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Setting the Hysteresis Constant to Zero in Adaptive Switching Control

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Abstract

In this paper, we derive conditions under which one may safely set the hysteresis constant to zero in the well-known hysteresis switching algorithm of adaptive control without the risk of chattering instability. The case of continuously parameterized controllers is examined and conditions for global convergence are derived.

1. INTRODUCTION

Adaptive control is usually used to control imprecisely known plants. The main goal of adaptive control is to achieve improved performance by choosing a controller k from given set of candidate controllers $\mathbf{K}$ using real-time data and prior information. The general architecture of an adaptive control system is shown in Figure 1. Two different techniques of choosing among the candidate controllers $K \in \mathbf{K}$ have been used: continuous adaptive tuning and logic-based switching. In both methods, a primary goal of adaptive control is to ensure stability and convergence to a controller that achieves optimum performance.

One challenge facing a switching adaptive systems is the type of instability called chattering in which the supervisor cycles endlessly among two or more of the candidate controllers without converging, even when there is no change in the plant. This undesirable phenomenon was a motivation for several studies ([11],[12],[8],[13],[10]). A fundamental contribution to the solution to the chattering problem was made by Morse *et. al.* [13], who introduced the hysteresis switching algorithm of adaptive control along with the celebrated "hysteresis convergence lemma" which guarantees convergence after a finite number of switches under very mild assumptions.

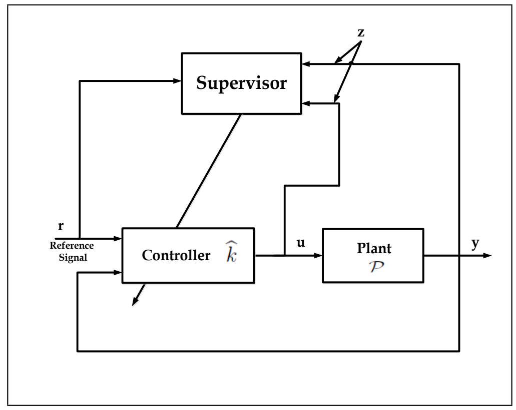

Figure 1. Adaptive control system $\Sigma(\mathscr{P},\widehat{k})$

In the hysteresis switching algorithm, the adaptive control systems is assumed to have the general architecture shown in Figure 1. The task of supervisor is to select at each time a controller $K \in \mathbf{K}$ based on the value of a real-valued performance function

$$V(k,z,t) \tag{1}$$

where $k \in \mathbf{K}$, $t \in [0,\infty]$ is time and

$$z = \begin{bmatrix} u \\ y \end{bmatrix} \tag{2}$$

where u, y are the plant input and output signals respectively. Given an initial controller $\hat{k}(0)$, the task of the supervisor is to determine at each time $t > 0$ which controller $\hat{k}(t) \in \mathbf{K}$ to insert into the feedback loop. In the original formulation of Morse [13], the hysteresis algorithm is permitted to switch controllers only at times t_i $(i = 1, 2, \dots)$ at which another controller improves the performance by a given constant $h > 0$ called the *hysteresis constant*; i.e.,

$$V(k_L(t_i), z, t_i) \leq V(\widehat{k}(t_i), z, t_i) - h \tag{3}$$

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where for all $t > 0$

$$k_L(t) = \underset{k \in \mathbf{K}}{\operatorname{argmin}} \{V(k,z,t)\}. \quad (4)$$

At each switching time t_i, the active controller $\widehat{k}(t)$ is changed to

$$\widehat{k}(t_i) = k_L(t_i). \quad (5)$$

A key result is the Morse-Mayne-Goodwin hysteresis switching convergence lemma, which may be stated as follows.

Lemma 1.1 (Convergence Lemma [13]) *Suppose that both of the following hold:*

1) *Monotonicity: For all k and z it holds that*

$$V(k,z,t) \geq V(k,z,\tau) \text{ for all } t > \tau$$

2) *Feasibility: There exists a controller $k_{robust} \in \mathbf{K}$ for which the performance function is uniformly bounded*

$$\sup_{z,t} V(k_{robust},z,t) < \infty.$$

Then, if $h > 0$ and if the number of controllers in the set $\mathbf{K}$ is finite, the hysteresis switching algorithm converges after at most finitely many controller switches. □

When the conditions of Lemma 1.1 hold, then we denote the final switching time and final controller respectively as

$$t_f \triangleq \max_i t_i \quad (6)$$

$$k_f \triangleq \widehat{k}(t_f). \quad (7)$$

With additional assumptions on the plant beyond feasibility, Morse *et. al.* [13] was also able to show that the final controller stabilizes the plant. Wang *et. al.* [17] showed that such additional plant assumptions are actually unnecessary, provided that the performance function $V(k,z,t)$ is selected to have a property called *cost-detectability*. Stefanovic *et. al.* [16] proved that the requirement that the controller set be finite can also be removed if the cost function $V(k,z,t)$ is *equi-continuous* (i.e., uniformly continuous in k for all t). A number of successful applications of hysteresis switching have been reported for both finite and infinite (or continuum) sets of candidate controllers (e.g., [13],[10],[9],[7],[16]). But, all of these results require that the hysteresis constant h be strictly positive.

