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Robust Model Predictive Control for LPV Systems with Delayed State using Relaxation Matrices

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Abstract— **This paper proposes a model predictive control (MPC) algorithm for Linear Parameter Varying (LPV) systems with unknown bounded delay, subject to input constraints. To deal with the delay, an optimization problem is formulated by applying the equivalence property. Sufficient conditions are derived in terms of linear matrix inequalities (LMIs) using relaxation matrices, satisfying the terminal inequality. The proposed MPC algorithm with the conditions guarantees the asymptotic stability of the closed-loop system. A numerical example is presented to illustrate the effectiveness of the proposed method.**

I. INTRODUCTION

Linear Parameter Varying (LPV) systems have received considerable attention in recent years due to their applicability in many practical situations [1]-[7]. One of advantages of LPV system is the capability to express nonlinear systems into linear systems with the time varying parameters of affine functions. Many researchers proposed various control methods to express these parameters as polytopic [2]-[4] or structured feedback representation [5]-[7]. With these representations, model predictive control (MPC), also known as moving or receding horizon control, has been suggested as an effective technique to control LPV systems due to its ability to describe their time-varying behaviors. Moreover, in company with many practical applications especially in chemical process control, such as petrochemical, pulp, and paper control, many studies of MPC for LPV systems have been published, continuously [3]-[7].

Although MPC technique has some powerful properties, for example, guaranteed stability, good tracking performance, and input-output constraint handling, the time delay existing in a system is an essential factor which should be considered. This is why the time delay is inevitable in practical as well as can be a main factor of performance degradation and instability. However, there exist only a few MPC methods that consider time-delayed LPV system [9]-[10]. In [9], authors proposed a parameter-dependent state-feedback controller for LPV system with parameter-varying time delay and proved the stability by using parameter-dependent Lyapunov functionals. Also, a delay-scheduled state-feedback controller was designed in [10] by introducing a new model transformation turning a time-delay system into the uncertain LPV systems. Even though they showed significant efficiencies of their controller, there are still some drawbacks as follows; First, both assumed that the delay is represented as a known function of a parameter vector or is approximately known in real-time. But it is difficult to know the prior information of the delay in practical situations. Second, in their methods, there exists the limitation to apply input constraints which is demanded when controlling many practical plants. Hence, such a controller that can overcome the drawbacks mentioned above is needed with MPC method that enables to effectively handle constraints.

Motivated by these discussions, in this paper, we present an MPC method for polytopic state delayed LPV system having unknown bounded delay and input constraint. To deal with the unknown delay, we represent two different optimization problems, and then we solve original one by using the equivalence property between two different problems. The infinite-horizon min-max optimization problem is formulated as a minimization of the upper bound of cost function to design a state feedback MPC law. At the end, we derive sufficient conditions which guarantee the asymptotic stability of the closed loop system, including not only the input constraint but also a new condition for the cost monotonicity which is derived by using relaxation matrices.

The rest of this paper is organized as follows. Section II provides the problem statement and preliminary. Section III presents main results for the derived sufficient conditions in terms of LMIs, proof of feasibility, and stability for proposed MPC algorithm. Section IV shows simulation results to demonstrate its effectiveness of the proposed method. Finally, Section V concludes this paper with a summarization.

Throughout this paper, R^n and $R^{n\times m}$ denote the n-dimensional Euclidean space and the set of all $n\times m$ matrices, respectively. $*$ in symmetric matrices represents the abbreviated off-diagonal block. For symmetric matrices X and Y, the notation $X \geq Y$ and $X > Y$ mean that $X-Y$ is positive semidefinite and positive definite, respectively. Finally, we denote $\|x\|_{\mathscr{W}} = x^T \mathscr{W} x$ for a vector x.

II. PROBLEM STATEMENT

Consider the following discrete polytopic LPV systems whose system matrices $A(p(k)),\bar{A}(p(k)),B(p(k))$ are affine functions of a parameter vector $p(k)$:

$$\begin{aligned} x(k+1) &= A(p(k))x(k)+\bar{A}(p(k))x(k-d)+B(p(k))u(k) \\ x(k) &= \Phi(k), \quad k\in[-d^*,0] \end{aligned} \tag{1}$$

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subject to input constraints

$$-\bar{u} \leq u(k) \leq \bar{u}, \quad \bar{u} > 0, \quad \text{for all} \quad k \in [0, \infty) \tag{2}$$

where $A(p(k)) = \sum_{l=1}^{L} p_l(k) A_l$, $\bar{A}(p(k)) = \sum_{l=1}^{L} p_l(k) \bar{A}_l$, $B(p(k)) = \sum_{l=1}^{L} p_l(k) B_l$. $x(k) \in R^n$ is the state vector, $u \in R^m$ is the control input, and $\Phi(k) \in R^n$ is the initial condition. d is an unknown constant integer representing the number of delay units in the state, but assumed $0 \leq d \leq d^*$ with a known integer d^*, and the time-varying parameter vectors $p_j(k)$ belong to a convex polytope, i.e.

