function of amplitude a satisfying

$$0 < N(a) \le \frac{2h}{\pi\Phi}\left(\sin^{-1}\frac{\Phi}{a} + \frac{\Phi}{a}\sqrt{1-\frac{\Phi^2}{a^2}}\right), \tag{13}$$

$$N(0) = h/\Phi \tag{14}$$

For the nonlinearity given by (7), $\left.\frac{\mathrm{d}\,f(\sigma)}{\mathrm{d}\,\sigma}\right|_{\sigma=0} = N(0) = h/\varepsilon$ that falls into the definition (8)-(10) when $\varepsilon = \Phi$. The condition of the absence of chattering as self-excited oscillations in the closed-loop system that includes the plant (with possible parasitic dynamics), the sliding surface, and the considered nonlinear controller can be formulated as the absence of solution of the harmonic balance equation:

$$\forall \omega \geq 0, a \geq 0 \quad W_l(j\omega) \neq -1/N(a), \tag{15}$$

where $W_l(j\omega)$ is the frequency response of the linear part (plant and sliding surface). Condition (15) can be depicted in the complex plane as in Fig. 2, as the condition of the absence of intersection between the Nyquist plot of the linear part and the negative reciprocal of the DF of the nonlinearity.

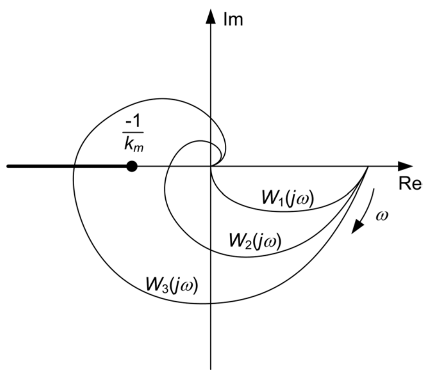


Fig. 2. Condition of chattering existence

The negative reciprocal DF of the nonlinearity defined by (8)-(10) is always located on the real axis left of the point ($-\Phi/h$, $j0$)= ($-1/k_m$, $j0$). From Fig. 2, one can see that the system with continuous approximation of control in the boundary layer may or may not have chattering – depending on the contribution of parasitic dynamics. If parasitic dynamics are not present and the Nyquist plot approaches the origin from below along the imaginary axis, so that there is no point of intersection (plot $W_1(j\omega)$), chattering does not occur. If parasitic dynamics are present and there is a point of intersection of $W_l(j\omega)$ with the real axis but this point of intersection is located right of the point ($-1/k_m$, $j0$) (plot $W_2(j\omega)$), then chattering does not occur either. Yet if parasitic dynamics are present and there is a point of intersection of $W_l(j\omega)$ with the negative reciprocal DF of the nonlinearity (plot $W_3(j\omega)$) then chattering occurs.

The frequency and the amplitude of chattering (when it occurs) can be easily found from the harmonic balance equation. Because negative reciprocal of the DF for any $f(\sigma)$ from the considered class is located on the real axis, the frequency of chattering Ω will correspond to -180° phase lag of the linear part: $\arg W_l(j\Omega) = -180^\circ$. The amplitude of chattering is found from the same harmonic balance equation but will depend on the nonlinearity $f(\sigma)$.

It is also worth noting that the presented DF analysis fully agrees with analysis of this system as of a linear one subject to the consideration of only saturation nonlinearity and the assumption that the motion occur in the linear zone of the nonlinearity. Indeed, the Nyquist stability criterion would state that the frequency response plot of the open-loop system should not encircle the point (-1,$j0$) or, alternatively, the plot $W_l(j\omega)$ should not encircle the point ($-1/k_m$, $j0$), which coincides with the DF analysis results for the same nonlinearity. Yet, the DF analysis is valid for the whole class of nonlinearities.

Therefore, chattering suppression by the continuous approximation of control in the boundary layer is possible only if the value of k_m is sufficiently small. This, however, would reduce the system performance, which reveals a trade-off between the possibility of chattering suppression and system performance.

