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A Small-Gain Theorem and Construction of Sum-Type Lyapunov Functions for Networks of iSS Systems

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***Abstract*— This paper gives a solution to the problem of verifying stability of networks consisting of integral input-to-state stable (iISS) subsystems. The iISS small-gain theorem developed recently has been restricted to interconnection of two subsystems. For large-scale systems, stability criteria relying only on gain-type information have been successful only in dealing with input-to-state stable stable (ISS) subsystems. To address the stability problem involving iISS subsystems interconnected in general structure, this paper shows how to construct Lyapunov functions of the network by means of nonlinear sum of individual Lyapunov functions of subsystems given in a dissipation formulation under an appropriate small-gain condition.**

I. Introduction

The notion of input-to-state stability (ISS) introduces the concept of nonlinear gain between input and state in order to deal with systems which do not admit finite linear gain [25]. This notion is useful in stability and robustness analysis of large-scale systems since system components are often incompatible with linear-like properties. Decomposition of a system into subsystems allowing for infinite linear gain sometimes reduces conservativeness arising in stability and robustness analysis [19], [28]. However, requiring bounded state for arbitrary magnitude of input is still restrictive. For instance, modules of biological networks are often not ISS [11], [21]. The notion of iISS is a way to remove this limitation [26], and considering networks of iISS subsystems broadens the horizon of stability theory. Difficulties of dealing with non-ISS systems have pushed forward the development of new theoretical tools [3], [10], [1], [4], [21].

In contrast to networks of ISS systems for which a number of small-gain-type results have become available recently, e.g. [7], [20], [22], [8], [6], only a few attempts have been made for networks of iISS systems. Most of Lyapunov-based studies on small-gain criteria have employed the max-type construction for networks whose Lyapunov function V is defined as the weighted maximum of Lyapunov functions of individual subsystems V_i:

$$V(x) = \max_i W_i(V_i(x_i)) \tag{1}$$

This function was first employed for interconnected ISS systems in [18]. The weights are represented by the nonlinear functions W_i. In contrast, there have been only a few results on the construction of sum-type Lyapunov functions for networks whose Lyapunov function is defined as the nonlinearly-weighted sum of Lyapunov functions of individual subsystems:

$$V(x) = \sum_i W_i(V_i(x_i)) \tag{2}$$

A problem of constructing a function of the form (2) was posed for general networks consisting of ISS subsystems in [6] although no solution was derived. Recently, it has been proved in [13] that the max-type construction (1) is never able to yield a Lyapunov function[1] if the network contains a subsystem which is not ISS. The sum-type construction (2) has some clear advantages over the max-type construction since it yields smooth Lyapunov functions directly and it is applicable to networks involving iISS subsystems which are not ISS. Historically, the max-type construction belongs to the idea of vector Lyapunov functions, while the sum-type construction belongs to the idea of scalar Lyapunov functions [23], [24]. The class of networks for which the sum-type construction is solved has been limited to trivial cases exhibiting explicit energy-type conservation or finite linear gain systems such as finite L_p gain systems (See , e.g., [9], [6]). It was found recently that the sum-type construction could give Lyapunov functions explicitly for feedback and cascade connection of two iISS subsystems [10], [17], [12]. In the presence of more than two subsystems, the technique proposed there could be extended to only a specific structure of networks [15], i.e., cactus graphs.

An attempt to tackle iISS networks was made in [5] and their investigation agrees that new tools are needed when the network involves non-ISS subsystems. The problem of guaranteeing stability of such a network remains unsolved [13]. In [21], as a useful idea of circumventing the difficulty of tackling the direct iISS formulation, a time embedded formulation aiming at verifying input-to-output stability is introduced in a trajectory-based setup. The ISS small-gain condition can be still used for non-ISS subsystems by assuming that the behavior of the subsystems is ISS after a

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[1]If there exists a max-type Lyapunov function guaranteeing the stability of all networks whose subsystems fulfill prescribed iISS properties, all the subsystems are ISS.

transient period and that a trajectory estimate of the network during the period is available in a desired manner.

In these circumstances, the purpose of this paper is to present a small-gain criterion for networks consisting of iISS subsystems interconnected in general graph structure. To the best of our knowledge, a methodology leading to the construction of ISS or iISS sum-type Lyapunov functions for general networks is presented for the first time in this paper.

In this paper, the symbol $|x|$ denotes the Euclidean norm of a real vector $x \in \mathbb{R}^n$. A continuous function $\omega : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}_+$ is said to be positive definite if it satisfies $\omega(0) = 0$ and $\omega(s) > 0$ holds for all $s > 0$. A continuous function ω is of class $\mathcal{K}$ and written as $\omega \in \mathcal{K}$ if it is positive definite and strictly increasing; of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and is unbounded. The symbol $\mathbf{Id}$ denotes the identity function on $\mathbb{R}_+$. The symbols $\vee$ and $\wedge$ denote logical sum and logical product, respectively. All proofs are omitted due to the space limitation.

II. Network of iISS Systems

Consider a network Σ consisting of n subsystems Σ_i, $i = 1, 2, ..., n$ where $n \geq 2$. Let $x = [x_1^T, \dots, x_n^T]^T \in \mathbb{R}^N$ be the state vector of Σ, where the state vector of each subsystem is $x_i \in \mathbb{R}^{N_i}$, and $N := \sum N_i$ holds. Suppose that the dynamics of the i-th subsystem Σ_i is governed by

$$\Sigma_i : \ \dot{x}_i = f_i(x_1, \dots, x_n, r), \tag{3}$$

where $r \in \mathbb{R}^M$ and $f_i : \mathbb{R}^{N+M} \to \mathbb{R}^{N_i}$. For each $i \in \{1, 2, ..., n\}$, the subsystem (3) is assumed to have a unique maximal solution $x_i(t)$ for any given initial condition $x_i(0) \in \mathbb{R}^{N_i}$ and any locally L^∞-inputs $x_j : [0, \infty) \to \mathbb{R}^{N_j}, j \neq i$, and $r : [0, \infty) \to \mathbb{R}^M$. For instance, this can be guaranteed by the local Lipschitz condition on f_i. Using $f = [f_1^T, \dots, f_n^T]^T : \mathbb{R}^{N+M} \to \mathbb{R}^N$, the overall network Σ is written as

$$\Sigma : \ \dot{x} = f(x, r) . \tag{4}$$

The knowledge of f is not assumed. Instead, this paper assumes that a dissipation inequality of each subsystem Σ_i is known as follows:

Assumption 1: For each $i = 1, 2, ..., n$, there exist a $\mathbf{C}^1$ function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ and continuous functions $\hat{\alpha}_i \in \mathcal{K}$, $\hat{\sigma}_{i,j}, \hat{\kappa}_i \in \mathcal{K} \cup \{0\}$ and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_i(|x_i|) \leq V_i(x_i) \leq \overline{\alpha}_i(|x_i|), \quad x_i \in \mathbb{R}^{N_i} \tag{5}$$

$$\dot{V}_i(x_i) \leq -\hat{\alpha}_i(|x_i|) + \sum_{j=1}^{n} \hat{\sigma}_{i,j}(|x_j|) + \hat{\kappa}_i(|r|) \tag{6}$$

hold along the trajectories $x_i(t)$ for all $x_j \in \mathbb{R}^{N_j}, j \neq i$ and all $r \in \mathbb{R}^M$, where $\hat{\sigma}_{i,i} \equiv 0$, $i = 1, 2, ..., n$.

The inequality (6) is called a dissipation inequality and means that each subsystem Σ_i with the inputs x_j, $j \neq i$ and r is integral input-to-state stable (iISS), and that V_i is an iISS Lyapunov function for the disconnected Σ_i [2]. Under a stronger assumption $\hat{\alpha}_i \in \mathcal{K}_\infty$, the subsystem Σ_i is guaranteed to be input-to-state stable (ISS), and V_i is an ISS Lyapunov function [27]. By definition [26], the set of ISS systems is a strict subset of the set of iISS systems. The goal of this paper is to construct an iISS Lyapunov function $V(x)$ of the network Σ with respect to input r and state x, and to find a condition under which such construction is possible.

Remark 1: The function V_i is an iISS Lyapunov function of Σ_i even when $\hat{\alpha}_i$ is only positive definite [2]. To allow each Σ_i to form cycles in Σ, this paper assumes $\hat{\alpha}_i \in \mathcal{K}$ which is a strict subset of positive definite functions. It is proved in [12] that a feedback interconnection of iISS systems defined with the dissipation inequalities (6) is guaranteed to be iISS only if for each i the function $\hat{\alpha}_i$ can be bounded from below by a class $\mathcal{K}$ function. For cascade connection, $\hat{\alpha}_i \in \mathcal{K}$ is not necessary. Such relaxation is not covered by this paper.

Remark 2: If a subsystem Σ_i is ISS, the existence of β_i, $\chi_{i,j}$, $\chi_i \in \mathcal{K}$ ($\chi_{i,i} = 0$) satisfying the implication of the form

$$|x_i| \geq \sum_{j=1}^{n} \chi_{i,j}(|x_j|) + \chi_i(|r|) \ \Rightarrow \ \dot{V}_i(x_i) \leq -\beta_i(|x_i|) \tag{7}$$

is an alternative to (6) [27]. The characterization (7) referred to as the implication formulation is used for ISS networks in [20], [22], [8] with some equivalent variations in the conditional $|x_i| \geq \sum_{j=1}^{n} \chi_{i,j}(|x_j|) + \chi_i(|r|)$. If subsystems are not ISS, we do not have such implication formulation.

III. Sum-Type Lyapunov Functions

Define $\hat{A}, \hat{S}, \hat{D}, \Lambda$: $\mathbf{s} = [s_1, s_2, ..., s_n]^T \in \mathbb{R}_+^n \mapsto \mathbf{z} \in \mathbb{R}_+^n$ by

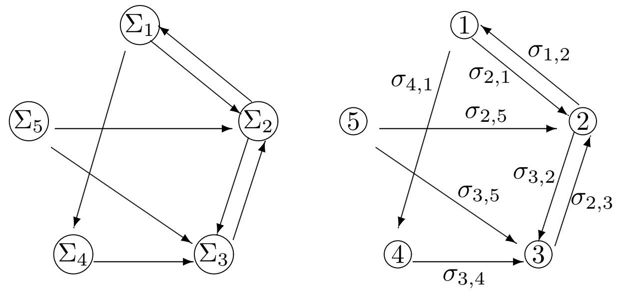
$$\mathbf{z} = \hat{A}(\mathbf{s})$$
$$= \begin{bmatrix} \hat{\alpha}_1 \circ \overline{\alpha}_1^{-1}(s_1) \\ \hat{\alpha}_2 \circ \overline{\alpha}_2^{-1}(s_2) \\ \vdots \\ \hat{\alpha}_n \circ \overline{\alpha}_n^{-1}(s_n) \end{bmatrix} = \begin{bmatrix} \hat{\alpha}_1 & 0 & \cdots & 0 \\ 0 & \hat{\alpha}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \hat{\alpha}_n \end{bmatrix} \begin{pmatrix} \overline{\alpha}_1^{-1}(s_1) \\ \overline{\alpha}_2^{-1}(s_2) \\ \vdots \\ \overline{\alpha}_n^{-1}(s_n) \end{pmatrix}$$

$$\mathbf{z} = \hat{S}(\mathbf{s})$$
$$= \begin{bmatrix} \sum_{j \neq 1} \hat{\sigma}_{1,j} \circ \underline{\alpha}_j^{-1}(s_j) \\ \sum_{j \neq 2} \hat{\sigma}_{2,j} \circ \underline{\alpha}_j^{-1}(s_j) \\ \vdots \\ \sum_{j \neq n} \hat{\sigma}_{n,j} \circ \underline{\alpha}_j^{-1}(s_j) \end{bmatrix} = \begin{bmatrix} 0 & \hat{\sigma}_{12} & \cdots & \hat{\sigma}_{1,n} \\ \hat{\sigma}_{21} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \hat{\sigma}_{n-1,n} \\ \hat{\sigma}_{n,1} & \cdots & \hat{\sigma}_{n,n-1} & 0 \end{bmatrix} \begin{pmatrix} \underline{\alpha}_1^{-1}(s_1) \\ \underline{\alpha}_2^{-1}(s_2) \\ \vdots \\ \underline{\alpha}_n^{-1}(s_n) \end{pmatrix}$$

$$\hat{D}(\mathbf{s}) = \begin{bmatrix} s_1 + \hat{\delta}_1(s_1) \\ s_2 + \hat{\delta}_2(s_2) \\ \vdots \\ s_n + \hat{\delta}_n(s_n) \end{bmatrix}, \quad \Lambda(\mathbf{s}) = \begin{bmatrix} \lambda_1(s_1) \\ \lambda_2(s_2) \\ \vdots \\ \lambda_n(s_n) \end{bmatrix}.$$