$$\sum_{l=1}^{L} p_l(k) = 1, \quad 0 \leq p_l(k) \leq 1, \quad l = 1, 2, \cdots, L. \tag{3}$$

Then, it is clear that as $p(k)$ varies inside its polytope, the LPV system matrices vary inside a corresponding polytope Ω whose vertices consist of L local system matrices.

$$\begin{aligned}[A(p(k)), &\bar{A}(p(k)), B(p(k))] \in \Omega \\ &= C_o\{[A_1, \bar{A}_1, B_1], [A_2, \bar{A}_2, B_2], \ldots, [A_L, \bar{A}_L, B_L]\}\end{aligned} \tag{4}$$

where C_o denotes the convex hull.

Our goal is to design a state-feedback controller $u(k) = K(k)x(k+j|k)$ by MPC strategy to stabilize (1). To obtain such a controller, we consider the following optimization problem at each time instant.

$$\min_{u(k+j|k), j \geq 0} \quad \max_{[A(p(k)), \bar{A}(p(k)), B(p(k))] \in \Omega} J_\infty(k) \tag{5}$$

subject to

$$J_\infty(k) \triangleq \sum_{j=0}^{\infty} \|x(k+j|k)\|_{\mathscr{Q}} + \|u(k+j|k)\|_{\mathscr{R}}, \tag{6}$$

$$\begin{aligned} x(k+j+1|k) = &A(k+j)x(k+j|k) + \bar{A}(k+j)x(k+j-d|k) \\ &+ B(k+j)u(k+j|k), \end{aligned} \tag{7}$$

$$-\bar{u} \leq u(k+j|k) \leq \bar{u}, \quad j \in [0, \infty) \tag{8}$$

where $J_\infty(k)$ is the infinite horizon quadratic cost function with the given symmetric constant matrices $\mathscr{Q} > 0$ and $\mathscr{R} > 0$; $x(k+j|k)$ and $u(k+j|k)$ denote predicted variables of the state and input at time instant k, respectively. We define a quadratic function to compute the control input $u(k)$

$$\begin{aligned} &V(x(k+j|k), P(k), P_d(k), d) \\ &\triangleq \|x(k+j|k)\|_{P(k)} + \sum_{i=1}^{d} \|x(k+j-i|k)\|_{P_d(k)}, \end{aligned} \tag{9}$$

$j \geq 0$ where $P(k)$ and $P_d(k)$ are any positive definite symmetric matrices.

At each instant k, it is assumed that (9) satisfies the following robust stability constraint for all states and control inputs of the system (1) as follows:

$$\begin{aligned} \triangle V = &V(x(k+j+1|k), P(k), P_d(k), d) \\ &- V(x(k+j|k), P(k), P_d(k), d) \\ &< -(\|x(k+j|k)\|_{\mathscr{Q}} + \|u(k+j|k)\|_{\mathscr{R}} \end{aligned} \tag{10}$$

This equation (10) is called the terminal inequality [11] and is used to derive the sufficient condition for cost monotonicity in the next section.

III. MAIN RESULTS

In this section, deriving a new sufficient condition for cost monotonicity through relaxation matrix, we propose an MPC algorithm which can make the closed loop system of the LPV system (1) be asymptotically stabilized.

A. LMI condition for cost monotonicity using Relaxation Matrices

Since the relaxation matrices allow the extended solution range of LMIs conditions by reducing the upper bound of worst-case functions [13], it can be a good method to obtain sufficient conditions to stabilize LPV system [5]. The condition is presented in the following *Theorem 1*.

Theorem 1: The terminal inequality (10) is satisfied for any polytope $[A(p(k)), \bar{A}(p(k)), B(p(k))] \in \Omega$, if there exist $X(k)$, $Y(k)$, $Z(k)$, $H(k)$, $Q^T(k) = Q(k) > 0$, and $Q_d(k) = Q_d^T(k) > 0$ satisfying the following LMI:

$$\begin{bmatrix} -X(k)-X(k)^T & \bar{A}_l Q_d(k) - Y(k) & (A_l - I)Q(k) + B_l H(k) - Z(k) + X^T(k) & X^T(k) & 0 & 0 & 0 \\ * & -Q_d(k) & Y^T(k) & Y^T(k) & 0 & 0 & 0 \\ * & * & Z(k) + Z^T(k) & Z^T(k) & Q^T(k) & Q^T(k)\mathscr{Q}^{T/2} & H^T(k)\mathscr{R}^{T/2} \\ * & * & * & -Q(k) & 0 & 0 & 0 \\ * & * & * & * & -Q_d(k) & 0 & 0 \\ * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & -(k)I \end{bmatrix} < 0 \tag{11}$$

where $Q(k) \triangleq P^{-1}(k)$, $Q_d(k) \triangleq P_d^{-1}(k)$, and $H(k) \triangleq K(k)Q(k)$ for each polytope $[A_l, \bar{A}_l, B_l]$ with $l = 1, \ldots, L$.