IV. Analysis of SM Systems with Boundary Layer Continuous Control Based on Popov's Stability Criterion

It is known that V.M. Popov's criterion of absolute stability [8] allows one to establish sufficient conditions of stability of an equilibrium point for some class of nonlinear functions. This property becomes very convenient with respect to the analysis of the control approximation in the boundary layer.

Again, we consider the class of single-valued odd-symmetric nonlinear functions that define control u, given by (8)-(10). These formulas, as was mentioned earlier, define the class of nonlinearities satisfying the so-called sector condition that fully falls in the scope of the conditions of the Popov's theorem [8]. It says that for the system comprising the nonlinearity satisfying the sector condition [0;k_m] (see Fig. 2), and the stable linear part with controllable pair (**A**;**B**), the point of origin will be globally asymptotically stable if there exists a strictly positive number α, such that

$$\forall \omega \geq 0 \qquad \mathrm{Re}\left[(1 + j\alpha\omega) W_l(j\omega)\right] + \frac{1}{k_m} \geq e, \tag{16}$$

for an arbitrarily small e>0.

Considering the constraints on the class of nonlinearities (8)-(10) we can see that for this class k_m is $k_m = h/\Phi$ for every nonlinearity, and, therefore, the stability analysis again arrives at checking the location of the point $(-1/k_m, j0) = (-\Phi/h, j0)$ relative to a certain frequency-domain characteristic in the complex plane. This frequency-domain characteristic is the Popov's curve (modified frequency response), which is formed from the frequency response $W_l(j\omega)$ by multiplying the imaginary part by frequency ω: $W_m(\omega) = \mathrm{Re}\, W_l(j\omega) + j\omega\, \mathrm{Im}\, W_l(j\omega)$.

The sufficient condition of absolute stability of the origin, and therefore, the condition of the elimination of chattering, is the possibility of drawing a straight line through the point ($-1/k_m$, $j0$) that would not intersect the plot $W_m(\omega)$ (Fig. 3). This is possible only if the value of k_m is sufficiently small, which in turn would reduce the system performance. Therefore, as it was noted earlier, there is a trade-off between the possibility of chattering suppression and the system performance.

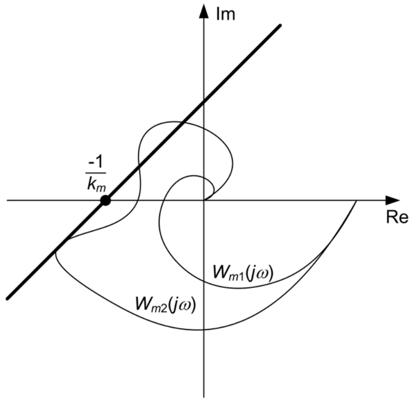


Fig. 3. Condition of chattering elimination (per Popov's criterion)

V. Analysis of Chattering in SM Systems with Boundary Layer Continuous Control, Based on Poincare Map

We now approach the problem being studied assuming that chattering exists and derive Poincare map for the system being analyzed. We find analytical solution for every part of the piece-wise linear control, and join the solutions. Consider solutions for the following controls:

$$u = \begin{cases} h & \text{if } \sigma \ge b \\ K\sigma & \text{if } -b < \sigma < b \\ -h & \text{if } \sigma \le -b \end{cases}, \tag{17}$$

Assume the existence of a symmetric limit cycle and find parameters of this limit cycle. Let θ_1 be the duration of motion under control *u*=*h*, and θ_2 be the duration of motion under control *u*=*K*σ, so that the period of the oscillation is $T = 2(\theta_1 + \theta_2)$. We assume that time *t*=0 corresponds to the starting point of control *u*=*h*, so that the following relationships hold: $y(0) = b = h/K$, $\dot{y}(0) > 0$, $y(\theta_1) = b = h/K$, $\dot{y}(\theta_1) < 0$, $y(\theta_1+\theta_2) = -b = -h/K$, $\dot{y}(\theta_1+\theta_2) < 0$ (see Fig. 4). Considering the following response of the linear part to the constant control *u*:

$$\mathbf{x}(t) = \mathrm{e}^{\mathbf{A}t}\,\mathbf{x}(0) - \mathbf{A}^{-1}\left(\mathrm{e}^{\mathbf{A}t} - \mathbf{I}\right)\mathbf{B}u\,, \tag{18}$$

we can find the following mappings: $\boldsymbol{\rho} = \mathbf{x}(0) \to \boldsymbol{\eta} = \mathbf{x}(\theta_1)$ and $\boldsymbol{\eta} = \mathbf{x}(\theta_1) \to \boldsymbol{\rho}' = \mathbf{x}(\theta_1+\theta_2)$, the fixed point of which will be defined by the identity $\boldsymbol{\rho}' = -\boldsymbol{\rho}$ (considering the condition of a symmetric motion). The mapping $\boldsymbol{\rho} = \mathbf{x}(0) \to \boldsymbol{\eta} = \mathbf{x}(\theta_1)$ is given by the following formula (on the time interval $t \in [0;\theta_1]$ the control *u*=*h*):

$$\boldsymbol{\eta} = \mathrm{e}^{\mathbf{A}\theta_1}\,\boldsymbol{\rho} - \mathbf{A}^{-1}\left(\mathrm{e}^{\mathbf{A}\theta_1} - \mathbf{I}\right)\mathbf{B}h\,. \tag{19}$$

The mapping $\boldsymbol{\eta} = \mathbf{x}(\theta_1) \to \boldsymbol{\rho}' = \mathbf{x}(\theta_1+\theta_2)$ can be derived as follows, considering that on the time interval $t \in [\theta_1;\theta_1+\theta_2]$ the control *u*=*K*σ. The equations of the system can be considered as equations of the free (unforced) motion in the closed-loop system (follows from (1) and (17):

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}K\sigma = \mathbf{A}\mathbf{x} - K\mathbf{B}\mathbf{C} = \mathbf{A}'\mathbf{x}\,, \tag{20}$$

where $\mathbf{A}' = \mathbf{A} - K\mathbf{B}\mathbf{C}$. Therefore,

$$\boldsymbol{\rho}' = \mathrm{e}^{\mathbf{A}'\theta_2}\,\boldsymbol{\eta}\,. \tag{21}$$

Now solve the equation $\boldsymbol{\rho}' = -\boldsymbol{\rho}$ that defines the fixed point.

$$-\boldsymbol{\rho} = \mathrm{e}^{\mathbf{A}'\theta_2}\left(\mathrm{e}^{\mathbf{A}\theta_1}\,\boldsymbol{\rho} - \mathbf{A}^{-1}\left(\mathrm{e}^{\mathbf{A}\theta_1} - \mathbf{I}\right)\mathbf{B}h\right)$$

From the last equation, we find the following solution:

$$\boldsymbol{\rho} = \left(\mathbf{I} + \mathrm{e}^{\mathbf{A}T/2}\mathrm{e}^{-\mathbf{B}\mathbf{C}K\theta_2}\right)^{-1}\left(\mathrm{e}^{\mathbf{A}T/2} - \mathrm{e}^{\mathbf{A}\theta_2}\right) \times \mathrm{e}^{-\mathbf{B}\mathbf{C}K\theta_2}\,\mathbf{A}^{-1}\mathbf{B}h \tag{22}$$

Considering the following relationship between $\boldsymbol{\rho}$ and $\boldsymbol{\eta}$

$$\boldsymbol{\eta} = -\mathrm{e}^{-\mathbf{A}'\theta_2}\,\boldsymbol{\rho}$$

we can find $\boldsymbol{\eta}$ as follows:

$$\boldsymbol{\eta} = \left(\mathbf{I} + \mathrm{e}^{\mathbf{A}T/2}\mathrm{e}^{-\mathbf{B}\mathbf{C}K\theta_2}\right)^{-1}\left(\mathbf{I} - \mathrm{e}^{\mathbf{A}T/2}\mathrm{e}^{-\mathbf{A}\theta_2}\right)\mathbf{A}^{-1}\mathbf{B}h\,. \tag{23}$$