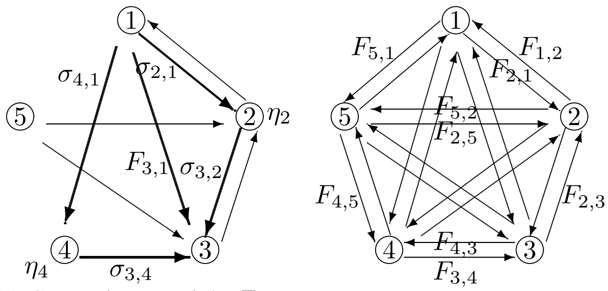
Note that the last identity in both $\hat{A}$ and $\hat{S}$ shown above is not a matrix operation since their entries are functions. The matrix-like representation helps us see the structure of the operators. Indeed, $\hat{A}$, $\hat{D}$ and Λ have the same diagonal structure while $\hat{S}$ is not diagonal. The functions λ_i and $\hat{\delta}_i$ have yet to be determined. The following is a result in [13].

Theorem 1: Suppose that there exist continuous functions



(a) Network Σ. (b) Weighted graph G.

Fig. 1. An example of a general network.



(a) Computing a weight $F_{3,1}$ of a fictitious arc. (b) Complete graph with $F_{i,j}$.

Fig. 2. Rearrangement of weights to obtain a fictitious complete graph.

$\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, ..., n$, such that

$$\lambda_i(s) > 0, \quad \forall s \in (0, \infty), \qquad i = 1, 2, ..., n \tag{8}$$

$$\int_1^\infty \lambda_i(s) ds = \infty, \qquad i = 1, 2, ..., n \tag{9}$$

$$\{\lim_{s\to\infty} \hat{\alpha}_i(s) = \infty \ \vee \ \limsup_{s\to\infty} \lambda_i(s) < \infty\}, \qquad i = 1, 2, ..., n \tag{10}$$

$$\Lambda(\mathbf{s})^T[-\hat{D}^{-1} \circ \hat{A}(\mathbf{s}) + \hat{S}(\mathbf{s})] \leq 0, \quad \forall \mathbf{s} \in \mathbb{R}^n_+ \tag{11}$$

hold for some $\hat{\delta}_1$, $\hat{\delta}_2$,..., $\hat{\delta}_n \in \mathcal{K}_\infty$. Then the network Σ is iISS with respect to input r and state x. If

$$\hat{\alpha}_i \in \mathcal{K}_\infty, \qquad i = 1, 2, ..., n \tag{12}$$

$$\liminf_{s_i\to\infty} \lambda_i(s_i) > 0, \quad i = 1, 2, ..., n \tag{13}$$

are satisfied additionally, then the network Σ is ISS. Furthermore, an iISS (ISS) Lyapunov function is

$$V(x) = \sum_{i=1}^n \int_0^{V_i(x_i)} \lambda_i(s) ds \ . \tag{14}$$

In this paper, the form (14) is referred to as the sum-type construction of Lyapunov functions. This sum form is the key to the success in constructing a Lyapunov function of the overall network involving non-ISS subsystems [13].

Remark 3: The global asymptotic stability of $x = 0$ for $r(t) \equiv 0$ needs neither (10) nor the class $\mathcal{K}_\infty$ property of $\hat{\delta}_i$.

IV. General Networks

This section presents the main results which give a solution $\Lambda(\mathbf{s})$ to the problem formulated in Theorem 1. Without any loss of generality, we can assume the following:

Assumption 2: There are no isolated subsystems, i.e.,

$$\left\{ \sum_{j=1}^n \hat{\sigma}_{i,j} \neq 0 \ \vee \ \sum_{j=1}^n \hat{\sigma}_{j,i} \neq 0 \right\}, i = 1, 2, ..., n.$$

Since isolated subsystems are iISS or ISS, we can focus on the rest of the network. Or alternatively, to avoid Assumption 2, one can add small $\hat{\sigma}_{j,i} \neq 0$ fictitiously. The existence of such a sufficiently small $\hat{\sigma}_{j,i}$ is guaranteed whenever the original problem in Theorem 1 has a solution.

We shall introduce basic concepts of weighted directed graphs [29]. The terms "weighted" and "directed" are omitted when they are clear from the context. The vertex set and the arc set of a directed graph G are denoted by $\mathcal{V}(G)$ and $\mathcal{A}(G)$, respectively. In this paper, a walk is an alternating sequence of vertices and connecting arcs, beginning and ending with a vertex. A walk is a path if it has no repeated vertices. A walk is a cycle if it starts and ends at the same vertex but otherwise has no repeated vertices. Given a path or a cycle U of length k, we employ the following notation:

$$|U| = k, \quad U = (u(1), u(2), ..., u(k), u(k+1)),$$

where $u(i)$'s listed above are "all" the vertices comprising U and they are listed in the "reversed" order of appearance. If U is a cycle, we have $u(1) = u(k+1)$. The starting vertex of the path U is $u(k+1)$, and the ending vertex is $u(1)$. When an arc is represented by the ordered pair (i, j), it is directed away from the j-th vertex and directed toward the i-th vertex. Let $\mathcal{C}(G)$ denote the set of all directed cycles contained in the graph G. Let $\mathcal{P}(G)$ denote the set of all directed paths contained in the graph G. The length of cycles (resp. paths) is larger than or equal to two (resp. one).