A concern with the strictly positive h required by Lemma 1.1 is that it inherently tends to slow supervisor's adaptive response and it limits the accuracy with which the supervisor is able to minimize the performance function $V(k,z,t)$ to $\pm h$. Using a smaller h can partially address these concerns, but as h is decreased toward zero the number of controller switches usually tends to increase and chattering instability may sometimes occur in the limit as $h \to 0$ — though not always.

In present paper we derive sufficient conditions under which we can set $h = 0$ without chattering instability, thereby allowing supervisor to respond instantaneously and continuously using the *zero-hysteresis* optimal adaptive law

$$\widehat{k}(t) = k_L(t) \quad (8)$$

where $k_L(t)$ is given by (4). The paper is organized as follows. In Section 2 preliminary facts are given. Section 3 contains the main result. A simple example of the performance function satisfying sufficient conditions for convergence is provided in Section 4. Conclusions follow in Section 5.

2. Preliminaries

The linear truncation operator P_τ is a projection operator that truncates the signal at $t = \tau$, $\tau \in \mathbb{R}_+$, and the L_2-norm of x is given as $||x|| = \sqrt{\int_0^\infty x(t)^T x(t) dt}$. For brevity, we also denote $x_\tau = P_\tau x$ and $\|x\|_\tau = ||P_\tau x||$.

Definition 2.1 (Stability [20, 19, 18]) *We say a system Σ with input v and output z is stable if there exists constants $\beta, \alpha \geq 0$ such that*

$$\|z\|_\tau < \beta \|v\|_\tau + \alpha, \ \forall \tau > 0, v \in L_2. \quad (9)$$

*Otherwise, it is said to be unstable. Furthermore, if (9) holds, then the system Σ is said to be finite-gain stable, the least β for which (9) holds is called the L_2-*gain *of Σ.* □

Definition 2.2 *Suppose that $f : R^n \to R$ is twice differentiable on $X \subset R^n$ and that, for some $\alpha > 0$,*

$$\nabla^2 f(x) \geq \alpha I \ \forall x \in X. \quad (10)$$

Then, we say f is strongly convex (or uniformly convex) *on X.* □

One implication of strong convexity is that if $f(x)$ is strongly convex on a connected set $X \subset R^n$, then for every $\alpha > 0$ satisfying (10) we have [3, Prop. A.23]

$$f(y)-f(x) = (\nabla f(x))^T(y-x) + \int_0^1\int_0^1 (y-x) \nabla^2 f(x+\tau y)(y-x)d\tau dt \geq (\nabla f(x))^T(y-x) + \frac{\alpha}{2}\|y-x\|^2 \quad (11)$$

for any $\alpha > 0$ satisfying (10).

Definition 2.3 *A function $\nu(k,z,t)$ is said to be* equi-quasi-positive definite (EQPD) *in k if for some continuous monotone function $\phi : [0,\infty) \mapsto [0,\infty]$ with $\phi(0) = 0$ and $\phi(x) > 0\ \forall x > 0$, it holds for all $z \neq 0$ and all t sufficiently large that*

$$\widehat{k}(t) \quad = \quad \underset{k}{\operatorname{argmin}}\{\nu(k,z,t)\} \text{ exists, and} \quad (12)$$

$$\nu(k,z,t) \ - \ \nu(\widehat{k}(t),z,t) \ \geq \phi(||k-\widehat{k}(t)||) > 0 \quad (13)$$

□

A sufficient condition for $\nu(k,z,t)$ to be EQPD is that it be uniformly positive definite in k for all z and all t sufficiently large. A uniformly positive definite function is a special case of an EQPD in which the minimum occurs at $\widehat{k} = 0$.

The following lemmas will be used in proving our main result.

