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H_2 and H_∞ low-gain theory

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Abstract— **For stabilization of linear systems subject to input saturation, there exist four different approaches of low-gain design all of which are independently proposed in the literature, namely direct eigenstructure assignment, H_2 and H_∞ algebraic Riccati equation (ARE) based methods, and parametric Lyapunov equation based method. We show here all these methods are rooted in and can be unified under two fundamental control theories, H_2 and H_∞ theory. Moreover, both the H_2 and H_∞ ARE based methods are generalized to consider systems where all input channels are not subject to saturation, and explicit design methods are developed.**

I. INTRODUCTION

The low-gain feedback design methodology was first developed in [6], [7] to achieve semi-global stabilization of linear systems subject to input saturation. Since then, it has been widely employed in various control problems, such as output regulation with constraints, H_2 and H_∞ optimal control etc [10], [11]. The low-gain feedback can be constructed using four different approaches, namely direct eigenstructure assignment [6], [7], H_2 and H_∞ algebraic Riccati equation (ARE) based methods [10], [19], and parametric Lyapunov equation based method [20], [21]. Although these four methodologies were independently proposed in literature, we shall show in this paper that they are all rooted in and can be unified under two fundamental control theories, H_2 and H_∞ theory.

Moreover, all these designs of low-gain consider only the case where low gains are demanded by all input channels, and consequently require the asymptotic null controllability with bounded input (ANCBC) of the given system. In this note, we introduce the concept of H_2 and H_∞ low-gains in a general setting where partial or all input channel are engaged with low-gain. We provide explicit existence conditions and design methods which yield the classical ANCBC condition and the four design methods as special cases.

Standard notations are used in this paper. $\mathbb{C}^-$, $\mathbb{C}^\bigcirc$ and $\mathbb{C}^+$ denote open left half complex plane, the imaginary axis and open right half complex plane respectively. For $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm and x' denotes the transpose of x. For $X \in \mathbb{R}^{n\times m}$, $\|X\|$ denotes its induced 2-norm and X' denotes the transpose of X. For a vector-valued continuous-time signal y, $\|y\|_{\mathscr{L}_p}$ denotes the $\mathscr{L}_p$ norm of y. For a continuous-time system Σ having a $q\times\ell$ stable transfer function G, $\|G\|_2$ and $\|G\|_\infty$ denote respectively the standard H_2 and H_∞ norm of G.

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II. Definition of H_2 and H_∞ low-gain sequences

Consider the linear time invariant continuous-time system,

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ z = Du \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $z \in \mathbb{R}^{m_0}$. Without loss of generality, we assume that $D = \begin{pmatrix} I_{m_0} & 0 \end{pmatrix}$.

In what follows, a state feedback gain such as F_ε parameterized in a parameter ε is called a gain sequence since as ε changes one obtains a sequence of gains. We define below formally what we mean by H_2 and H_∞ low-gain sequences.

Definition 1: For the system Σ in (1), the H_2 **low-gain sequence** is a sequence of parameterized static state feedback gains F_ε for which there exists an ε^* such that the following properties hold:

1) There exists a M such that $\|F_\varepsilon\| \le M$ for any $\varepsilon \in (0,\varepsilon^*]$;
2) $A + BF_\varepsilon$ is Hurwitz stable for any $\varepsilon \in (0,\varepsilon^*]$;
3) For any $x(0) \in \mathbb{R}^n$, the closed-loop system with $u = F_\varepsilon x$ satisfies $\lim_{\varepsilon\to 0} \|z\|_{\mathscr{L}_2} = 0$.

The H_∞ low-gain sequence will depend on an a priori given data γ, hence we define it as the γ-level H_∞ low-gain sequence. Whenever we refer to the H_∞ low-gain sequence, we always imply the γ-level H_∞ low-gain sequence.

Definition 2: For Σ in (1) and for an arbitrary $E \in \mathbb{R}^{n\times p}$, define an auxiliary system

$$\Sigma_\infty : \begin{cases} \dot{x} = Ax + Bu + E\omega \\ z = Du, \end{cases} \tag{2}$$

and the infimum

$$\gamma^* = \inf_F \left\{ \|DF(sI - A - BF)^{-1}E\|_\infty \mid \lambda(A+BF) \in C^- \right\}. \tag{3}$$

For a given $\gamma > \gamma^*$, the **γ-level** H_∞ **low-gain sequence** is a sequence of parameterized static state feedback gains $F_\varepsilon(E,\gamma)$ for which there exists an ε^* such that

1) There exists a M such that $\|F_\varepsilon(E,\gamma)\| \le M$ for any $\varepsilon \in (0,\varepsilon^*]$;
2) $A + BF_\varepsilon(E,\gamma)$ is Hurwitz stable for any $\varepsilon \in (0,\varepsilon^*]$;
3) For system Σ_∞ with $u = F_\varepsilon(E,\gamma)$ and any $x(0) \in \mathbb{R}^n$,

$$\lim_{\varepsilon\to 0} \left\{ \sup_{\omega\in\mathscr{L}_2} \left(\|z\|^2_{\mathscr{L}_2} - \gamma\|\omega\|^2_{\mathscr{L}_2} \right) \right\} = 0.$$

III. Properties of H_2 and H_∞ low-gain sequences

Theorem 1: For the system Σ in (1) with a given $E \in \mathbb{R}^{n\times p}$ and a $\gamma > \gamma^*$ where γ^* is defined in (3), a sequence of feedback gains $F_\varepsilon(E,\gamma)$ is a γ-level H_∞ low-gain sequence only if it is an H_2 low-gain sequence.

Proof: By setting $\omega = 0$ in the definition of H_∞-γ-level low-gain sequence, we immediately conclude this result. ■

Remark 1: The converse of Theorem 1 is not true. For any given E, we can always construct a γ_1-level H_∞ low-gain sequence with $\gamma_1 > \gamma$ which, according to Theorem 1, is a H_2 low-gain sequence but not a γ-level H_∞ low-gain sequence.

The next theorem shows that for the closed-loop system Σ in (1) with either an H_2 low-gain controller $u = F_\varepsilon x$ or an H_∞ low-gain controller $u = F_\varepsilon(E,\gamma)x$, the magnitude of z and DF_ε or $DF_\varepsilon(E,\gamma)$ can be made arbitrarily small.

Theorem 2: The closed-loop system (1) with either $u = F_\varepsilon x$ or $u = F_\varepsilon(E,\gamma)x$ satisfies the following properties:

1) $\lim_{\varepsilon\to 0}\|z\|_{\mathscr{L}_\infty} = 0$,
2) $\lim_{\varepsilon\to 0} DF_\varepsilon = 0$ and $\lim_{\varepsilon\to 0} DF_\varepsilon(E,\gamma) = 0$.