Let G denote the directed graph of the network Σ. The vertices of G are the subsystems Σ_i, while the arcs are the signal flows between the subsystems. In other words, the zero-nonzero structure of $\hat{S}(\mathbf{s})$ introduced in Section III is the transpose of the adjacency matrix of the directed graph G. In order to associate the network Σ with a "weighted" directed graph, we introduce $\alpha_i \in \mathcal{K}$, $\sigma_{i,j} \in \mathcal{K} \cup \{0\}$ satisfying

$$\begin{aligned}
&\sum_{i=1}^n \Big[-\hat{\alpha}_i(s_i) + \sum_{j=1}^n \hat{\sigma}_{i,j}(s_j) \Big] = \\
&\quad \sum_{U\in\mathcal{C}(G)} K(U) \sum_{i=1}^{|U|} \Big[-\alpha_{u(i)}(s_{u(i)}) + \sigma_{u(i),u(i+1)}(s_{u(i+1)}) \Big] \\
&\quad + \sum_{T\in\mathcal{P}(G)} K(T) \Bigg\{ \sum_{i=1}^{|T|} \Big[-\alpha_{t(i)}(s_{t(i)}) + \sigma_{t(i),t(i+1)}(s_{t(i+1)}) \Big] \\
&\qquad - \alpha_{t(|T|+1)}(s_{t(|T|+1)}) \Bigg\}.
\end{aligned} \tag{15}$$

for $[s_1, s_3, ..., s_n]^T \in \mathbb{R}^n_+$, where $K : \mathcal{C}(G) \cup \mathcal{P}(G) \to \mathbb{R}_+$. The above decomposes the problem formulated in Theorem 1 into cycles and paths. However, notice that the functions λ_i interlace multiple cycles and paths in (11). The function $K : \mathcal{C}(G) \cup \mathcal{P}(G) \to \mathbb{R}_+$ determines whether the sum taken along a particular cycle U or a particular path T appears in (11) and how large its contribution to the sum is. Since there are no isolated vertices, the network Σ with an arbitrary structure always admits a decomposition of the form (15) with

$$\hat{\alpha}_i(s) = w_i\alpha_i(s), \quad \hat{\sigma}_{i,j}(s) = w_{i,j}\sigma_{i,j}(s), \tag{16}$$

where w_i and $w_{i,j}$ are positive real numbers satisfying

$$w_i = \sum_{j=1}^{n} w_{i,j} = \sum_{U\in\{W\in\mathcal{C}(G)\cup\mathcal{P}(G):\mathcal{V}(W)\ni i\}} K(U) \tag{17}$$

$$w_{i,j} = \sum_{U\in\{W\in\mathcal{C}(G)\cup\mathcal{P}(G):\mathcal{A}(W)\ni(i,j)\}} K(U). \tag{18}$$

The decomposition (15) of $\hat{\alpha}_i$ and $\hat{\sigma}_{i,j}$ into α_i and $\sigma_{i,j}$ is not unique. We define the weight of the arc (i,j) of G as the function $\sigma_{i,j}(s)$. An example of a general network and its weighted directed graph is illustrated in Fig.1. Due to the non-uniqueness of the decomposition (15), the weighted graph is not uniquely determined from the network Σ although there always exists a weighted graph for each Σ. We employ the following convention for $i = 1, 2, ..., n$:

$$\eta_i(s) = \begin{cases} \alpha_i^{-1}(s) & \text{if } \lim_{\tau\to\infty}\alpha_i(\tau) > s \\ \infty & \text{otherwise} \end{cases} \tag{19}$$

which is a slightly abused notation of inverse operation on α_i. The benefit of the notation is discussed in Remark 9.

We now build a weighted complete directed graph from G by adding fictitious arcs and weights. Consider $F_{i,j} \in \mathcal{K}$, $i,j = 1, 2, ..., n$, satisfying the following properties:

$$F_{i,j}(s) \geq \max\left\{ \max_{\substack{1\leq q\leq n \\ q\neq i, q\neq j}} F_{i,q}\circ\underline{\alpha}_q^{-1}\circ\overline{\alpha}_q\circ\eta_q\circ\tau_q F_{q,j}(s),\ \sigma_{i,j}(s)\right\}, \quad \forall s\in\mathbb{R}_+,\ i,j = 1,2,...,n,\ i\neq j \tag{20}$$

$$F_{i,i}(s) \geq \max_{\substack{1\leq q\leq n \\ q\neq i}} F_{i,q}\circ\underline{\alpha}_q^{-1}\circ\overline{\alpha}_q\circ\eta_q\circ\tau_q F_{q,i}(s), \quad \forall s\in\mathbb{R}_+,\ i = 1,2,...,n \tag{21}$$

$$\underline{\alpha}_i^{-1}\circ\overline{\alpha}_i\circ\eta_i\circ c_i F_{i,i}(s) \leq s,\ \forall s\in\mathbb{R}_+, i = 1,2,...,n \tag{22}$$

$$\lim_{s\to\infty} F_{i,j}(s) < \infty \ \vee \ \lim_{s\to\infty}\max\left\{ \max_{\substack{1\leq q\leq n \\ q\neq i, q\neq j}} F_{i,q}\circ\underline{\alpha}_q^{-1}\circ\overline{\alpha}_q\circ\eta_q\circ\tau_q F_{q,j}(s),\ \sigma_{i,j}(s)\right\} = \infty, \quad i,j = 1,2,...,n,\ i\neq j \tag{23}$$

$$\lim_{s\to\infty} F_{i,i}(s) < \infty \ \vee \ \lim_{s\to\infty}\max_{\substack{1\leq q\leq n \\ q\neq i}} F_{i,q}\circ\underline{\alpha}_q^{-1}\circ\overline{\alpha}_q\circ\eta_q\circ\tau_q F_{q,i}(s) = \infty, \quad i = 1,2,...,n \tag{24}$$

The parameters $\tau_i, c_i \in \mathbb{R}_+$, $i = 1, 2, ..., n$ will be chosen later so that the above "class $\mathcal{K}$" functions $F_{i,j}$ exist. Using $F_{i,j}$ given for all pairs (i,j) of $i,j = 1, 2, ..., n$, we can define a weighted complete directed graph[2]. The weight functions $F_{i,j}$ are assigned to individual arcs connecting all possible pairs in $\mathcal{V}(G)$. Some arcs are fictitious so that they are not present in the original graph G. The function $F_{i,j}$ replaces the original weight $\sigma_{i,j}$ if the arc (i,j) exists in the original graph G. The rearrangement of the weights to define the complete graph is illustrated by Fig.2. This rearrangement allows us to find the functions λ_i which are compatible with all the cycles and paths comprising G. Now, we are in position to state the main result which gives the functions λ_i solving (11). The proof is based on the tools developed in Section V.