Lemma 2.1 *Let $\mathbf{K} \subseteq \mathbb{R}^n$ and the function $V(k,z,t)$ is twice differentiable in k. If $\nabla_k^2 V(k,z,t) > \alpha I$ for $\alpha > 0$. Then $V(k,z,t)$ is EQPD function.*

□

Lemma 2.2 *Let $\nu(k,z,t)$ be monotonically increasing in t for all z,k and suppose a minimizing value $\widehat{k}(t,z) = \underset{k}{\operatorname{argmin}}\{\nu(k,z,t)\}$ exists for all t. Then,*

$$\nu(\widehat{k}(t_2),z,t_2) \geq \nu(\widehat{k}(t_1),z,t_1) \text{ for all } t_2 \geq t_1.$$

□

Proof
By monotonicity

$$\nu(k,z,t_2) \geq \nu(k,z,t_1)\ \forall t_2 \geq t_1 \quad (14)$$

Also, since $\widehat{k}(t)$ minimizes $\nu(k,z,t)$

$$\nu(\widehat{k}(t),z,t) \leq \nu(k,z,t)\ \forall k \in \mathbf{K}. \quad (15)$$

From (14) $\nu(\widehat{k}(t_2),z,t_2) \geq \nu(\widehat{k}(t_2),z,t_1)$ and from (15) $\nu(\widehat{k}(t_2),z,t_1) \geq \nu(\widehat{k}(t_1),z,t_1)$. Hence,

$$\nu(\widehat{k}(t_2),z,t_2) \geq \nu(\widehat{k}(t_1),z,t_1)\ \forall\ t_2 \geq t_1.$$

□

Lemma 2.3 *Let $\nu(k,z,t)$ be monotonically increasing in t for all z,k and suppose a minimizing value $\widehat{k}(t,z) = \underset{k}{\operatorname{argmin}}\{\nu(k,z,t)\}$ exist for all t. If $\nu(k,z,t)$ is EQPD in k then, $\nu(\widehat{k}(t_2),z,t_2) - \nu(\widehat{k}(t_1),z,t_1) \geq \phi(||\widehat{k}(t_2)-\widehat{k}(t_1)||) \quad \forall\ t_2 \geq t_1$.*

Proof
Since $\nu(k,z,t)$ is an EQPD in k,

$$\nu(\widehat{k}(t_2),z,t_1) \ - \ \nu(\widehat{k}(t_1),z,t_1) \ \geq \ \phi(||\widehat{k}(t_2)-\widehat{k}(t_1)||)$$

and, since $\nu(k,z,t)$ is monotonic in t, we have

$$\nu(\widehat{k}(t_2),z,t_2) \geq \nu(\widehat{k}(t_2),z,t_1)\ \forall\ t_2 \geq t_1.$$

Therefore,

$$\nu(\widehat{k}(t_2),z,t_2) - \nu(\widehat{k}(t_1),z,t_1) \geq \nu(\widehat{k}(t_2),z,t_1) - \nu(\widehat{k}(t_1),z,t_1) \geq \phi(||\widehat{k}(t_2)-\widehat{k}(t_1)||)\ \forall\ t_2 \geq t_1$$

and hence for all $t_2 \geq t_1$

$$\nu(\widehat{k}(t_2),z,t_2) \ - \ \nu(\widehat{k}(t_1),z,t_1) \ \geq \phi(||\widehat{k}(t_2)-\widehat{k}(t_1)||).$$

□

3. Main Result

The following theorem establishes that if one replaces the requirement that hysteresis constant h be strictly positive in the Morse-Mayne-Goodwin Convergence Lemma with a requirement that the performance function be equi-quasi-positive definite in k, one still obtains convergence of the controller $\widehat{k}(t)$ as $t \to \infty$.

Theorem 3.1 (Main Result) *Consider the feedback adaptive control system $\Sigma(\mathscr{P},\widehat{k})$ in Figure 1.*
Suppose that both of the following hold:

1) *Monotonicity: For all k and z it holds that*

$$V(k,z,t) \geq V(k,z,\tau) \text{ for all } t > \tau$$

2) *Feasibility: There exists a controller $k_{robust} \in \mathbf{K}$ for which the performance function is uniformly bounded*

$$\sup_{z,t} V(k_{robust},z,t) < \infty.$$

If $V(K,z,t)$ is an EQPD function of k, then the zero-hysteresis adaptive law (8) converges as t increases to infinity to a point in the closure of the set $\mathbf{K}$.