Proof: At first, let us define the difference between $x(k+1)$ and $x(k)$, i.e. $\delta x(k) \triangleq x(k+1) - x(k)$. Then, we can rewrite the system equation (1) for a polytope $[A_l, \bar{A}_l, B_l]$, applying state feedback controller $u(k) = K(k)x(k)$, as follows

$$\begin{bmatrix} A_l + B_l K(k) - I & \bar{A}_l & -I \end{bmatrix} \begin{bmatrix} x(k) \\ x(k-d) \\ \delta x(k) \end{bmatrix} = 0. \tag{12}$$

The following equality is always satisfied for nonzero relaxation matrices $\Theta_1(k)$, $\Theta_2(k)$, and $\Theta_3(k) \in R^{n \times n}$.

$$\begin{bmatrix} x(k) \\ x(k-d) \\ \delta x(k) \end{bmatrix}^T \begin{bmatrix} \Theta_1(k) \\ \Theta_2(k) \\ \Theta_3(k) \end{bmatrix} \begin{bmatrix} A_{cl} & \bar{A}_l & -I \end{bmatrix} \begin{bmatrix} x(k) \\ x(k-d) \\ \delta x(k) \end{bmatrix} = 0 \tag{13}$$

where $A_{cl}(k) = A_l + B_l K(k) - I$. By using (13), we derive a new sufficient condition satisfying the terminal inequality (10) for LPV system (1).

The terminal inequality (10) can be rewritten as

$$\begin{aligned}\triangle V(k) =&\|x(k+j+1|k)\|_{P(k)} - \|x(k+j|k)\|_{P(k)} \\ &+\sum_{i=1}^{d}\|x(k+j+1-i)\|_{P_d(k)} - \sum_{i=1}^{d}\|x(k+j-i)\|_{P_d(k)} \\ =&\|x(k+j+1|k)\|_{P(k)} - \|x(k+j|k)\|_{P(k)} \\ &-\|x(k+j-d|k)\|_{P_d(k)} + \|x(k+j|k)\|_{P_d(k)} \end{aligned} \tag{14}$$

For notational simplicity, let us define $x = x(k+j|k)$, $x_d = x(k+j-d|k)$, and $\delta x = x(k+j+1|k) - x(k+j|k)$. Then, (14) can be rewritten as

$$\begin{aligned}\triangle V(k) =&(x+\delta x)^T P(k)(x+\delta x) \\ &-x^T P(k)x - x_d^T P(k)x_d + x^T P_d(k)x \\ =&\begin{bmatrix} x \\ x_d \\ \delta x \end{bmatrix}^T \begin{bmatrix} P_d(k) & 0 & P(k) \\ 0 & -P_d(k) & 0 \\ P(k) & 0 & P(k) \end{bmatrix} \begin{bmatrix} x \\ x_d \\ \delta x \end{bmatrix} \\ <&-\begin{bmatrix} x \\ x_d \\ \delta x \end{bmatrix}^T \begin{bmatrix} \mathscr{Q}+K^T(k)\mathscr{R}K(k) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_d \\ \delta x \end{bmatrix}\end{aligned} \tag{15}$$

Combining (13) with (15) through addition, we obtain the following inequality,

$$\begin{aligned}&\begin{bmatrix} P_d(k) & 0 & P(k) \\ 0 & -P_d(k) & 0 \\ P(k) & 0 & P(k) \end{bmatrix} + \begin{bmatrix} \Theta_1(k) \\ \Theta_2(k) \\ \Theta_3(k) \end{bmatrix} \begin{bmatrix} A_{cl} & \bar{A}_l & -I \end{bmatrix} \\ &+\begin{bmatrix} A_{cl} & \bar{A}_l & -I \end{bmatrix}^T \begin{bmatrix} \Theta_1(k) \\ \Theta_2(k) \\ \Theta_3(k) \end{bmatrix}^T \\ &<-\begin{bmatrix} \mathscr{Q}+K^T(k)\mathscr{R}K(k) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.\end{aligned} \tag{16}$$