Proof: Owing to Theorem 1, we only need to prove these two properties for an H_2 low-gain sequence. The fact that $\|z\|_{\mathscr{L}_2} \to 0$ as $\varepsilon \to 0$ for any $x(0)$ implies that

$$\lim_{\varepsilon\to 0}\|F_\varepsilon e^{(A+BF_\varepsilon)t}\| = 0.$$

Since $\|F_\varepsilon\|$ is bounded for all $\varepsilon \in (0,\varepsilon^*]$, $\|A+BF_\varepsilon\|$ is also bounded for all $\varepsilon \in (0,\varepsilon^*]$. We have

$$\lim_{\varepsilon\to 0}\|F_\varepsilon e^{(A+BF_\varepsilon)t}(A+BF_\varepsilon)\| = 0.$$

This implies that $\dot{z} \in \mathscr{L}_2$ and moreover $\lim_{\varepsilon\to 0}\|\dot{z}\|_{\mathscr{L}_2} = 0$. Applying Cauchy-Schwartz inequality, we can show that

$$\big|\|z(t)\|^2 - \|z(0)\|^2\big| \le 2\|\dot{z}\|^{[0,t]}_{\mathscr{L}_2}\|z\|^{[0,t]}_{\mathscr{L}_2}. \tag{4}$$

Let ε be fixed and $t\to\infty$. Since $A+BF_\varepsilon$ is Hurwitz, $\|z(t)\| \to 0$. We have then $\|z(0)\|^2 \le 2\|\dot{z}\|_{\mathscr{L}_2}\|z\|_{\mathscr{L}_2}$. Then let $\varepsilon \to 0$. We conclude that for any $x(0) \in \mathbb{R}^n$,

$$\lim_{\varepsilon\to 0}\|z(0)\|^2 = \lim_{\varepsilon\to 0}\|DF_\varepsilon x(0)\|^2 = 2\lim_{\varepsilon\to 0}\|\dot{z}\|_{\mathscr{L}_2}\|z\|_{\mathscr{L}_2} = 0,$$

and hence $\lim_{\varepsilon\to 0} DF_\varepsilon = 0$. On the other hand, (4) also yields

$$\|z(t)\|^2 \le 2\|\dot{z}\|^{[0,t]}_{\mathscr{L}_2}\|z\|^{[0,t]}_{\mathscr{L}_2} + \|z(0)\|^2 \le 2\|\dot{z}\|_{\mathscr{L}_2}\|z\|_{\mathscr{L}_2} + \|z(0)\|^2.$$

Therefore, $\lim_{\varepsilon\to 0}\|z\|_{\mathscr{L}_\infty} = 0$. ■

We emphasize that if F_ε is not bounded, the above theorem is not true in general.

Theorem 2 enables us to connect to the literature and explain why the H_2 and γ-level H_∞ sequences as defined in Definitions 1 and 2 are termed as '*low-gain*' sequences. As we alluded to in introduction to this paper, the name *low-gain* sequence arose or has roots in one of the classical problems, namely the problem of semi-globally stabilizing a linear system subject to actuator saturation. (For readers not familiar with the saturation literature, we refer to [1], [3], [4], [12], [18] for more details.) To be precise, let

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\sigma(\bar{u}) \tag{5}$$

where the function $\sigma(\cdot)$ denotes a standard saturation; that is, $\sigma(\bar{u}) = \text{sign}(\bar{u})\min\{1,|\bar{u}|\}$. Let the pair $(\bar{A},\bar{B})$ be stabilizable and $\bar{A}$ has all its eigenvalues in the closed left half plane.

Consider a state feedback controller, $\bar{u} = \bar{F}_\varepsilon\bar{x}$ where $\bar{F}_\varepsilon$ is a parameterized sequence with the parameter as ε. If the feedback sequence $\bar{F}_\varepsilon$ satisfies all the three conditions posed in Theorem 3.1 of [7], it is known as a '*low-gain*' feedback in the context of stabilization of linear systems subject to saturation (see also [5]). In fact, the state feedback controller $\bar{u} = \bar{F}_\varepsilon\bar{x}$ where $\bar{F}_\varepsilon$ is such a *low-gain* sequence semi-globally stabilizes (5) for a small enough value of ε. That is, there exists an ε^* such that for all $\varepsilon \in (0,\varepsilon^*)$, the closed-loop system comprising (5) and $\bar{u} = \bar{F}_\varepsilon\bar{x}$ is semi-globally stable with *a priori* given (arbitrarily large) bounded set Ω being in the region of attraction, and moreover the smaller the value of ε the larger can be the *a priori* prescribed set Ω.

Having recalled above the classical semi-global stabilization problem of a linear system with saturating linear feedbacks, we can now emphasize its connection to Theorem 2. As is done in classical semi-global stabilization problem, let us first assume that all the control channels are subject to saturation. Then, to see the connection between such a semi-global stabilization problem and Theorem 2, set $D = I_m$ and thus take $z = u$ as the constrained variable subject to saturation. Then, Theorem 2 shows that the H_2 and γ-level H_∞ sequences as defined in Definitions 1 and 2 satisfy all the three conditions posed in Theorem 3.1 of [7], and hence they can appropriately be termed as *low-gain* sequences. Furthermore, as is evident from Theorem 2, they can readily achieve semi-global stabilization of a continuous-time linear system where all control inputs are subject to saturation whenever it is achievable.

For the general setting when $D = \begin{bmatrix} I_{m_0} & 0 \end{bmatrix}$ for some $m_0 < m$, in the scenario of a linear system subject to input saturation, all the input channels are not necessarily constrained. To be precise, let

$$\dot{\xi} = A\xi + B_0\sigma(u_0) + B_1u_1 \tag{6}$$

where $\xi \in \mathbb{R}^n$, $u_0 \in \mathbb{R}^{m_0}$, $u_1 \in \mathbb{R}^{m-m_0}$ and $B = \begin{bmatrix} B_0 & B_1 \end{bmatrix}$. Partial inputs as represented by u_0 are subject to saturation. In another word, we have the constrained variable $z = Du = u_0$. In this case, property 1 of Theorem 2 implies that for an initial condition x_0 in a given set and a pre-specified saturation level Δ, there exists an ε^* such that for all $\varepsilon \in (0,\varepsilon^*)$ the closed-loop system satisfies $\|z(t)\| = \|u_0(t)\| = \|DF_\varepsilon e^{(A+BF_\varepsilon)t}x_0\| \le \Delta$ for all $t \ge 0$. This implies that the saturation can be made inactive for all time, and hence the closed-loop system can in fact be linear. Therefore, the stability of the closed-loop system directly follows from Definitions 1 and 2.