Proof:
By feasibility $V_L(z) = \sup_t V(\widehat{k}(t), z, t)$ *exists and, by Lemma 2.2,* $V(\widehat{k}(t), z, t)$ *is monotonic in* t *and, by feasibility, it is bounded above. Hence,*

$$\begin{aligned} V_L(z) &= \lim_{t\to\infty} V(\widehat{k}(t), z, t) & (16) \\ &\geq V(\widehat{k}(t), z, t)\ \forall t & (17) \end{aligned}$$

Since $V(k,z,t)$ *is EQPD, it follows from Lemma 2.3) that for all* $t_2 \geq t_1$

$$V(\widehat{k}(t_2), z, t_2) \ - \ V(\widehat{k}(t_1), z, t_1) \ \geq \ \phi(||\widehat{k}(t_2) - \widehat{k}(t_1)||). \quad (18)$$

So, for all $t_2 \geq t_1$ *it holds that*

$$\begin{aligned} V_L(z) - V(\widehat{k}(t_1), z, t_1) &\geq V(\widehat{k}(t_2), z, t_2) - V(\widehat{k}(t_1), z, t_1) \\ &\geq \phi(||\widehat{k}(t_2) - \widehat{k}(t_1)||) \end{aligned}$$

Thus, for every $\varepsilon > 0$ *there exists* t_ε *such that for all* $t_1, t_2 \geq t_\varepsilon$

$$\varepsilon \geq V(\widehat{k}(t_2), z, t_2) - V(\widehat{k}(t_1), z, t_1) \geq \phi(||\widehat{k}(t_2) - \widehat{k}(t_1)||).$$

and hence $\phi(||\widehat{k}(t_2) - \widehat{k}(t_1)||) \to 0$ *as* $t \to \infty$. *Since,* ϕ *is nondecreasing continuous function satisfies* $\phi(0) = 0$ *and* $\phi(x) > 0$ *for* $x > 0$, *it follow that for every* $\delta > 0$, *there exists a* t_δ *such that* $||\widehat{k}(t_2) - \widehat{k}(t_1)|| < \delta$ *for all* $t_1, t_2 \geq t_\delta$. *Therefore, the sequence* $\{\widehat{k}(t)\}_{t=0}^{\infty}$ *is Cauchy. Since every Cauchy sequence converges [14], it follows that* $\widehat{k}(t)$ *converges as* $t \to \infty$ *to a point in the closure of the set* $\mathbf{K}$.

□

4. Performance Function Example

An example of the performance function and the conditions under which it ensures convergence according to the previous theorem may be constructed as follows. Consider a model reference adaptive control algorithm shown in Figure 2 with integral norms of estimation errors as our performance function.

$$V(K, z(t), \tau) = \frac{1}{2}\int_0^{\tau} ||\tilde{e}(z;K)||^2 dt \quad (19)$$

where

$$\begin{aligned} \tilde{e}(z;K) &= \tilde{y}_m(z;K) - y & (20) \\ \tilde{y}_m(z;K) &= W_m * \tilde{r}(z;K) & (21) \\ \tilde{r}(z;K) &= (1 + \theta^T Q) * u + y & (22) \end{aligned}$$

where Q is a given proper stable vector of transfer functions and $*$ denotes convolution..

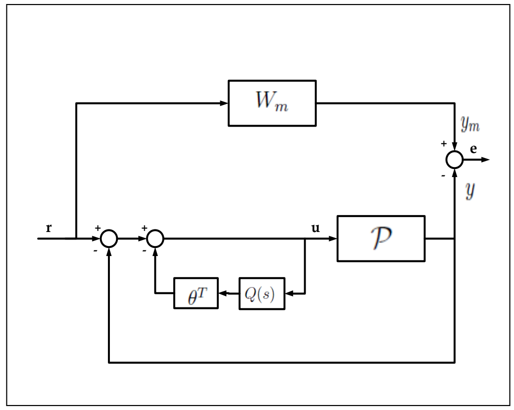

Figure 2. Model reference adaptive control

Then,

$$\nabla_\theta V(k, z(t), \tau) = \int_0^{\tau} ((W_m(y + (1 + \theta^T Q)u)^T u^T Q^T \, dt \quad (23)$$

and

$$\nabla_\theta^2 V(k, z(t), \tau) = \int_0^{\tau} Q \, u \, u^T Q^T dt \quad (24)$$

Definition 4.1 *We say that the system is* persistently excited *if* $\nabla_\theta^2 V(k, z(t), \tau) \geq \alpha I$ *for some* $\alpha > 0$*).*

□

Under the persistent excitation assumption, the function $V(k_i, z(t), t)$ is uniformly convex function in k for sufficiently large time t.
Therefore, whenever the systems is persistently excited, this performance function has the uniform convexity property. The persistent excitation (PE) property defined here is crucial in many adaptive schemes where parameter convergence is one of the objectives and is closely related to convergence conditions of [15, 6, 5, 4, 2, 1].