Then, the inequality (16) can be rearranged as

$$\begin{aligned}&\begin{bmatrix} \mathscr{Q}+P_d(k) & 0 & 0 \\ 0 & -P_d(k) & 0 \\ 0 & 0 & P(k) \end{bmatrix} \\ &+\begin{bmatrix} \Theta_1(k) & 0 & P(k) \\ \Theta_2(k) & P_d(k) & 0 \\ \Theta_3(k) & 0 & 0 \end{bmatrix} \begin{bmatrix} A_{cl} & \bar{A}_l & -I \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \\ &+\begin{bmatrix} A_{cl} & \bar{A}_l & -I \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}^T \begin{bmatrix} \Theta_1(k) & 0 & P(k) \\ \Theta_2(k) & P_d(k) & 0 \\ \Theta_3(k) & 0 & 0 \end{bmatrix}^T \\ &+\begin{bmatrix} K^T(k) \\ 0 \\ 0 \end{bmatrix} \mathscr{R} \begin{bmatrix} K(k) & 0 & 0 \end{bmatrix} < 0\end{aligned} \tag{17}$$

Let us define

$$\Xi(k) \triangleq \begin{bmatrix} 0 & 0 & Q(k) \\ 0 & Q_d(k) & 0 \\ X(k) & Y(k) & Z(k) \end{bmatrix} = \begin{bmatrix} \Theta_1(k) & 0 & P(k) \\ \Theta_2(k) & P_d(k) & 0 \\ \Theta_3(k) & 0 & 0 \end{bmatrix}^{-T}.$$

After pre- and post-multiplying $\Xi^T(k)$ and $\Xi(k)$ in the left-hand side of (17), respectively, then we obtain

$$\begin{aligned}&\begin{bmatrix} X^T(k) \\ Y^T(k) \\ Z^T(k) \end{bmatrix} P(k) \begin{bmatrix} X(k) & Y(k) & Z(k) \end{bmatrix} \\ &+\begin{bmatrix} -X(k)-X^T(k) & \bar{A}_l Q_d(k) - Y(k) & A_{cl}Q(k) - Z(k) + X^T(k) \\ * & -Q_d^T(k)P_d(k)Q_d(k) & Y^T(k) \\ * & * & Z(k)+Z^T(k)+Q^T(k)[\bar{Q}+P_d(k)]Q(k) \end{bmatrix} < 0\end{aligned} \tag{18}$$

where $\bar{Q} = \mathscr{Q} + K^T(k)\mathscr{R}K(k)$. Applying the Schur complements [14] to (18), the inequality can be equivalent to (11). This completes the proof. ■

B. Model Predictive Controller Design

To obtain the upper bound of the cost function (6), the terminal inequality (10) is added from $j = 1$ to $j = \infty$, requiring $x(\infty|k) = 0$ or $V(x(\infty|k)) = 0$, as follows

$$J_\infty(k) \le V(x(k|k), P(k), P_d(k), d). \tag{19}$$

Then, the original min-max problem (5) can be transformed into the following problem

$$\begin{aligned}\mathscr{P}(P(k),P_d(k),d):\quad &\min_{K(k),P(k),P_d(k)} V(x(k|k),P(k),P_d(k),d) \\ &\text{s.t.}\quad (7),(8),\text{and } (11).\end{aligned} \tag{20}$$

However, this problem cannot directly solved due to the unknown delay d. We present a fact and an assumption to deal with the delay.

Fact 1: Let us consider the following two optimization problems:

$$\begin{aligned}(Q_1,X_1,Y_1) = &arg \min_{Q,X,Y} \|x\|_Q + \alpha \\ &\text{s.t.}\quad 0 \ge F_1(Q,X,Y),...,0 \ge F_n(Q,X,Y)\end{aligned} \tag{21}$$

$$\begin{aligned}(Q_2,X_2,Y_2) = &arg \min_{Q,X,Y} \|x\|_Q + \beta \\ &\text{s.t.}\quad 0 \ge F_1(Q,X,Y),...,0 \ge F_n(Q,X,Y)\end{aligned} \tag{22}$$

where Q, $X$and Y denote optimization variables; α and β denote constant terms; $F_i(Q,X,Y)$ denote functions of Q, $X$and Y. The difference of the two optimization problems is only α and β. Then, if one of the two problems is solvable, so is the other. Moreover, minimizing arguments of the two problems are identical, that is, $Q_1 = Q_2$, $X_1 = X_2$ and $Y_1 = Y_2$.

Assumption 1: The matrix $P_d(k) > 0$ in $\mathscr{P}(P(k),P_d(k),d)$ is fixed to a constant matrix $\bar{P}_d$ at all time instants k.

Remark 1: This assumption will make the solution of the optimization problem more conservative. But it will be relaxed in the next section.