From formula (22), one can see that if intervals duration θ_1 and θ_2 are treated as independent variables the periodic motion of period *T* is established in the system, with the value of the state vector at *t*=0 given by (22). However, durations θ_1 and θ_2 are not independent variables but the result of the switching conditions (from one control to the other one). Therefore, θ_1 and θ_2 are found from the following two equations: $\mathbf{C}\boldsymbol{\rho} = b$ and $\mathbf{C}\boldsymbol{\eta} = b$, which are complemented by the inequalities $\dot{y}(0) > 0$ and $\dot{y}(\theta_1) < 0$ that need to be checked. Considering also that $b = h/K$, we write the following two equations for θ_1 and θ_2.

$$\mathbf{C}\left(\mathbf{I} + \mathrm{e}^{\mathbf{A}T/2}\mathrm{e}^{-\mathbf{B}\mathbf{C}K\theta_2}\right)^{-1}\left(\mathrm{e}^{\mathbf{A}T/2} - \mathrm{e}^{\mathbf{A}\theta_2}\right)\mathrm{e}^{-\mathbf{B}\mathbf{C}K\theta_2}\,\mathbf{A}^{-1}\mathbf{B} = \frac{1}{K}\,, \tag{24}$$

$$\mathbf{C}\left(\mathbf{I} + \mathrm{e}^{\mathbf{A}T/2}\mathrm{e}^{-\mathbf{B}\mathbf{C}K\theta_2}\right)^{-1}\left(\mathbf{I} - \mathrm{e}^{\mathbf{A}T/2}\mathrm{e}^{-\mathbf{A}\theta_2}\right)\mathbf{A}^{-1}\mathbf{B} = \frac{1}{K}\,. \tag{25}$$

Solution of (24), (25) can be better presented as the solution for the frequency $\Omega = \pi/(\theta_1+\theta_2)$ (or period) of oscillations and for the *relative interval duration* $\gamma = \theta_1/(\theta_1+\theta_2)$.

It is worth noting that (24) and (25) do not depend on such parameter of the control as *h* but depend only on *K*, which results in the *invariance of chattering frequency and relative interval duration with respect to the control amplitude h* (as far as the value of *K* is constant). Analysis of the derived relationships is also interesting from the point of view of the dependence of the solution on the value of gain *K*. We can legitimately assume that the increase of gain *K* will result in the increase of θ_1 and in the decrease of θ_2, so that when $K \to \infty$ the control transforms into the relay control with an ideal relay that corresponds to $\theta_2 = 0$ and $\theta_1 = T/2$. This limiting case results in the following relationship:

$$\boldsymbol{\rho}_r = \left(\mathbf{I} + e^{\mathbf{A}T/2}\right)^{-1}\left(e^{\mathbf{A}T/2} - \mathbf{I}\right)\mathbf{A}^{-1}\mathbf{B}h,$$

where the subscript "r" refers to the relay system.

For the practical solution of (24) and (25), it is reasonable to assume that the frequency $\Omega = \pi/(\theta_1 + \theta_2)$ of chattering belongs to the interval $[\Omega_r ; \Omega_l]$, where Ω_r is the frequency of chattering in the relay system with the same plant, which corresponds to the case of $\theta_1 \to T/2$, $\theta_2 \to 0$, and Ω_l is the frequency of oscillations (that are only theoretically possible) in the marginally stable linear system with the same plant, which corresponds to the case of $\theta_1 \to 0$, $\theta_2 \to T/2$. These two frequencies are close and from the point of view of the describing function analysis are even the same. This fact significantly simplifies the solution of (24), (25), as solution of (24), (25) is complex and involves sorting out the solutions that do not provide a fixed point of Poincare map (equation (24) or (25) is the condition of the equality of the plant output to the value of b, which may be provided by a non-periodic signal). The proposed solution schema involves variation of parameters Ω and γ within the intervals:

$$\Omega \in [\Omega_r ; \Omega_l], \quad \gamma \in [0;1] \tag{26}$$

The practical aspects of finding θ_1 and θ_2 are illustrated by the example in the following section.