Theorem 2: Consider $\alpha_i \in \mathcal{K}$, $\sigma_{i,j} \in \mathcal{K}\cup\{0\}$ and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$, $i,j = 1, 2, ..., n$, satisfying (15) and

$$\textit{(H1)}\ \left\{\lim_{s\to\infty}\alpha_j(s) = \infty \vee \lim_{s\to\infty}\sum_{i=1}^{n}\sigma_{i,j}(s) < \infty\right\},\quad j = 1, 2, ..., n.$$

Suppose that there exist $c_i > 1$, $i = 1, 2, ..., n$ such that

$$\underline{\alpha}_{u(1)}^{-1}\circ\overline{\alpha}_{u(1)}\circ\eta_{u(1)}\circ c_{u(1)}\sigma_{u(1),u(2)}\circ \underline{\alpha}_{u(2)}^{-1}\circ\overline{\alpha}_{u(2)}\circ\eta_{u(2)}\circ c_{u(2)}\sigma_{u(2),u(3)}\circ\cdots\circ \underline{\alpha}_{u(k)}^{-1}\circ\overline{\alpha}_{u(k)}\circ\eta_{u(k)}\circ c_{u(k)}\sigma_{u(k),u(k+1)}(s) \leq s,\ \forall s\in\mathbb{R}_+ \tag{25}$$

holds for all cycles $U\in\mathcal{C}(G)$, where $k \in \{2, 3, ..., n\}$ denotes the length of each cycle U. Let τ_i and $\psi \geq 0$ be such that

$$1 < \tau_i < c_i, \quad i = 1, 2, ..., n \tag{26}$$

$$\left(\frac{\tau_i}{c_i}\right)^{\psi} \leq \tau_i - 1, \quad i = 1, 2, ..., n \tag{27}$$

are satisfied. Pick class $\mathcal{K}$ functions $F_{i,j}$, $i,j = 1, 2, ..., n$, such that (20)-(24) are satisfied. Define class $\mathcal{K}$ functions $\overline{\lambda}_i$, $i = 1, 2, ..., n$, by

$$\overline{\lambda}_i(s) = \left[\frac{1}{\tau_i}\alpha_i(\overline{\alpha}_i^{-1}(s))\right]^{\psi}\prod_{j\in\mathcal{V}(G)-\{i\}}\left[F_{j,i}(\underline{\alpha}_i^{-1}(s))\right]^{\psi+1}. \tag{28}$$

Let ν_i: $(0,\infty)\to\mathbb{R}_+$, $i = 1, 2, ..., n$, be continuous functions fulfilling

$$0 < \nu_i(s) < \infty, \quad s\in(0,\infty) \tag{29}$$

$$\lim_{s\to\infty}\alpha_i(s) = \infty \ \vee \ \lim_{s\to\infty}\nu_i(s) < \infty \tag{30}$$

$$\overline{\lambda}_i(s)\nu_i(s) : \text{non-decreasing continuous for } s\in(0,\infty) \tag{31}$$

and

$$\nu_{u(j)}\circ\overline{\alpha}_{u(j)}\circ\eta_{u(j)}\circ\tau_{u(j)}\sigma_{u(j),u(j+1)}(s) \leq \nu_{u(j+1)}\circ\underline{\alpha}_{u(j+1)}(s), \quad \forall s\in(0,\infty),\ j = 1, 2, ..., |U| \tag{32}$$

for all cycles $U \in \mathcal{C}(G)$. Then non-decreasing continuous functions $\lambda_i : \mathbb{R}_+\to\mathbb{R}_+$, $i = 1, 2, ..., n$, defined by

$$\lambda_i(s) = \overline{\lambda}_i(s)\nu_i(s), \quad s\in(0,\infty), \quad i = 1, 2, ..., n \tag{33}$$

$$\lambda_i(0) = \lim_{s\to 0^+}\overline{\lambda}_i(s)\nu_i(s) \tag{34}$$

satisfy (8), (9) and (10), and achieve (11) with $\hat{\delta}_i(s) = b_i s$, $i = 1, 2, ..., n$, for some $b_i > 0$.

It is stressed that the constants τ_i, ψ satisfying (26)-(27) and the functions ν_i, $i = 1, 2, ..., n$, satisfying (29)-(32) always exist and can be chosen easily (See Remark 5 for ν_i). The following proves the existence of the desired $F_{i,j}(s)$.

Lemma 1: Consider $\alpha_i \in \mathcal{K}$, $\sigma_{i,j} \in \mathcal{K}\cup\{0\}$ and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$, $i,j = 1, 2, ..., n$, satisfying *(H1)*. Suppose that there

[2] A complete directed graph is a directed graph in which each ordered pair of vertices is connected by an arc.

exist $c_i > 1$, $i = 1, 2, ..., n$ such that (25) holds for all cycles $U \in \mathcal{C}(G)$, where $k \in \{2, 3, ..., n\}$ denotes the length of each cycle U. Let τ_i be such that (26) is satisfied. Then there exist class $\mathcal{K}$ functions $F_{i,j}$, $i, j = 1, 2, ..., n$, satisfying (20)-(24) and

$$\{\lim_{s\to\infty} \alpha_j(s) = \infty \ \vee \ \lim_{s\to\infty} \sum_{i=1}^{n} F_{i,j}(s) < \infty\}, \ j = 1, 2, ..., n \tag{35}$$

The set of functions λ_i given in (33) yields an iISS Lyapunov function in the form of (14). Since λ_i are of class $\mathcal{K}$, the network is guaranteed to be ISS in the case of (12). Note that *(H1)* is fulfilled by $\alpha_1, ..., \alpha_n \in \mathcal{K}_\infty$. Theorem 2 demonstrates that the collection of the inequalities in (25) is a sufficient condition for the iISS and ISS of the network Σ. It is the property that small-gain conditions are satisfied for all cycles in the "original" graph G. The "fictitious" complete graph (Fig.2(b)) allows us to write a systematic formula of λ_i in terms of $F_{i,j}$ as in (28). By virtue of the notation of η_i's, the conditions (25) are invariant under cyclic shifting of vertices. The reader might be concerned that the left hand side of (25) would not be well-defined for all $s \in \mathbb{R}_+$ due to the inverse maps in η_i's. The assumption *(H1)* ensures that the left hand side is well-defined for all $s \in \mathbb{R}_+$.