IV. Existence of H_2 and H_∞ low-gain sequences

Theorem 3: For the system Σ in (1) with an arbitrarily given $E \in \mathbb{R}^{n\times p}$ and $\gamma > \gamma^*$ where γ^* is defined in (3), the H_2 and γ-level H_∞ low-gain sequences exist if and only if

1) (A,B) is stabilizable;

2) $(A,B,0,D)$ is at most weakly non-minimum phase, i.e it has all its invariant zeros are in $\mathbb{C}^- \cup \mathbb{C}^{\bigcirc}$.

Remark 2: In the special case of $D = I_m$, the invariant zeros of $(A,B,0,I)$ coincide with the eigenvalues of A. Hence Condition 2 requires all the eigenvalues of A are in closed left half plane. In this case, a system that satisfies Conditions 1 and 2 is said to be asymptotically null controllable with bounded control (ANCBC), see [17].

Proof: For the case of H_2 low-gain sequence, let $\gamma_2^* = \sqrt{\text{trace}(P)}$ where P is the unique semi-stabilizing solution to the continuous-time linear matrix inequality (CLMI),

$$\begin{pmatrix} A'P + PA & PB \\ B'P & D'D \end{pmatrix} \geq 0. \tag{7}$$

It is evident from [11] that H_2 low-gain sequence exists if and only if $\gamma_2^* = 0$, i.e. $P = 0$. This is equivalent to the conditions that (A,B) is stabilizable and

$$\text{rank}\begin{pmatrix} sI - A & -B \\ 0 & D \end{pmatrix} = \text{normrank}\begin{pmatrix} sI - A & -B \\ 0 & D \end{pmatrix}$$

for any $s \in \mathbb{C}^+$, i.e. $(A,B,0,D)$ is at most weakly non-minimum phase.

For the case of H_∞ low-gain sequence, we can easily verify [15] that given $\gamma > \gamma^*$ the γ-level H_∞ low-gain sequences exist if and only if, $P = 0$ is a semi-stabilizing solution to the continuous-time quadratic matrix inequality (CQMI),

$$\begin{pmatrix} A'P + PA + \gamma^{-2}PEE'P & PB \\ B'P & D'D \end{pmatrix} \geq 0,$$

which is equivalent to the conditions that (A,B) is stabilizable and that the matrix pencil

$$\begin{pmatrix} sI - A & -B \\ 0 & D \end{pmatrix}$$

does not have any zeros on the open right half plane, i.e. the system is at most weakly non-minimum phase. ■

Remark 3: As shown in the foregoing discussion, the low-gain sequences achieve semi-global stabilization of linear systems subject to input saturation. In order to design a low-gain sequence for the system (5), one can choose $D = I_m$ in (1). The above theorem then shows that the necessary and sufficient conditions for semi-global stabilization are that (A,B) is stabilizable and all the invariant zeros of $(A,B,0,I_m)$ are in the closed left half plane. It is known that the invariant zeros of $(A,B,0,I_m)$ coincide with eigenvalues of A. Hence Conditions 2 implies that all the eigenvalues of A are in the closed left half plane. Note that in this case of $D = I_m$, conditions 1 and 2 are well known to the saturation community as classical ANCBC conditions, see [17].

However, in general all the system inputs may not have to be subject to saturation as shown in (6). To design a low-gain feedback sequence for this type of system, we can choose $D = \begin{bmatrix} I_{m_0} & 0 \end{bmatrix}$ in (1). Then the necessary and sufficient conditions as required in Theorem 3 are that (A,B) is stabilizable and the invariant zeros of $(A,B,0,D)$ are in the closed left half plane. It can be shown that the invariant zeros of $(A,B,0,D)$ in this case are a subset of eigenvalues of A (see [13]). Therefore, only some eigenvalues of A have to be constrained while the others can be completely free. Moreover, Theorem 2 identifies those eigenvalues that need to be restricted. To illustrate this, consider a linear system with a partial input subject to saturation,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \sigma(u_0) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_1.$$

Clearly (A,B) is stabilizable. Matrix A has eigenvalues $(j,-j,2,3)$. It can be identified that $(j,-j)$ are the invariant zeros of $(A,B,0,D)$, which are on the imaginary axis. Hence the two conditions in Theorem 3 are still satisfied while the two eigenvalues $(2,3)$ are in the right half plane.

V. Design of H_2 low-gain sequences

The H_2 low-gain design procedures developed here yield the classical low-gain design methods as special cases. We note that the H_2 low-gain sequence as defined in Definition 1 for the system Σ in (1) is equivalent to a bounded H_2 sub-optimal sequence of controller for the following auxiliary system,

$$\Sigma_2 \begin{cases} \dot{x} = Ax + Bu + \omega \\ z = Du. \end{cases}$$

Such an H_2 sub-optimal controller for Σ_2 can be constructed using either direct eigenstructure assignment method or perturbation method, see [9], [11].

A. Direct eigenstructure assignment method

The design basically follows the SOSFGS algorithm developed in [8], [9]. There exists a nonsingular state transformation $[x_a', x_c']' = T_1 x$ such that the system Σ_2 can be transformed into a compact Special Coordinate Basis (SCB) form:

$$\bar{\Sigma}_2 : \begin{cases} \begin{pmatrix} \dot{x}_a \\ \dot{x}_c \end{pmatrix} = \begin{bmatrix} \bar{A}_a & 0 \\ \star & A_c \end{bmatrix} \begin{pmatrix} x_a \\ x_c \end{pmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} u_1 + \begin{bmatrix} \bar{B}_a \\ B_{ac} \end{bmatrix} u_0 + E\omega \\ \quad z \quad = u_0, \end{cases} \tag{8}$$

where $x_a \in \mathbb{R}^{n_a}$, $x_c \in \mathbb{R}^{n_c}$, $u_0 \in \mathbb{R}^{m_0}$, $u_c \in \mathbb{R}^{m_c}$, $n_a + n_c = n$ and $m_0 + m_c = m$, and $\star$ denotes matrix of less interest. The eigenvalues of A_a are the invariant zeros of system Σ. Theorem 3 implies that (A_a, B_a) is stabilizable and A_a has all its eigenvalues in the closed left half plane. Moreover, (A_c, B_c) is controllable. Details of SCB can be found in [13].