Using *Fact 1* and *Assumption 1*, the problem $\mathscr{P}(P(k),P_d(k),d)$ is equivalent to $\mathscr{P}(P(k),\bar{P}_d,d^*)$, which is called the equivalence property of $\mathscr{P}(P(k),P_d(k),d)$ and

$\mathscr{P}(P(k),\bar{P}_d,d^*)$. In addition, we can observe the feasibility of $\mathscr{P}(P(k),P_d(k),d)$ and the closed-loop stability.

(Feasibility) Under *Assumption 1*, suppose that a feasible control sequence $u(k+j|k),\ j\geq 0$ is a feasible control of $\mathscr{P}(P(k),\bar{P}_d,d^*)$ at time k. Then, at the next time $k+1$, the following control sequence can be chosen as

$$u(k+j+1|k+1)=u^*(k+j+1|k),\quad j\geq 0. \tag{23}$$

This shows the problem $\mathscr{P}(P(k),\bar{P}_d,d^*)$ has a feasible solution at time $k+1$, satisfying the input constraint (8) at time $k+1$. Hence, by induction, if the problem $\mathscr{P}(P(k),\bar{P}_d,d^*)$ is feasible at time k, then the problem is feasible at all time instants to be greater than k. Also, $u(k+j|k),\ j\geq 0$ becomes a feasible control sequence for the problem $\mathscr{P}(P(k),\bar{P}_d,d)$ at time k. Moreover, by the equivalence property, the solution of $\mathscr{P}(P(k),\bar{P}_d,d^*)$ is one of $\mathscr{P}(P(k),P_d(k),d)$.

(Closed-loop Stability) $P^*(k)$ and $P^*(k+1)$ are also optimal for $\mathscr{P}(P(k),\bar{P}_d,d)$ by the equivalence, where $P^*(k)$ and $P^*(k+1)$ are the optimal values of $\mathscr{P}(P(k),\bar{P}_d,d^*)$ at time k and $k+1$, respectively.

Since $P^*(k+1)$ is optimal while $P^*(k)$ is only feasible at time $k+1$, the following inequality is obtained from optimality

$$\begin{aligned}&V(x(k+1|k+1),P^*(k+1),\bar{P}_d,d)\\&\qquad\leq V(x(k+1|k+1),P^*(k),\bar{P}_d(k),d).\end{aligned} \tag{24}$$

Then, using the terminal inequality (10) and $x(k+1-i|k+1)=x(k+1-i|k)$ with $i=1,...,d$, we have

$$\begin{aligned}&V(x(k+1|k+1),P^*(k+1),\bar{P}_d,d)-V(x(k|k),P^*(k),\bar{P}_d,d)\\&\qquad\leq -(\|x(k|k)\|_{\mathscr{Q}}+\|u(k|k)\|_{\mathscr{R}}).\end{aligned} \tag{25}$$

After summing (25) from $k=0$ to $k=i-1$, it yields

$$\begin{aligned}&V(x(i|i),P^*(i),\bar{P}_d,d)+\sum_{k=0}^{i-1}[\|x(k|k)\|_{\mathscr{Q}}+\|u(k|k)\|_{\mathscr{R}}]\\&\qquad\leq V(x(0|0),P^*(0),\bar{P}_d,d)\end{aligned} \tag{26}$$

Therefore, $x(i)=x(i|i)$ and $u(i)=u(i|i)$ must go to zero as i goes to infinity, since the left hand side of (26) is bounded above by the constant $V(x(0|0),P^*(0),\bar{P}_d,d)$. This means the feasible MPC from $\mathscr{P}(P(k),\bar{P}_d,d^*)$ robustly asymptotically stabilizes the closed-loop system.

C. MPC algorithm for polytopic time-delayed LPV system

Although we showed the asymptotic stability of the closed-loop system, it is still conservative due to the fixed $\bar{P}_d$. Therefore, we relax the conservatism by presenting an MPC algorithm, which can update $\bar{P}_d$, for the polytopic time-delayed LPV system.

The algorithm is summarized as follows;

Step 1: (Initialization) at time $k=0$, find $K(k)$ and $P(k)$ by solving the optimization problem $\mathscr{P}(P(k),P_d(k),d^*)$, and set $\bar{P}_d=P_d(0)$ and flag=1.

Step 2: (Generic) at time $k\geq 0$, find $K(k)$ and $P(k)$ by solving the optimization problem $\mathscr{P}(P(k),P_d(k)=\bar{P}_d,d^*)$, and set $K^*(k)=K(k)$ and $P^*(k)=P(k)$.

Step 3: Find $K(k)$ and $P_d(k)$ by solving the optimization problem $\mathscr{P}(P(k)=P^*(k),P_d(k),d^*)$. If $P_d(k)<\bar{P}_d$, set $K^*(k)=K(k)$, $\bar{P}_d=P_d(k)$, flag=flag+1. Otherwise, go to *Step 5*.