It is verified easily that the small-gain condition (25) with $c_i > 1$ implies the existence of $i \in \{1, 2, ..., |U|\}$ satisfying

$$\lim_{s\to\infty} \alpha_{u(i)}(s) = \infty \ \vee \ \lim_{s\to\infty} \alpha_{u(i)}(s) > \lim_{s\to\infty} \sigma_{u(i),u(i+1)}(s) \tag{36}$$

for each cycle $U \in \mathcal{C}(G)$. Using (16) and (17), we can also prove the existence of $i \in \{1, 2, ..., n\}$ satisfying

$$\lim_{s\to\infty} \hat{\alpha}_i(s) = \infty \ \vee \ \lim_{s\to\infty} \hat{\alpha}_i(s) > \lim_{s\to\infty} \sum_{j=1}^{n} \hat{\sigma}_{i,j}(s) \tag{37}$$

Hence, the small-gain condition (25) implies that at least one subsystem Σ_i should be ISS with respect to the combined input consisting of all the signals from the other subsystems. This fact conforms to a necessity condition derived in [14] for the stability of iISS networks. In the case of $r(t) \equiv 0$, the strict inequality signs in (36) and (37) can be replaced by non-strict ones (See Section VI).

Remark 4: The functions $F_{i,j}$, $i \neq j$, can be computed by evaluating arcs with $\sigma_{p,q}$'s and vertices with η_p's in all paths from j to i in G. We do not have to evaluate walks which are not paths, thanks to (22). The functions $F_{i,i}$ can be determined by evaluating $\sigma_{p,q}$'s and η_p's along all cycles starting and ending at i in G.

Remark 5: The simplest choice of continuous functions $\nu_i : (0, \infty) \to \mathbb{R}_+$, $i = 1, 2, ..., n$ fulfilling (29)-(32) is

$$\nu_1(s) = \nu_2(s) = ... = \nu_n(s) = \text{constant} > 0. \tag{38}$$

A non-constant choice of ν_i, $i = 1, 2, ..., n$ is

$$\nu_i(s) = \nu \circ F_{l,i} \circ \underline{\alpha}_i^{-1}(s), \ i = 1, 2, ..., n \tag{39}$$

defined with any $l \in \mathcal{V}(G)$ and any non-decreasing continuous function ν satisfying $0 < \nu(s) < \infty$ for $s \in (0, \infty)$. Indeed, the properties (29)-(32) follow from $F_{l,i} \in \mathcal{K}$, (35), (20), (21) and (22). The above examples of ν_i's are non-decreasing functions. It is worth noting that decreasing functions are also eligible. This feature contrasts with the previous results [16], [17], [15]. To see this point, consider the network Σ which is a cycle graph and satisfies $\gamma_j \alpha_j(s) = \sigma_{i,j}(s)$, $\gamma_j > 0$ and $\underline{\alpha}_i(s) = \overline{\alpha}_i(s) = s$ for $i = 1, 2, ..., n$, Then the choice

$$\nu_i(s) = g_i[\alpha_i(s)]^{-n\psi-n+1}, \quad i = 1, 2, ..., n \tag{40}$$

fulfills (29)-(32) for appropriate constants $g_i > 0$. In this case, the functions λ_i, $i = 1, 2, ..., n$, become positive constants and the small-gain condition is $\gamma_1\gamma_2 \cdots \gamma_n < 1$.

Remark 6: A condition similar to (25) has been developed for networks of ISS subsystems within an implication formulation of Σ_i [20], [22], [8]. It is stressed that even if the subsystems are restricted to ISS ones, the implication formulation and this paper take different definitions of gains appearing in the small-gain conditions. Although the implication formulation naturally leads to a stability test in the form of a set of multiple small-gain conditions, the implication formulation is valid only for ISS subsystems. In contrast, the dissipation formulation used in Assumption 1 is "applicable" to iISS as well as ISS subsystems. The applicability, however, does not automatically guarantee the capability to establish the stability of networks containing non-ISS subsystems [13]. For assuring the capability, this paper employs the sum-type Lyapunov function (14) and it is linked to the multiple small-gain conditions as in (25).

Remark 7: If the graph G contains no cycle, the problem (11) is always solvable. Since the small-gain condition (25) is required for only cycles, the functions $F_{i,j}$, $i, j = 1, 2, ..., n$ are guaranteed to exist and the functions in (33) satisfy (8), (9) and (10), and solve (11). It is stressed that this holds true under the assumption of $\alpha_i \in \mathcal{K}$, $i = 1, 2, ..., n$, and *(H1)*. This fact is consistent with the $n = 2$ result in [12].

V. Cycle Networks

This section has two objectives. One is to solve the stability problem posed by Theorem 1 for cycle networks. The other is to provide the key tools to obtain the main results presented in Section IV. Consider the network Σ which admits the following representation:

$$\sum_{i=1}^{n}\Big[-\hat{\alpha}_i(s_i) + \sum_{j=1}^{n} \hat{\sigma}_{i,j}(s_j)\Big] = \sum_{i=1}^{n}\Big[-\alpha_i(s_i) + \sigma_{i,q}(s_q)\Big],$$
$$q(i) = (i \mod n) + 1, \quad \forall \mathbf{s} \in \mathbb{R}_+^n, \tag{41}$$

where $\alpha_i \in \mathcal{K}$ and $\sigma_{i,q} \in \mathcal{K}$. The identity (41) implies that the weighted directed graph G of Σ is a cycle of length n. An example for $n = 5$ is shown in Fig.3(a). For a cycle network of arbitrary length, the identity (41) follows immediately from $\hat{\alpha}_i = \alpha_i$ and $\hat{\sigma}_{i,j} = \sigma_{i,j}$.

The next lemma generates weighting functions $F_{i,j}$ for all possible ordered pairs of the vertices in the graph G, which plays a key role in computing a solution Λ to (11).