In order to use the eigenstructure assignment method, we need to perform another transformation $[\bar{x}_a', x_c']' = T_2 [x_a', x_c']'$ such that the system can be further converted into:

$$\bar{A}_a = \begin{pmatrix} A_1 & A_{12} & \cdots & A_{1\ell} & 0 \\ 0 & A_2 & \cdots & A_{2\ell} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_\ell & 0 \\ 0 & 0 & 0 & 0 & A_o \end{pmatrix},$$

$$\bar{B}_a = \begin{pmatrix} B_1 & 0 & \cdots & 0 & B_{1,o} \\ 0 & B_2 & \cdots & 0 & B_{2,o} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_\ell & B_{\ell,o} \\ B_{o,1} & B_{o,2} & \cdots & B_{o,\ell} & B_o \end{pmatrix},$$

and where A_o is Hurwitz stable, (A_i,B_i) is controllable, and A_i has all its eigenvalues on the imaginary axis. Moreover, (A_i,B_i) is in the controllability canonical form as given by

$$A_i = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & 1 & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_{i,0} & -\alpha_{i,1} & \cdots & -\alpha_{i,n_i-2} & -\alpha_{i,n_i-1} \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

For each pair (A_i,B_i), let the feedback gain $F_i(\varepsilon)$ be such that $\lambda(A_i+B_iF_i(\varepsilon)) = -\varepsilon - \lambda(A_i)$. Define

$$F_{a,\varepsilon} = \begin{bmatrix} \text{blkdiag}\{\bar{F}_i(\varepsilon)\}_{i=1}^{\ell} \\ & 0 \end{bmatrix}, \quad \bar{F}_i(\varepsilon) = F_i(\varepsilon^{2^{\ell-i}(r_{i+1}+1)\ldots(r_\ell+1)})$$

where r_i is the largest algebraic multiplicity of eigenvalues of A_i. Since (A_c,B_c) is controllable, we can choose a bounded F_c such that $A_c+B_cF_c$ is stable and has a desired set of eigenvalues. The sequence of feedback gains for the system Σ_2 can then be constructed as

$$F_\varepsilon = \begin{pmatrix} F_{a,\varepsilon} & 0 \\ 0 & F_c \end{pmatrix} T_2 T_1.$$

Clearly, F_ε is bounded and $A+BF_\varepsilon$ is Hurwitz. It follows from [9] that F_ε also satisfies Property 3 in Definition 1. Therefore, F_ε is an H_2 low-gain sequence.

Remark 4: For $D = I_m$, the above design procedure recovers the direct eigenstructure assignment method in the classical low-gain design of [6] for linear systems subject to input saturation.

B. Perturbation methods

The philosophy of perturbation methods used in H_2 low-gain design is the same as in classical H_2 sub-optimal controller design, that is to perturb the data of the system so that an H_2 optimal controller exists for the perturbed system and then based on continuity argument, we can obtain a sequence of H_2 low-gains for the original system utilizing H_2 optimal control design techniques developed in [11].

For a given quadruple (A,B,C,D), let a sequence of perturbed data $(A_\varepsilon,B_\varepsilon,C_\varepsilon,D_\varepsilon)$ be such that $A_\varepsilon \to A$, $B_\varepsilon \to B$, $\bar{Q}_\varepsilon \to \bar{Q}_0$ as $\varepsilon \to 0$ and $\bar{Q}_\varepsilon$ is continuous at $\varepsilon = 0$ where

$$\bar{Q}_0 = \begin{bmatrix} C & D \end{bmatrix}' \begin{bmatrix} C & D \end{bmatrix}, \quad \bar{Q}_\varepsilon = \begin{bmatrix} C_\varepsilon & D_\varepsilon \end{bmatrix}' \begin{bmatrix} C_\varepsilon & D_\varepsilon \end{bmatrix}. \quad (9)$$

For this perturbation $(A_\varepsilon,B_\varepsilon,C_\varepsilon,D_\varepsilon)$ to be admissible for H_2 low-gain design, it has to satisfy the following conditions:

1) The positive semi-definite semi-stabilizing solution P_ε to the CLMI ,
$$\begin{bmatrix} A_\varepsilon' P_\varepsilon + P_\varepsilon A_\varepsilon & P_\varepsilon B_\varepsilon + C_\varepsilon' D_\varepsilon \\ B_\varepsilon' P_\varepsilon + D_\varepsilon' C_\varepsilon & D_\varepsilon' D_\varepsilon \end{bmatrix} \geq 0, \quad (10)$$
converges to 0.
2) An H_2 optimal state feedback controller F_ε exists for the perturbed system characterized by $(A_\varepsilon,B_\varepsilon,C_\varepsilon,D_\varepsilon,I)$ and can, for instance, be constructed using the $(COGFMDZ)$ or $(COGFMDZ)_{nli}$ algorithm in [11].

Moreover, the obtained F_ε should satisfy:

3) F_ε is bounded.
4) F_ε is such that $A+BF_\varepsilon$ is Hurwitz.
5) $\lim_{\varepsilon\to 0} \|(C+DF_\varepsilon)(sI-A-BF_\varepsilon)^{-1}\|_2 = 0$.

If $(A_\varepsilon,B_\varepsilon,C_\varepsilon,D_\varepsilon)$ and the corresponding F_ε satisfy the 5 conditions stated above, then F_ε is an H_2 low-gain sequence. Specifically in our problem, for the system Σ in (1) characterized by (A,B,C,D) with $C=0$, we can use two perturbation methods to design an H_2 low-gain sequence.

a) Perturbation method I: The classical perturbation that is used in H_2 sub-optimal control is in the form $(A,B,C_\varepsilon,D_\varepsilon)$ where C_ε and D_ε are such that $(A,B,C_\varepsilon,D_\varepsilon)$ has neither invariant zeros nor infinite zeros, and

$$\bar{Q}_\varepsilon \to \bar{Q}_0 \text{ as } \varepsilon \to 0, \quad \bar{Q}_{\varepsilon_1} \leq \bar{Q}_{\varepsilon_2} \text{ with } 0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \beta. \quad (11)$$

for some $\beta > 0$ and $\bar{Q}_\varepsilon$ and $\bar{Q}_0$ are defined in (9). This leads to a perturbed system

$$\Sigma_2^\varepsilon : \begin{cases} \dot{x} &= Ax + Bu + w \\ z_\varepsilon &= C_\varepsilon x + D_\varepsilon u \end{cases}$$

For this perturbation, we have:

- since C_ε and D_ε satisfy (11), condition 1 follows from Theorem 5 in Appendix.
- since the quadruple $(A,B,C_\varepsilon,D_\varepsilon)$ has neither finite invariant zeros nor infinite zeros, condition 2 follows from Lemma 5.6.3 in [11].
- since we do not perturb A and B, condition 4 is obvious.
- since $u = F_\varepsilon x$ is an H_2 optimal state feedback for the perturbed system and $P_\varepsilon \to 0$, we have that $\|(C_\varepsilon + D_\varepsilon F_\varepsilon)(sI-A-BF_\varepsilon)^{-1}\|_2 \to 0$. Then (11) implies that
$$\|(C+DF_\varepsilon)(sI-A-BF_\varepsilon)^{-1}\|_2 \leq \|(C_\varepsilon+D_\varepsilon F_\varepsilon)(sI-A-BF_\varepsilon)^{-1}\|_2.$$
Therefore, $\|(C+DF_\varepsilon)(sI-A-BF_\varepsilon)^{-1}\|_2 \to 0$ as $\varepsilon \to 0$.

We find that conditions 1, 2, 4, 5 are always satisfied by this type of perturbation. It remains to verify condition 3. We note that since $C=0$ in our problem, we can always find a $(C_\varepsilon,D_\varepsilon)$ such that an bounded F_ε can be constructed following $(COGFMDZ)$ or $(COGFMDZ)_{nli}$ algorithm in [11]. In what follows, we give two examples for this type of perturbation.

Example 1: One choice of perturbation for system Σ_2 is given by $(A,B,C_\varepsilon,D_\varepsilon)$ where

$$C_\varepsilon' = \begin{bmatrix} 0 & 0 & \sqrt{Q_\varepsilon}' \end{bmatrix}, \quad D_\varepsilon' = \begin{bmatrix} D' & \varepsilon I & 0 \end{bmatrix}',$$

and $Q_\varepsilon \in \mathbb{R}^{n\times n}$ is such that

$$Q_\varepsilon > 0 \text{ and } \lim_{\varepsilon\to 0} Q_\varepsilon = 0. \quad (12)$$

Clearly, $(A,B,C_\varepsilon,D_\varepsilon)$ does not have any zero structure (that is, neither invariant zeros nor infinite zeros), and $(C_\varepsilon,D_\varepsilon)$ satisfies (11). Hence we only need to check condition 3. Let X_ε be the positive definite solution of H_2 ARE,

$$A'X_\varepsilon + X_\varepsilon A + Q_\varepsilon - X_\varepsilon B'(D_\varepsilon' D_\varepsilon)^{-1} B X_\varepsilon = 0, \quad (13)$$

The H_2 optimal static state feedback for the perturbed system can then be constructed as

$$F_\varepsilon = -(D'_\varepsilon D_\varepsilon)^{-1} B' X_\varepsilon.$$

When $m_0 = m$, i.e. $D = I_m$, F_ε is bounded for $\varepsilon \in [0,1]$ and hence is an H_2 low-gain sequence. Moreover, it recovers the standard H_2-ARE based low-gain design for linear systems subject to input saturation [10]. However, when $m_0 < m$, the boundedness of F_ε needs to be proved. In the next example, we present an alternative perturbation of $(C_\varepsilon, D_\varepsilon)$ which automatically generates a bounded F_ε for any $m_0 \le m$.

Example 2: We can also perturb the auxiliary system $\bar{\Sigma}_2$ in its compact SCB form (8) as:

$$\bar{\Sigma}^\varepsilon_{2,I} : \begin{cases} \begin{bmatrix} \dot{x}_a \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_a & 0 \\ \star & A_c \end{bmatrix} \begin{bmatrix} x_a \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} u_c + \begin{bmatrix} B_a \\ B_{ac} \end{bmatrix} u_0 + T_1 w \\ \begin{bmatrix} z \\ z_{\varepsilon,1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \sqrt{Q_\varepsilon} & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_c \end{bmatrix} + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_c \end{bmatrix}, \end{cases}$$

where Q_ε satisfies (12). In this case,

$$C_\varepsilon = \begin{bmatrix} 0 & 0 \\ \sqrt{Q_\varepsilon} & 0 \end{bmatrix}, \quad D_\varepsilon = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}.$$

The perturbed system does not have zero structure (that is, neither invariant zeros nor infinite zeros) and $(C_\varepsilon, D_\varepsilon)$ satisfies (11). We proceed to check condition 3.

Let X_ε be the positive definite solution of H_2 ARE,

$$A'_a X_\varepsilon + X_\varepsilon A_a + Q_\varepsilon - X_{\varepsilon,1} B_a B'_a X_\varepsilon = 0,$$

and choose a bounded F_c such that $A_c + B_c F_c$ is Hurwitz. An H_2 optimal static state feedback gain F_ε for the perturbed system can be constructed as

$$F_\varepsilon = \begin{bmatrix} -B'_a X_\varepsilon & 0 \\ 0 & F_c \end{bmatrix} T_1.$$

F_ε is bounded for any $m_0 \le m$ and $\varepsilon \in [0,1]$. Therefore, it is an H_2 low-gain sequence. When $m_0 = m$, i.e. $D = I_m$, we recover the standard H_2-ARE low-gain design of [10].

b) Perturbation method II: In perturbation method I, we add fictitious outputs to completely remove zero dynamics. However, we can also directly perturb system dynamics to move those invariant zeros on the imaginary axis without adding outputs. Consider a perturbation $(A_\varepsilon, B_\varepsilon, C, D_\varepsilon)$ which leads to the following perturbed system

$$\bar{\Sigma}^\varepsilon_{2,II} : \begin{cases} \dot{\bar{x}} = A_\varepsilon \bar{x} + B_\varepsilon u + \omega \\ \bar{z} = D_\varepsilon u \end{cases}$$

where $A_\varepsilon = (1+\varepsilon)A$, $B_\varepsilon = (1+\varepsilon)B$, $D_\varepsilon = (1+\varepsilon)D$ and ε small enough such that $((1+\varepsilon)A, (1+\varepsilon)B)$ is stabilizable. For the sake of clarity, we focus on this particular choice of perturbation. The conditions required for perturbation can be verified as follows:

- Since both $(A_\varepsilon, B, 0, D)$ and $(A, B, 0, D)$ have the same normal rank, condition 1 follows from Theorem 4 in Appendix.
- since $(A_\varepsilon, B, 0, D)$ does not have any invariant zeros on the imaginary axis and has no infinite zeros, condition 2 follows from Lemma 5.6.3 in [11].
- Note that $DF_\varepsilon e^{(A+BF_\varepsilon+\frac{\varepsilon}{2}I)t} = e^{\frac{\varepsilon}{2}t} DF_\varepsilon e^{(A+BF_\varepsilon)t}$. This implies that $\|DF_\varepsilon(sI - A - BF_\varepsilon)\|_2 \le \|DF_\varepsilon(sI - A - \frac{\varepsilon}{2}I - BF_\varepsilon)\|_2$. Therefore, $\|DF_\varepsilon(sI - A - BF_\varepsilon)\|_2 \to 0$ if $\|DF_\varepsilon(sI - A - \frac{\varepsilon}{2}I - BF_\varepsilon)\|_2 \to 0$. We find that conditions 5 is satisfied.
- Obviously, $A + BF$ is Hurwitz stable if $A + BF + \frac{\varepsilon}{2}I$ is Hurwitz stable. Therefore, condition 4 is satisfied.