VI. Relation to Stability of Linear Systems

If the value of K is decreased to the level of $K \to K_{cr}$, where K_{cr} is the gain of the proportional control that makes the system marginally stable, then the parameters of the oscillations approach the following limits: $\theta_1 \to 0$, $\theta_2 \to T/2$. In fact, the value of K_{cr} can be found as the limiting case of (24), (25) when $\theta_1 \to 0$, $\theta_2 \to T/2$. In this case, the nonlinear system with saturation transforms into the marginally stable linear system revealing an undamped oscillation. The value of the saturation can serve as a natural measure of the size of the domain in the definition of stability. Because we are considering a linear system, the behavior of the system output σ can characterize the stability of the system. Therefore considering the definition of stability that says that the equilibrium point $\mathbf{x} = 0$ is stable if for any $R > 0$ there exists $r > 0$, such that if $\|\mathbf{x}(0)\| < r$, then $\|\mathbf{x}(t)\| < R$ for all $t \ge 0$, and analyzing the behavior of σ, we can conclude that if $|\sigma(0)| < r$ and $|\sigma(t)| < R = b = h/K$ for all $t \ge 0$, the system will be stable. This in turn happens if $K < K_{cr}$. We, therefore, can formulate the following statement.

Theorem (robust stability). Linear system with plant given by (1), (2) and control $u = K\sigma$ is stable for all K, $0 < K < K_{cr}$, if matrix $\mathbf{H} = \mathbf{I} + e^{(\mathbf{A} - \mathbf{BC}K_{cr})T/2}$ is singular, where T is the period of oscillations in the marginally stable system with state matrix being $\mathbf{A} - \mathbf{BC}K_{cr}$, and K_{cr} being the smallest real positive number that makes $\mathbf{H}$ singular.

Proof. Consider the limiting case of (24) for $\theta_1 \to 0$, $\theta_2 \to T/2$:

$$\mathbf{C}\left(\mathbf{I} + e^{\mathbf{A}T/2} e^{-\mathbf{BC}K_{cr}T/2}\right)^{-1}\left(e^{\mathbf{A}T/2} - \lim_{\theta_2 \to T/2} e^{\mathbf{A}\theta_2}\right)$$

$$\times e^{-\mathbf{BC}K_{cr}T/2}\,\mathbf{A}^{-1}\mathbf{B} = \frac{1}{K_{cr}}$$

One can see that $e^{\mathbf{A}T/2} - \lim_{\theta_2 \to T/2} e^{\mathbf{A}\theta_2} = 0$, which requires that $\mathbf{H} = \mathbf{I} + e^{(\mathbf{A} - \mathbf{BC}K_{cr})T/2}$ should be singular (for K_{cr} to be finite), which in turn means that the following should hold: $\det \mathbf{H} = 0$. Because the matrix $\mathbf{A} - \mathbf{BC}K_{cr}$ is the state matrix of the closed-loop system and this system is marginally stable, this matrix should have a pair of eigenvalues with *zero* real parts. The matrix $e^{(\mathbf{A} - \mathbf{BC}K_{cr})T/2}$, as being the state transition matrix with the time being the half-period, has at least two eigenvalues equal to –1, and other eigenvalues within the unit disk. (Note: $e^{(\mathbf{A} - \mathbf{BC}K_{cr})T/2}$ can be considered a Poincare map in the marginally stable system having a periodic process.) This leads to $\det \mathbf{H} = 0$.■

VII. Example

Consider the system given by the following transfer function $W_p(s) = 1/(s^2 + s + 1)$, with the sliding surface given by $W_{ss}(s) = s + 1$ and the parasitic dynamics is given by $W_{pd}(s) = 1/(0.0001s^2 + 0.01s + 1)$. These models give the following transfer function of the linear part:

$$W_l(s) = \frac{s+1}{0.0001s^4 + 0.0101s^3 + 1.0101s^2 + 1.01s + 1}$$

and the state space model with the following matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10000 & -10100 & -10101 & -101 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10000 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$$