Step 4: If flag $\leq r_n$, go to *Step 2*. Otherwise, go to *Step 5*. Here r_n is a given fixed integer that represents the maximum repetition number of *Step 2* and *Step 3*.

Step 5: Apply the state-feedback control $u(k)=K^*(k)x(k)$ to the system.

Step 6: At the next time, set flag $=1$ and repeat *Step 2* to *Step 5*.

Updating P_d in *Step 3* with the closed-loop stability mentioned above guarantees that once the problem $\mathscr{P}(P(k),\bar{P}_d,d^*)$ is feasible at $k=0$, the MPC from the previous algorithm robustly asymptotically stabilizes the system.

Despite the guarantee of the stability, the size of r_n can cause computational burden. Hence, the selection of r_n can be an important factor in terms of trade-off between performance and computational burden.

Finally, we complete the algorithm with *Theorem 2*.

Theorem 2: The optimization problem $\mathscr{P}(P(k),P_d(k),d^*)$ can be solved by this semidefinte program;

$$\min_{\gamma(k),H(k),Q(k),Q_d(k),X(k),Y(k),Z(k)}\gamma(k) \tag{27}$$

subject to

$$\begin{bmatrix} -X(k)-X(k)^T & \bar{A}_l\bar{Q}_d(k)-Y(k) & (A_l-I)\bar{Q}(k)+B_lH(k)-Z(k)+X^T(k) & X^T(k) & 0 & 0 & 0\\ * & -\bar{Q}_d(k) & Y^T(k) & Y^T(k) & 0 & 0 & 0\\ * & * & Z(k)+Z^T(k) & Z^T(k) & \bar{Q}^T(k) & \bar{Q}^T(k)\mathscr{Q}^{T/2} & \bar{H}^T(k)\mathscr{R}^{T/2}\\ * & * & * & -\bar{Q}(k) & 0 & 0 & 0\\ * & * & * & * & -\bar{Q}_d(k) & 0 & 0\\ * & * & * & * & * & -\gamma(k)I & 0\\ * & * & * & * & * & * & -\gamma(k)I \end{bmatrix}<0, \tag{28}$$

$$\begin{bmatrix} 1 & x^T(k|k) & x^T(k-1|k) & \cdots & x^T(k-d^*|k)\\ x(k|k) & \bar{Q}(k) & 0 & \cdots & 0\\ x(k-1|k) & 0 & \bar{Q}_d(k) & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ x(k-d^*|k) & 0 & 0 & \cdots & \bar{Q}_d(k)\end{bmatrix}>0, \tag{29}$$

$$\text{and} \quad \begin{bmatrix} G(k) & H(k) \\ H^T(k) & \bar{Q}(k) \end{bmatrix} \geq 0, \quad G_{ii}(k) \leq \bar{u}_i^2(k). \tag{30}$$

where, for each polytope $[A_l, \bar{A}_l, B_l]$ with $l = 1, ..., L$, $\bar{Q}(k) \triangleq \gamma(k)P^{-1}(k) > 0$, $\bar{Q}_d(k) \triangleq \gamma(k)P_d^{-1}(k) > 0$, $\bar{H}(k) \triangleq K(k)\bar{Q}(k)$, $G_{ii}(k)$ is the ith diagonal entry of $G(k)$, and $\bar{u}_i$ is the ith element of $\bar{u}$.

Proof: Minimizing $V(x(k|k), P(k), P_d(k), d^*)$ is equivalent to

$$\min_{\gamma(k), K(k), P(k), P_d(k)} \gamma(k)$$

$$\text{subject to } \|x(k|k)\|_{P(k)} + \sum_{i=1}^{d^*} \|x(k-i|k)\|_{P_d(k)} \leq \gamma(k). \tag{31}$$

Let us define $\bar{Q}(k) = \gamma P^{-1}(k) > 0$ and $\bar{Q}_d(k) = \gamma P_d^{-1}(k) > 0$. Then, by using Schur complement, the conditions (28) and (29) are derived with performing some procedure as in Theorem 1 and from the constraints of (31), respectively.