Therefore, the conditions 1, 2, 3 can be satisfied. For this perturbation, we can always construct a bounded H_2 optimal controller following $(COGFMDZ)_{nli}$ algorithm. This can be done as follows. We first find a nonsingular state transformation independent of ε, $\begin{pmatrix} x_a^{-'} & x_a^{\bigcirc'} & x'_c \end{pmatrix} = T_2 x'$, such that the perturbed system can be transformed into its SCB form,

$$\bar{\Sigma}^\varepsilon_{2,II} : \begin{cases} \begin{pmatrix} \dot{x}_a^- \\ \dot{x}_a^\bigcirc \\ \dot{x}_c \end{pmatrix} = \begin{bmatrix} A_a^- + \frac{\varepsilon}{2}I & 0 & 0 \\ 0 & A_a^\bigcirc + \frac{\varepsilon}{2}I & 0 \\ \star & \star & A_c + \frac{\varepsilon}{2}I \end{bmatrix} \begin{pmatrix} x_a^- \\ x_a^\bigcirc \\ x_c \end{pmatrix} \\ \qquad + \begin{bmatrix} 0 \\ 0 \\ B_c \end{bmatrix} u_c + \begin{bmatrix} B_a^- \\ B_a^\bigcirc \\ B_{ac} \end{bmatrix} u_0 + E\omega \\ z = u_0, \end{cases} \tag{14}$$

where A_a^- is Hurwitz stable, the pairs $(A_a^\bigcirc, B_a^\bigcirc)$ and (A_c, B_c) are controllable and the eigenvalues of $A_a^\bigcirc$ are on the imaginary axis. The eigenvalues of $(1+\varepsilon)A_a^\bigcirc$ and $(1+\varepsilon)A_a^-$ are the invariant zeros of the perturbed system. For a small ε, $(1+\varepsilon)A_a^-$ is also Hurwitz stable. Let X_ε be the positive definite solution of ARE,

$$(A_a^\bigcirc + \tfrac{\varepsilon}{2}I)' X_\varepsilon + X_\varepsilon (A_a^\bigcirc + \tfrac{\varepsilon}{2}I) - X_\varepsilon B_a^\bigcirc B_a^{\bigcirc'} X_\varepsilon = 0, \tag{15}$$

and choose a bounded F_c such that $A_c + B_c F_c$ is Hurwitz. The H_2 low-gain sequence F_ε can be constructed as

$$F_\varepsilon = \begin{bmatrix} 0 & -B_a^{\bigcirc'} X_\varepsilon & 0 \\ 0 & 0 & F_c \end{bmatrix} T_2.$$

Remark 5: In the special case when $D = I_m$, this method recovers the parametric Lyapunov approach to low-gain design as in [20] for linear systems subject to input saturation.

VI. Design of H_∞ low-gain sequences

Different alternate design procedures for γ-level H_∞ low-gain sequences we develop here recover the classical H_∞-ARE low-gain design methods in [19] as a special case.

A. Direct eigenstructure assignment method

The direct eigenstructure assignment method of γ-level H_∞ low-gain design can be found in [2]. In this paper, we focus on designing γ-level H_∞ low-gain sequences using perturbation methods.

B. Perturbation methods

The philosophy of the perturbation methods is similar to that in H_2 low-gain design. However, the conditions imposed on perturbations are more restrictive. For a given quintuple (A, B, C, D, E), let a sequence of perturbations $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$ be such that $A_\varepsilon \to A$, $B_\varepsilon \to B$, $E_\varepsilon \to E$ and $\bar{Q}_\varepsilon \to Q$ where Q and $\bar{Q}_\varepsilon$ are defined in (9).

$(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$ is admissible for γ-level H_∞ low-gain design if it satisfies the following conditions:

1) Define

$$\gamma_\varepsilon^* = \inf_F \left\{ \|(C_\varepsilon + D_\varepsilon F)(zI - A_\varepsilon - B_\varepsilon F)^{-1} E_\varepsilon\|_\infty \mid \lambda(A_\varepsilon + B_\varepsilon F) \in C^- \right\}. \quad (16)$$

For a sufficiently small ε, we have $\gamma_\varepsilon^* < \gamma$.

2) The positive semi-definite semi-stabilizing solution P_ε to CQMI,

$$\begin{bmatrix} A_\varepsilon' P_\varepsilon + P_\varepsilon A_\varepsilon + C_\varepsilon' C_\varepsilon + \gamma^{-2} P_\varepsilon E_\varepsilon E_\varepsilon' P_\varepsilon & P_\varepsilon B_\varepsilon + C_\varepsilon' D_\varepsilon \\ B_\varepsilon' P_\varepsilon + D_\varepsilon' C_\varepsilon & D_\varepsilon' D_\varepsilon \end{bmatrix} \geq 0,$$

satisfies $P_\varepsilon \to 0$ as $\varepsilon \to 0$.

3) $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon)$ has neither invariant zeros on the imaginary axis nor any infinite zeros.

Using the above, a γ-level H_∞ sub-optimal state feedback $F_\varepsilon(E,\gamma)$ with $\gamma > \gamma^*(\varepsilon)$ for the perturbed system can be easily constructed following [14]. Moreover, such an $F_\varepsilon(E,\gamma)$ should satisfy the next three conditions:

4) For ε sufficiently small, $\|(C + DF_\varepsilon(E,\gamma))(sI - A - BF_\varepsilon(E,\gamma))^{-1}E\|_\infty < \gamma$,
5) The $F_\varepsilon(E,\gamma)$ is bounded,
6) The $F_\varepsilon(E,\gamma)$ is such that $A + BF_\varepsilon(E,\gamma)$ is Hurwitz.