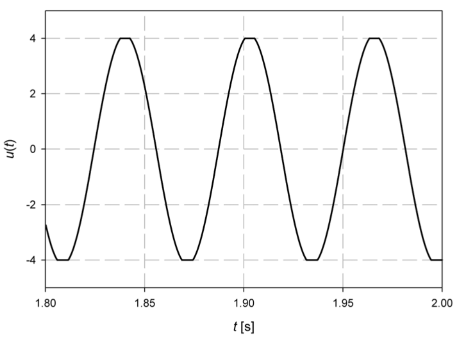
The task of analysis is to find the critical gain K_{cr} of the saturation nonlinearity that ensures stability of the system for $K \in [0; K_{cr}]$, and find parameters of periodic motions for a few different points of $K \in [K_{cr}; \infty)$.

The analysis of the linear system with the feedback gain K gives the critical gain value of $K_{cr} = 99.99$, with the frequency of theoretically possible oscillations in the marginally stable system $\Omega_l = 100.00 s^{-1}$. The eigenvalues of the matrix $\mathbf{A}-\mathbf{B}\mathbf{C}K_{cr}$ are $(-99.99, 0+j100, 0-j100, -1.01)$, and eigenvalues of the matrix $\mathrm{e}^{(\mathbf{A}-\mathbf{B}\mathbf{C}K_{cr})T/2}$ are $(0.9688, 0.0432, -1, -1)$. As a result, matrix $\mathbf{H}$ is singular because $\det\left(\mathbf{I}+\mathrm{e}^{(\mathbf{A}-\mathbf{B}\mathbf{C}K_{cr})T/2}\right)=0$.

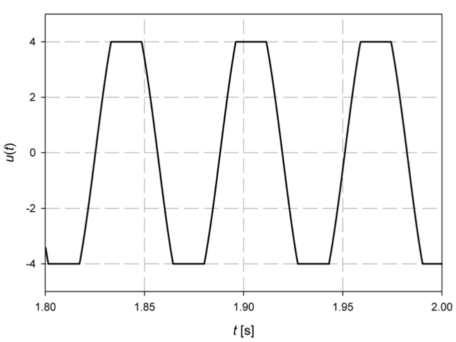
The results of the solution of equations (24), (25) for K, with treating parameters Ω and γ as independent variables satisfying (26) when solving equations (24) and (25), are presented in Table 1 (for a few different values of γ).

Table 1. Computed gain and frequency of chattering

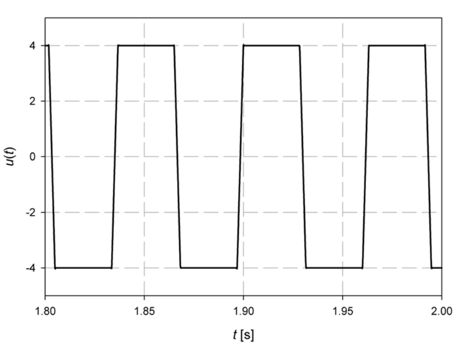
γ	0.1	0.3	0.5	0.7	0.9
K	100.75	103.65	121.06	176.77	493.30
Ω	99.93	99.78	99.64	99.49	99.35



(a) For K=100.75



(b) For K=121.06



(c) For K=493.30

Fig. 4. Periodic motions in system with saturation

The results of simulations of the system with saturation for the 3 different gains from Table 1 are presented in Fig. 4. In overall, the results of simulations provide a very good match to the predicted values of frequency Ω and relative interval duration γ.

As shown by the analysis presented above, Ω and γ do not depend on the control amplitude (saturation value) h. If we consider h=4 in our example (as in Fig. 4), we will conclude that chattering will be suppressed in the sliding mode system if the width of the boundary layer is less than 0.04. This value may be acceptable to ensure insignificant deterioration of the system performance.

VIII. Conclusion

A few approaches to analysis of the existence of chattering in sliding mode control systems having a continuous approximation of discontinuous control in the boundary layer is proposed in the paper. These approaches are based on the describing function method, the Popov's approach, and the Poincare map analysis. In the last case, analysis is provided only for the boundary layer approximation given by the saturation function. Analysis of parameters of chattering (frequency and shape of oscillations) is given for the boundary layer approximation given by the saturation function.

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