Lemma 2: Consider $\alpha_i \in \mathcal{K}$, $\sigma_{i,j} \in \mathcal{K} \cup \{0\}$ and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$, $i = 1, 2, ..., n$, $j = (i \mod n) + 1$, satisfying

(J1) $\{\lim_{s\to\infty} \alpha_j(s) = \infty \vee \lim_{s\to\infty} \sigma_{i,j}(s) < \infty\}$,
$j = 1, 2, ..., n, \ i = (j - 2 \mod n) + 1$.

Suppose that there exist $c_i > 1$, $i = 1, 2, ..., n$ such that

$$\underline{\alpha}_1^{-1} \circ \overline{\alpha}_1 \circ \eta_1 \circ c_1 \sigma_{1,2} \circ \underline{\alpha}_2^{-1} \circ \overline{\alpha}_2 \circ \eta_2 \circ c_2 \sigma_{2,3} \circ \cdots \circ \underline{\alpha}_{n-1}^{-1} \circ \overline{\alpha}_{n-1} \circ \eta_{n-1} \circ c_{n-1} \sigma_{n-1,n} \circ \underline{\alpha}_n^{-1} \circ \overline{\alpha}_n \circ \eta_n \circ c_n \sigma_{n,1}(s) \le s, \ \forall s \in \mathbb{R}_+ \quad (42)$$

holds. Let τ_i be such that (26) is satisfied. Then there exist class $\mathcal{K}$ functions $F_{i,j}$, $i, j = 1, 2, ..., n$, such that

$$F_{i,j}(s) \ge \sigma_{i,j}(s), \ \forall s \in \mathbb{R}_+, \quad i = 1, 2, ..., n, \ j = (i \mod n) + 1 \quad (43)$$

$$F_{i,j}(s) \ge F_{i,q} \circ \underline{\alpha}_q^{-1} \circ \overline{\alpha}_q \circ \eta_q \circ \tau_q \sigma_{q,j}(s), \quad \forall s \in \mathbb{R}_+, \quad i = 1, 2, ..., n, \ i + 2 \le h \le i + n, \quad q = (h - 2 \mod n) + 1, \quad j = (h - 1 \mod n) + 1 \quad (44)$$

$$\underline{\alpha}_i^{-1} \circ \overline{\alpha}_i \circ \eta_i \circ c_i F_{i,i}(s) \le s, \ \forall s \in \mathbb{R}_+, \ i = 1, 2, ..., n \quad (45)$$

$$\lim_{s\to\infty} F_{i,j}(s) < \infty \ \vee \ \lim_{s\to\infty} \sigma_{i,j}(s) = \infty, \quad i = 1, 2, ..., n, \ j = (i \mod n) + 1 \quad (46)$$

$$\lim_{s\to\infty} F_{i,j}(s) < \infty \ \vee \ \lim_{s\to\infty} F_{i,q} \circ \underline{\alpha}_q^{-1} \circ \overline{\alpha}_q \circ \eta_q \circ \tau_q \sigma_{q,j}(s) = \infty, \quad i = 1, 2, ..., n, \ i + 2 \le h \le i + n, \quad q = (h - 2 \mod n) + 1, \quad j = (h - 1 \mod n) + 1 \quad (47)$$

and (35) hold.

Figure 3(b) illustrates how $F_{i,j}$ is generated from the weighted graph G. Based on the functions $F_{i,j}$ defined for all $i, j = 1, 2, ..., n$, the following theorem gives a set of functions λ_i solving (11) for the cycle graph.

Theorem 3: Consider $\alpha_i \in \mathcal{K}$, $\sigma_{i,j} \in \mathcal{K} \cup \{0\}$ and $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$, $i = 1, 2, ..., n$, $j = (i \mod n) + 1$, satisfying (41) and *(J1)*. Suppose that there exist $c_i > 1$, $i = 1, 2, ..., n$ such that (42) holds. Let τ_i and $\psi \ge 0$ be such that (26) and (27) are satisfied. Pick $F_{i,j} \in \mathcal{K}$, $i, j = 1, 2, ..., n$, as in Lemma 2. Let $\mathcal{V}(G) = \{1, 2, ..., n\}$ and define class $\mathcal{K}$ functions $\bar{\lambda}_i$, $i = 1, 2, ..., n$ by (28). Let ν_i: $(0, \infty) \to \mathbb{R}_+$, $i = 1, 2, ..., n$, be continuous functions fulfilling (29), (30), (31) and

$$\nu_i \circ \overline{\alpha}_i \circ \eta_i \circ \tau_i \sigma_{i,j}(s) \le \nu_j \circ \underline{\alpha}_j(s), \ \forall s \in (0, \infty), \quad j = (i \mod n) + 1 \quad (48)$$

Then non-decreasing continuous functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2, ..., n$, defined by (33) and (34) satisfy (8), (9) and (10), and achieve (11) with $\hat{\delta}_i(s) = b_i s$, $i = 1, 2, ..., n$, for some $b_i > 0$.

Combining Theorem 3 with Theorem 1, the set of functions λ_i in (33) yields an iISS Lyapunov function of the network Σ in the form of (14). The existence of $c_i > 1$, $i = 1, 2, ..., n$, satisfying (42) is a sufficient condition for the iISS property of the cycle network. The network is guaranteed to be ISS in the case of (12).

Outlining the proof of Theorem 2: Applying Theorem 3 to all cycles U residing in G and satisfying $K(U) \neq 0$ allows us to arrive at (11). The flexibility of "ν_i in Theorem 3" is used for obtaining "λ_i in Theorem 2" so that the functions "λ_i in Theorem 3" computed for all the cycles containing the i-th vertex agree with each other.

Remark 8: The inequality (42) generalizes the iISS small-gain condition developed for the two subsystems case [17]. Although setting $n = 2$ reduces Theorem 3 to the result in [17], the formula of λ_i's given in this paper is different from the one given in [17]. The new formula renders the construction of Lyapunov functions amenable to the general structure of networks as in Section IV.