If $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$ and a constructed $F_\varepsilon(E,\gamma)$ satisfy all 6 conditions, this $F_\varepsilon(E,\gamma)$ is a γ-level H_∞ low-gain sequence.

In our problem, for a given 5-tuple (A,B,C,D,E) with $C = 0$ and the given $\gamma > 0$ satisfying $\gamma > \gamma^*$, two perturbation methods can be used for γ-level H_∞ low-gain design.

c) Perturbation method I: Similar to that in H_2 low-gain design, the first perturbation is in the form of $(A, B, C_\varepsilon, D_\varepsilon, E)$ where C_ε and D_ε satisfy (11). We give two examples.

Example 1: Consider a sequence of perturbations $(A, B, C_\varepsilon, D_\varepsilon, E)$ where

$$C_\varepsilon' = \begin{pmatrix} 0 & 0 & \sqrt{Q_\varepsilon}' \end{pmatrix}, \quad D_\varepsilon' = \begin{pmatrix} D' & \varepsilon I & 0 \end{pmatrix},$$

where Q_ε satisfies (12). We first verify below that this perturbation is admissible for H_∞ low-gain design.

- Suppose we apply any bounded F to the system (1) characterized by $(A,B,0,D,E)$ such that $A+BF$ is Hurwitz. Let $\gamma_F = \|DF(sI - A - BF)^{-1}E\|_\infty$. We have

$$(C_\varepsilon + D_\varepsilon F)(sI - A - BF)^{-1}E = \begin{bmatrix} DF(sI - A - BF)^{-1}E \\ \varepsilon F(sI - A - BF)^{-1}E \\ \sqrt{Q_\varepsilon}(sI - A - BF)^{-1}E \end{bmatrix}.$$

  Since $A+BF$ is Hurwitz, F is bounded, there exists a M such that

$$\gamma_F \leq \|(C_\varepsilon + D_\varepsilon F)(sI - A - BF)^{-1}E\|_\infty \leq \gamma_F + \max\{\lambda_{max}(Q_\varepsilon), \varepsilon\} M.$$

  This together with (12) implies that for a given γ, there exists an ε^* such that for $\varepsilon \in (0, \varepsilon^*]$ conditions 1 and 4 are satisfied.
- $(A, B, C_\varepsilon, D_\varepsilon)$ has neither invariant zeros nor infinite zeros. One can then design a γ-level H_∞ sub-optimal feedback $F_\varepsilon(E,\gamma)$ using the techniques from [14].
- It is easy to see that C_ε and D_ε satisfy (11). Then condition 2 follows from Theorem 5 in Appendix.
- Since we only perturb C and D and $F_\varepsilon(E,\gamma)$ is obtained using H_∞ control techniques, condition 6 is obvious.

Therefore, for $\varepsilon \in (0, \varepsilon^*]$, conditions 1, 2, 3, 4, and 6 are all satisfied. Next, we construct a γ-level H_∞ suboptimal controller using the techniques developed in [14]. Let X_ε be the positive definite solution of H_∞ ARE,

$$A'X_\varepsilon + X_\varepsilon A + C_\varepsilon' C_\varepsilon - X_\varepsilon B'(D_\varepsilon' D_\varepsilon)^{-1} B X_\varepsilon + \gamma^{-2} X_\varepsilon E E' X_\varepsilon = 0.$$

Then a γ-level H_∞ sub-optimal static state feedback can be constructed as $F_\varepsilon(E,\gamma) = -(D_\varepsilon' D_\varepsilon)^{-1} B' X_\varepsilon$.

When $D = I_m$, this $F_\varepsilon(E,\gamma)$ is bounded for $\varepsilon \in (0, \varepsilon^*]$. Therefore, the condition 5 is satisfied and $F_\varepsilon(E,\gamma)$ is a γ-level H_∞ low-gain sequence. Moreover, it recovers the H_∞-ARE based low-gain design for semi-global stabilization of linear systems subject to input saturation [19]. When $D = \begin{bmatrix} I_{m_0} & 0 \end{bmatrix}$ with some $m_0 < m$, the boundedness of F_ε needs to be proved. However, we present below an alternative perturbation $(C_\varepsilon, D_\varepsilon)$ which yields a bounded $F_\varepsilon(E,\gamma)$.

Example 2: First, we can transfer the system into the SCB form (8) with transformation $(x_a', x_c')' = T_1 x$. Then consider a perturbed system based on (8) as

$$\Sigma_{\infty,I}^\varepsilon : \begin{cases} \begin{pmatrix} \dot{x}_a \\ \dot{x}_c \end{pmatrix} = \begin{bmatrix} A_a & 0 \\ \star & A_c \end{bmatrix} \begin{pmatrix} x_a \\ x_c \end{pmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} u_c + \begin{bmatrix} B_a \\ B_{ac} \end{bmatrix} u_0 + \begin{bmatrix} E_a \\ E_c \end{bmatrix} \omega \\ \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ \sqrt{Q_\varepsilon} & 0 \end{bmatrix} \begin{pmatrix} x_a \\ x_c \end{pmatrix} + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ u_c \end{pmatrix}, \end{cases}$$

where Q_ε satisfies (12). For the same reasons as argued in the previous example, there exists an ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, conditions 1, 2, 3, 4 and 6 are satisfied. It remains to check condition 5. Next we construct a γ-level H_∞ sub-optimal feedback F_ε for the perturbed system following [14]. Let X_ε be the positive definite solution of H_∞ ARE,

$$A_a' X_\varepsilon + X_\varepsilon A_a + Q_\varepsilon - X_\varepsilon B_a B_a' X_\varepsilon + \gamma^{-2} X_\varepsilon E_a E_a' X_\varepsilon = 0,$$

and choose a bounded F_c such that $A_c + B_c F_c$ is Hurwitz. The $F_\varepsilon(E,\gamma)$ can be constructed as

$$F_\varepsilon(E,\gamma) = \begin{bmatrix} -B_a' X_\varepsilon & 0 \\ 0 & F_c \end{bmatrix} T_1.$$

Clearly, $F_\varepsilon(E,\gamma)$ is bounded for $\varepsilon \in (0, \varepsilon^*]$. Therefore, $F_\varepsilon(E,\gamma)$ is a γ-level low-gain sequence.

d) Perturbation method II: We can also directly perturb the system dynamics to move those invariant zeros on the imaginary axis. Consider the perturbation $(A_\varepsilon, B_\varepsilon, 0, D, E_\varepsilon)$ where

$$A_\varepsilon = (1+\varepsilon)A, \; B_\varepsilon = (1+\varepsilon)B, \; E_\varepsilon = (1+\varepsilon)E$$

and ε small enough such that $((1+\varepsilon)A, (1+\varepsilon)B)$ is stabilizable. We shall focus on this particular choice of perturbation.