The input constraint (8) can be easily casted into (30) by applying the so called invariant ellipsoid [11]. So details are omitted. ■

IV. Numerical Example

In this section, a numerical example is presented to illustrate the performance of the proposed MPC algorithm for LPV system. The example is called a backing up control of a computer simulated truck-trailer and well-known as a difficult problem due to its nonlinearities and unstability even without delay [12], [15]. Therefore, it is appropriate to show the effectiveness of our MPC algorithm. Using Euler's first-order approximation with sampling time $T = 0.1$, we transform the nonlinear time delay system with the time-delayed LPV system (1). The time-varying parameter of the LPV system is caused by the nonlinear function $sin(\cdot)$ of the original system. Then, the system is described as follows

$$x(k+1) = A(p(k))x(k) + \bar{A}(p(k))x(k-d) + B(p(k))u(k) \tag{32}$$

where

$$[A(p(k)), \bar{A}(p(k)), B(p(k))] \in \Omega \triangleq Co\{[A_1, \bar{A}_1, B_1], [A_2, \bar{A}_2, B_2]\}, \tag{33}$$

$$A_1 = \begin{bmatrix} 1.0509 & 0 & 0 \\ -0.0509 & 1 & 0 \\ 0.0509 & -0.4 & 1 \end{bmatrix}, \bar{A}_1 = \begin{bmatrix} 0.0218 & 0 & 0 \\ -0.0218 & 0 & 0 \\ 0.0218 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1.0509 & 0 & 0 \\ -0.0509 & 1 & 0 \\ 0.0810 & -0.6366 & 1 \end{bmatrix}, \bar{A}_2 = \begin{bmatrix} 0.0218 & 0 & 0 \\ -0.0218 & 0 & 0 \\ 0.0347 & 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.1429 \\ 0 \\ 0 \end{bmatrix}, \text{and } B_2 = B_1.$$

Simulation parameters are as follows; the initial value of the state is $x(0) = \begin{bmatrix} 0.5\pi & 0.75\pi & -5 \end{bmatrix}^T$, $\mathscr{Q} = diag(10, 10, 10)$, $\mathscr{R} = 1$, $d = 4$, $d^* = 10$, and $r_n = 1$. Even though larger r_n shows better performance, it causes the computational burden when applied to practical cases. Therefore, by setting $r_n = 1$, we relax the burden. The variables x_1, x_2, x_3, and u mean angle difference between the truck and the trailer, angle of the trailer, y-coordinate of the rear end of the trailer and steering angle, respectively; x_1 is assumed to be perturbed by time-delay; $\bar{u} = \pi/2.5714$.

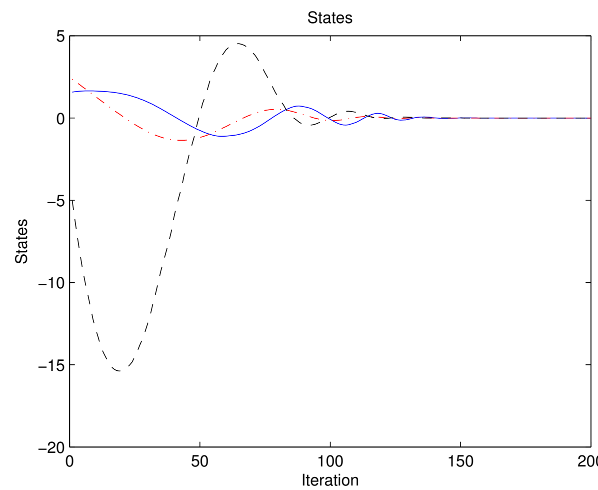

Fig. 1. States of the controlled polytopic delayed LPV system with proposed MPC method (x_1:solid, x_2:dashdotted, x_3:dashed)

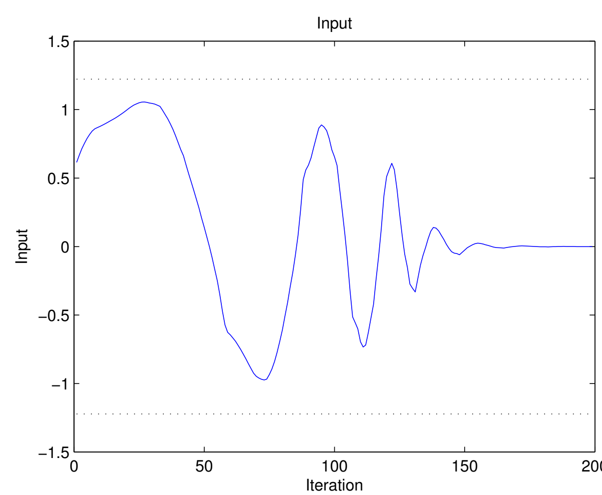

Fig. 2. Input trajectory (solid line) with input constraint $u(k) \leq \bar{u} = \pi/2.5714$ (dotted line)

From Fig. 1 to Fig. 3, we can observe that the proposed MPC algorithm for polytopic time-delayed LPV system works well to asymptotically stabilize the system. Fig. 1 shows that all system states go to zero despite the unknown bounded time delay. The control input is depicted in Fig. 2 and not able to exceed its upper bound $\bar{u}$, which means that the input constraint (8) is held. Moreover, we can easily see that the upper bound of infinite horizon cost function $\gamma(k)$ is monotonically decreasing (Fig. 3). It results from the