Comment 4.1 *The performance function in this example is not a cost-detectable performance function. The main difficulty with such performance function is that it needs prior information (i.e., standard adaptive control assumptions) to ensure feasibility. The question of whether there exist cost functions that are cost-detectable for which h can safely be set to zero remains open.*

□

5. Conclusion

In this paper we studied the problem of convergence in adaptive control for the case of continuum set of candidate controllers. We have examined the Morse-Mayne-Goodwin hysteresis switching algorithm for continuous adaptive control; our main result establishes that when a feasible controller exists, then the hysteresis constant may be set to zero if the cost function is monotone in time and, additionally, has a property that we call equi-quasi-positive definiteness (EQPD). The quadratic model-reference cost function without fading memory in our example has this property.

References

[1] B. D. O. Anderson. Exponential Stability of Linear Equations Arising in Adaptive Identification. *IEEE Transactions on Automatic Control*, 22:83–88, 1977.

[2] K. J. Astrom and T. Bohlin. Numerical identification of linear dynamic systems from normal operating records. In *Theory of Self-Adaptive Control Systems*, page 96, 1966.

[3] D. P. Bertsekas. *Nonlinear Programming*. Athena Scientific Belmont, MA, 1999.

[4] R. Bitmead. Persistence of excitation conditions and the convergence of adaptive schemes. *IEEE Transactions on Information Theory*, 30(2):183–191, 1984.

[5] S. Boyd and S. S. Sastry. Necessary and Sufficient Conditions for Parameter Convergence in Adaptive Control. *Automatica*, 22(6):629–639, 1986.

[6] P. Eykhoff. System Identification: Parameter and State Estimation. *Chichester, England*, 1974.

[7] J. Hespanha, D. Liberzon, A. S. Morse, B. D. O. Anderson, T. S. Brinsmead, and F. De Bruyne. Multiple model adaptive control. Part 2: Switching. *International Journal of Robust and Nonlinear Control*, 11(5):479–496, 2001.

[8] J. P. Hespanha and A. S. Morse. Stability of switched systems with average dwell-time. In *Proceedings of the 38th IEEE Conference on Decision and Control*, volume 3, pages 2655–2660 vol.3, 1999.

[9] D. Liberzon, J. P. Hespanha, and A. S. Morse. Hierarchical hysteresis switching. In *Proceedings of the IEEE Conference on Decision and Control*, volume 1, 2000.

[10] R. H. Middleton, G. C. Goodwin, D. J. Hill, and D. Q. Mayne. Design issues in adaptive control. *IEEE Transactions on Automatic Control*, 33(1):50–58, Jan 1988.

[11] A. S. Morse. Supervisory control of families of linear set-point controllers — Part I: Exact matching. *Automatic Control, IEEE Transactions on*, 41(10):1413–1431, Oct 1996.

[12] A. S. Morse. Supervisory control of families of linear set-point controllers — Part II: Robustness. *IEEE Transactions on Automatic Control*, 42(11):1500–1515, Nov 1997.

[13] A. S. Morse, D. Q. Mayne, and G. C. Goodwin. Applications of hysteresis switching in parameter adaptive control. *IEEE Transactions on Automatic Control*, 37(9):1343–1354, Sep 1992.

[14] J. R. Munkres. *Topology: A First Course*. Prentice-Hall, Englewood Cliffs, NJ, 1975.

[15] K.S. Narendra and A.M. Annaswamy. Persistent excitation in adaptive systems. *International Journal of Control*, 45(1):127–160, 1987.

[16] M. Stefanovic and M. G. Safonov. Safe adaptive switching control: Stability and convergence. *IEEE Transactions on Automatic Control*, 53(9):2012–2021, Oct. 2008.

[17] R. Wang, A. Paul, M. Stefanovic, and M. G. Safonov. Cost-detectability and stability of adaptive control systems. In *IEEE Conference on Decision and Control and European Control Conference*, pages 3584–3589, Dec. 2005.

[18] J. C. Willems. Mechanisms for the Stability and Instability in Feedback Systems. *Proceedings of the IEEE*, 64(1):24–35, 1976.

[19] J.C. Willems. Qualitative Behavior of Interconnected Systems. *Annals of Systems Research*, 3:61–80, 1973.

[20] G. Zames. On the input–output stability of time-varying nonlinear feedback systems — Part I: Conditions derived using concepts of loop gain, conicity, and positivity. AC-11(2):228–238, April 1966.