- Given $A + \frac{\varepsilon}{2} I + BF$ Hurwitz stable, we have $\|DF(sI - A - BF)^{-1}E\|_\infty \leq \|DF(sI - A - \frac{\varepsilon}{2} I - BF)^{-1}E\|_\infty$. This implies that conditions 1 and 4 are satisfied.

- Since $(A+\frac{\varepsilon}{2}I,B,0,D)$ always have the same normal rank as that of $(A,B,0,D)$, the condition 2 follows from Theorem 4 in Appendix.
- Since $(A+\frac{\varepsilon}{2}I,B,0,D)$ does not have any invariant zeros on the imaginary axis, the condition 3 is satisfied.
- $A+BF$ is Hurwitz if $A+\frac{\varepsilon}{2}I+BF$ is Hurwitz.

Therefore, Conditions 1, 2, 3, 4 and 6 are satisfied for sufficiently small ε. Moreover, one can always design a bounded γ-level H_∞ state feedback as in [14] as follows:

The perturbed system can be transformed into its compact SCB form using a nonsingular state transformation: $\begin{bmatrix} x_a' & x_a^{\bigcirc\prime} & x_c' \end{bmatrix}' = T_2 x$ as:

$$\bar{\Sigma}^{\varepsilon}_{\infty,II}: \begin{cases} \begin{pmatrix} \dot{x}_a^- \\ \dot{x}_a^{\bigcirc} \\ \dot{x}_c \end{pmatrix} = \begin{bmatrix} A_a^- + \frac{\varepsilon}{2}I & 0 & 0 \\ 0 & A_a^{\bigcirc} + \frac{\varepsilon}{2}I & 0 \\ \star & \star & A_c + \frac{\varepsilon}{2}I \end{bmatrix} \begin{pmatrix} x_a^- \\ x_a^{\bigcirc} \\ x_c \end{pmatrix} \\ \qquad + \begin{bmatrix} 0 \\ 0 \\ B_c \end{bmatrix} u_c + \begin{bmatrix} B_a^- \\ B_a^{\bigcirc} \\ B_{ac} \end{bmatrix} u_0 + \begin{bmatrix} E_a^- \\ E_a^{\bigcirc} \\ E_c \end{bmatrix} \omega \\ z = u_0, \end{cases} \tag{17}$$

where A_a^- is Hurwitz, (A_c,B_c) is controllable and $(A_a^{\bigcirc},B_a^{\bigcirc})$ is controllable. For a sufficiently small ε, $A_a^- + \frac{\varepsilon}{2}I$ is Hurwitz as well. Let X_ε be the positive definite solution of H_∞ ARE,

$$(A_a^{\bigcirc}+\tfrac{\varepsilon}{2}I)'X_\varepsilon + X_\varepsilon(A_a^{\bigcirc}+\tfrac{\varepsilon}{2}I) - X_\varepsilon B_a^{\bigcirc}B_a^{\bigcirc\prime}X_\varepsilon + \gamma^{-2}X_\varepsilon E_a^{\bigcirc}E_a^{\bigcirc\prime}X_\varepsilon = 0.$$

Let F_c be bounded and such that $A_c+B_cF_c$ is Hurwitz, and the γ-level H_∞ sub-optimal controller is given by

$$F_\varepsilon(E,\gamma) = \begin{bmatrix} 0 & -B_a^{\bigcirc\prime}X_\varepsilon & 0 \\ 0 & 0 & F_c \end{bmatrix} T_2.$$

Since X_ε is bounded, $F_\varepsilon(E,\gamma)$ is bounded. Therefore, $F_\varepsilon(E,\gamma)$ is a γ-level H_∞ low-gain sequence.

APPENDIX

Here our concern is the continuity of semi-stabilizing solution of the following CQMI associated with the 5-tuple (A,B,C,D,E) and $\gamma > \gamma^*$

$$\begin{bmatrix} A'P+PA+C'C+\gamma^{-2}PEE'P & PB+C'D \\ B'P+D'C & D'D \end{bmatrix} \geq 0, \tag{18}$$

where

$$\gamma^* := \inf_F \left\{ \|(C+DF)(sI-A-BF)^{-1}E\|_\infty \mid \lambda(A+BF) \in \mathbb{C}^- \right\}$$

We recall the following theorem from [16]:

Theorem 4: Consider a 5-tuple (A,B,C,D,E). Suppose (A,B) is stabilizable, (A,B,C,D) does not have any invariant zeros in $\mathbb{C}^+$, and $\gamma > \gamma^*$. Let a sequence of perturbed data $(A_\varepsilon,B_\varepsilon,C_\varepsilon,D_\varepsilon,E_\varepsilon)$ converges to (A,B,C,D,E). Moreover, assume that the normal rank of $C_\varepsilon(sI-A_\varepsilon)^{-1}B_\varepsilon+D_\varepsilon$ is equal to the normal rank of $C(sI-A)^{-1}B+D$ for all ε. Then, the smallest positive semi-definite semi-stabilizing solution of CQMI (18) associated with $(A_\varepsilon,B_\varepsilon,C_\varepsilon,D_\varepsilon,E_\varepsilon)$ converges to the smallest positive semi-definite semi-stabilizing solution of CQMI associated with (A,B,C,D,E).

In the perturbation method I of both H_2 and H_∞ low-gain design, we use perturbations which do not necessarily preserve the normal rank. In this case, we use the following:

Theorem 5: Consider a 5-tuple (A,B,C,D,E) and $\gamma > \gamma^*$. Suppose a sequence of perturbations $(C_\varepsilon,D_\varepsilon)$ converges to (C,D), and satisfies the following conditions:

1) $\bar{Q}_\varepsilon$ is continuous at $\varepsilon = 0$;
2) there exists a β such that for $0 \leq \varepsilon_1 \leq \varepsilon_2 \leq \beta$, we have $\bar{Q}_{\varepsilon_1} \leq \bar{Q}_{\varepsilon_2}$.

where $\bar{Q}_\varepsilon$ is defined in (9). Then the semi-stabilizing solution to CQMI (18) associated with $(A,B,C_\varepsilon,D_\varepsilon,E)$ converges to the semi-stabilizing solution of CQMI (18) associated with (A,B,C,D,E).

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