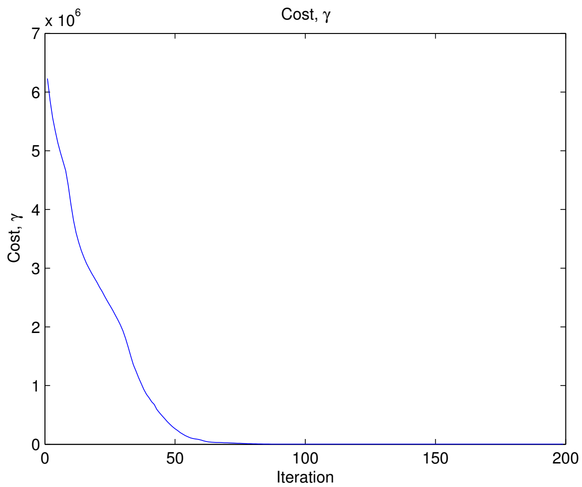

Fig. 3. Upper bound of cost, $\gamma(k)$

sufficient condition (29) derived by using relaxation matrices, satisfying the terminal inequality.

V. Conclusion

This paper proposed a robust MPC algorithm for polytopic state delayed LPV system having the unknown bounded delay and input constraints. We solved the original optimization problem by using the equivalence property between two different optimization problems to deal with the unknown delay. Moreover, we designed the state feedback controller from the min-max optimization problem which is formulated as a minimization of upper bound of infinite horizon cost function. With the newly proposed sufficient condition for the cost monotonicity using relaxation matrices, the proposed MPC algorithm asymptotically stabilized the polytopic delayed LPV system with input constraints. A numerical example showed its effectiveness.

References

[1] J. Shamma and M. Athans, "Guaranteed properties of gain scheduled control of linear parameter-varying plants," *Automatica*, vol. 27, 1991, pp 559-564.
[2] W. P. M. H. Heemels, J. Daafouz, and G. Millerioux, "Design of observer-based controllers for LPV systems with unknown parameters," *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*, 2006, pp 1836-1841.
[3] Y. Lu and Y. Arkun, "Quasi-Min-Max MPC algorithms for LPV systems," *Automatica*, vol. 36, 2000, pp 527-540.
[4] A. Casavola, D. Famularo, and G. Franze, "A feedback min-max MPC algorithm for LPV systems subject to bounded rates of change of parameters," *Automatic Control, IEEE Transactions on*, vol. 47, 2002, pp 1147-1153.
[5] S. M. Lee, J. H. Park, D. H. Ji, and S. C. Won, "Robust model predictive control for LPV systems using relaxation matrices," *IET Control Theory and Application*, vol. 1, 2007, pp 1567-1573.
[6] S. M. Lee and J. H. Park, "Output feedback model predictive control for LPV systems using parameter-dependent Lyapunov function," *Applied Mathematics and Computation*, vol. 190, 2007, pp 671-676.
[7] M. Sato, "Gain-scheduled open-loop system design for LPV systems using polynomially parameter-dependent Lyapunov functions." *Systems and Control Letters*, vol. 59, 2010, pp 265-276.
[8] M. Farhood and E. Feron, "A parameter-dependent Lyapunov approach for the control of nonstationary and hybrid LPV systems." *American Control Conference, 2009. ACC '09*, 2009, pp 3742-3747.
[9] F. Wu, K. M. Grigoriadis, "LPV Systems with parameter-varying time delays: analysis and control." *Automatica*, vol. 37, 2001, pp 221-229.
[10] C. Briat, O. Sename, and J. F. Lafay, "Delay-scheduled state-feedback design for time-delay systems with time-varying delays-A LPV approach." *Systems and Control Letters*, vol. 59, 2009, pp 265-276.
[11] M. V. Kothare, V. Balakrishnan, and M. Morari, "Robust constrained model predictive control using linear matrix inequalities," *Automatica*, vol. 32, 1996, pp 1361-1379.
[12] S. C. Jeong and P. G. Park, "Constrained MPC algorithm for uncertain time varying systems with state delay," *IEEE Trans. Autom. Control*, vol. 47, 2005, pp 257-263.
[13] R.E. Skelton, T. Iwasaki, and K.M. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design*, Taylor & Francis, 1997.
[14] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Philadelphia, SIAM, 1994.
[15] Y. Y. Cao, P. M. Frank, "Stability analysis and synthesis of nonlinear time-delay systems via linear Takagi-Sugeno fuzzy models," *Fuzzy Sets and Systems*, vol. 124, 2001, pp 213-229.