Remark 9: The notation (19) is helpful in avoiding listing the number of combinations of cases divided in accordance with the well-posedness of each α_i^{-1}, which becomes enormous as the number of subsystems increases. Another benefit of using (19) is to be able to render the small-gain condition (42) invariant under cyclic shifting of subsystems. In [10], [17] for feedback interconnection of two iISS subsystems, it is proved that the feedback system is stable if the stability property of one subsystem is strong enough to compensate the "weak stability" of the other subsystem. That is, one subsystem is not required to be ISS, which implies that α_i^{-1} of that subsystem is not defined on the whole $\mathbb{R}_+$. Therefore, the small-gain condition using α_i^{-1}'s becomes asymmetric since one α_i^{-1} is well-defined while the other α_i^{-1} is not. The key observation leading us to the employment of η_i's is that the other subsystem is required to have α_i^{-1} making the overall nonlinear loop gain less than the identity function [10], [17]. The employment of η_i's allows us to write the small-gain condition in a symmetric way even in the presence of iISS subsystems, which contrasts sharply with [15]. The necessity of a subsystem equipped with an ISS property which is strong enough to compensate the "weak stability" of other subsystems is elaborated for stability of networks of iISS systems in [14].

VI. Nonlinear Gaps in Small-Gain Conditions

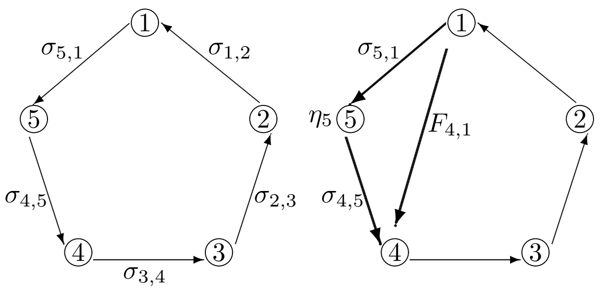
In (25) and (42), the constants $c_i - 1$ describe how much the loop gain is smaller than the identity function in a linear way. At the expense of some technical complexity in the formula for λ_i, the small-gain condition (25) can be relaxed into the existence of $\omega_i \in \mathcal{K}_\infty$, $i = 1, 2, ..., n$, satisfying

$$\underline{\alpha}_{u(1)}^{-1} \circ \overline{\alpha}_{u(1)} \circ \eta_{u(1)} \circ (\mathbf{Id} + \omega_{u(1)}) \circ \sigma_{u(1),u(2)} \circ \underline{\alpha}_{u(2)}^{-1} \circ \overline{\alpha}_{u(2)} \circ \eta_{u(2)} \circ (\mathbf{Id} + \omega_{u(2)}) \circ \sigma_{u(2),u(3)} \circ \cdots \circ \underline{\alpha}_{u(k)}^{-1} \circ \overline{\alpha}_{u(k)} \circ \eta_{u(k)} \circ (\mathbf{Id} + \omega_{u(k)}) \circ \sigma_{u(k),u(k+1)}(s) \le s, \ \forall s \in \mathbb{R}_+ \quad (49)$$

for all cycles $U \in \mathcal{C}(G)$. The property (25) just chooses the linear function $s + \omega_i(s) = c_i s$ in (49). All the results in this paper remain valid even if the nonlinear gap functions $\omega_i \in \mathcal{K}_\infty$ are used instead of $(c_i - 1)s$. The formula for λ_i is omitted due to the space limitation.

We can relax $\omega_i \in \mathcal{K}_\infty$ further and replace *(H1)* by another assumption as far as the global asymptotic stability of the equilibrium $x = 0$ for $r(t) \equiv 0$ is concerned. For instance, if

$$\left\{ \lim_{s\to\infty} \alpha_i(s) = \infty \vee \lim_{s\to\infty} \alpha_i(s) \ge \lim_{s\to\infty} \sigma_{i,j}(s), \ j = 1, 2, ..., n \right\}, \quad i = 1, 2, ..., n \quad (50)$$



(a) Weighted graph G.

(b) Computing a weight $F_{4,1}$ of a fictitious arc.

Fig. 3. Cycle network.

holds, we do not need *(H1)* and we can replace $c_i > 1$, $\leq$ and $s \in \mathbb{R}_+$ in (25) by $c_i = 1$, $<$ and $s \in (0, \infty)$, respectively. A technique in [17] can be used to replace τ_i by $\tau_i(s)$ for making the operation of η's well-defined in computing $F_{i,j} \in \mathcal{K}$. Note that the use of $c_i = 1$, $<$ and $s \in (0, \infty)$ in (25) cannot guarantee the global asymptotic stability of $x = 0$ unless we assume (50) or another appropriate property [1]. When the disturbance r is allowed to affect the network Σ, the assumption *(H1)* cannot be removed as long as we use (14). Recall that we use Assumption 1 instead of assuming a given function f in (4). We make no assumption on the functions $\hat{\kappa}_i \in \mathcal{K} \cup \{0\}$ describing how severely r disturbs the worst network in the set of Assumption 1. When $\lim_{s\to\infty} \alpha_j(s) < \infty$ and $\lim_{s\to\infty} \sigma_{i,j}(s) = \infty$ hold, we need $\lim_{s\to\infty} \lambda_j(s) = \infty$ for establishing the global asymptotic stability. In this case, since the term $\lambda_j(V_j(x_j))\kappa_j(|r|)$ cannot be bounded by any function of $|r|$, we do not have the iISS property of the overall network for that function V due to the zero-output dissipativity an iISS system must possess [2].

VII. Concluding Remarks

This paper has presented a solution to the stability analysis problem for general networks of iISS systems. Basically, the philosophy behind the pursuit has a lot in common with the work on ISS networks that has been rapidly developed in the past several years [7], [20], [22], [8]. However, going beyond ISS requires a substantial departure from their techniques. The ISS gain functions are not guaranteed to be defined for iISS subsystems [2]. The maximization in aggregating individual Lyapunov functions of subsystems never yield a Lyapunov function of the network when subsystems are not ISS [13]. The key idea of this paper is to compute a Lyapunov function of the entire network in the form of summation of individual Lyapunov functions of subsystems. This paper has developed techniques to make a breakthrough in constructing such a Lyapunov function.

Providing a less conservative criterion by getting rid of the proportional partitioning of $\hat{\alpha}_i$ and $\hat{\sigma}_{i,j}$ as in (16) or bypassing such explicit partitioning is a topic of future study.

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