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State estimation of spatially distributed processes using mobile sensing agents

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Abstract— **This work is intended as an attempt to develop a simple real-time guidance scheme for mobile sensors used to enhance estimation of a spatially distributed process described by a partial differential equation. Using Lyapunov stability arguments, a stable control law is provided for each of the mobile agents while taking account of the dynamics of sensor movements, collision avoidance conditions and various modes of inter-agent connectivity. Numerical simulations for a 1D diffusion equation with three sensor agents are included to demonstrate the effectiveness of such a mobile sensor network in improving the system performance.**

Index Terms— **Spatial processes; PDEs; state estimation; mobile sensors; collision avoidance; limited communication.**

I. Introduction

A fundamental problem underlying state estimation of spatially distributed process described by partial differential equations (i.e., distributed parameter systems, or DPSs) is the selection of sensor locations [1]. It seems clear that measurements at certain points in the spatial domain of the system may yield more information about the system than those at other points and, therefore, the accuracy of the state estimate depends on the number and locations of the sensors. In modern measurement systems, sensors can be located on various platforms and these platforms can be highly dynamic in motion. This results from recent advances in hardware, sensor and networking technologies which are enabling large-scale deployment of superior data acquisition systems with adjustable resolutions, called *sensor networks* [2], [3].

Endowing nodes in a sensor network with mobility drastically expands the spectrum of the network's capabilities. Naturally, mobility implies an additional layer of complexity [3]. For example, if communication connectivity is to be maintained, we must ensure that each node remains within the range of at least some other nodes. We must also take into account that mobility consumes a considerable amount of energy, which amplifies the need for various forms of power control. However, the complexity of the resulting sensor management problem is compensated by a number of benefits. Specifically, sensors are not assigned to fixed spatial positions, but are capable of tracking points which provide at a given time moment best information about the observed spatiotemporal process. In order to take advantage of these possibilities, sensors must be deployed and then guided so as to maximize the information extracted from the mission space while maintaining acceptable levels of communication and energy consumption.

The importance of mobile sensor trajectory design has already been recognized in numerous related application domains (for reviews, see papers [4], [5] and comprehensive monographs [6], [7]). They were widely investigated for the akin problem of parameter estimation, where various scalar measures of performance based on the Fisher information matrix (FIM) associated with the parameters to be identified were maximized. In the seminal article [8], the D-optimality criterion is considered and an optimal time-dependent measure is sought, rather than the trajectories themselves. On the other hand, Uciński [6], [9], [10], apart from generalizations, develops some computational algorithms based on the FIM. He reduces the problem to a state-constrained optimal-control one for which solutions are obtained via the method of successive linearizations, which is capable of handling various constraints imposed on sensor motions. In turn, the work [11] was intended as an attempt to properly formulate and solve the time-optimal problem for moving sensors which observe the state of a DPS so as to estimate some of its parameters. In [12], a similar technique was presented so as to make the Hessian of the parameter estimation cost well conditioned subject an additional constraint imposed on the achievable D-efficiency of the solutions. In [13] this approach was extended to trajectory design for state estimation.

As for state estimation of stochastic spatiotemporal systems from noisy observations, a common and powerful tool is the Kalman-Bucy filter [1], [14]. In this setting, the sensor location problem was most often formulated as minimization of the trace of the estimate error covariance matrix (its mean and/or terminal values) subject to a constraint in the form of the corresponding Riccati equation. In consequence, optimal control techniques could be employed to produce optimal sensor locations. Thus, using Athans' matrix minimum principle, Nakano and Sagara derived necessary conditions in the form of a TPBVP for the optimal sensor velocities [15], [16]. To avoid attendant computational difficulties, they developed a suboptimal solution based on an upper bound to the estmation error covariance matrix. In turn, Carotenuto *et al.* considered a general spatiotemporal process described by its mean and covariance kernel [17]. They minimized the energy of the sensor movements and the mean square estimation error at the terminal time with respect to the control input to the sensors and their initial positions, as well as to the kernel

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of a linear operator which was supposed to estimate the process. Numerical solutions were then sought using a quasi-Newton algorithm. Finally, a slightly different approach was proposed by Khapalov [18] whose attention was focused on minimax filtering for a general linear parabolic equation. However, although he provided necessary conditions for optimal control of the sensors, no reference was passed to computational procedures. It should be emphasized that all the above solutions are primarily off-line.

The interest in mobile observations for state estimation in DPSs seems to disappear in the mid 1990s. A renewed interest in this problem has been motivated by recent advances in hardware, sensor and networking technologies which enable large-scale deployment of superior sensor networks with adjustable resolutions. Consequently, the works [19]–[24] focused on on-line optimal guidance of sensor network nodes, which demonstrate that inclusion of a DPS model into the optimization setting can substantially improve the quality of the information collected by the network. Similar work in [25] examined the problem of incorporating the sensor trajectory dynamics into the state estimation problem and posed the requisite problem as an optimal control problem.

In spite of many efforts, the existing solutions still suffer from high complexity and only occasionally attempt to take account of the dynamics of sensor movements, collision avoidance conditions and various modes of inter-agent connectivity. This was our motivation to develop here a simple real-time guidance scheme for mobile sensors used to enhance estimation of a linear DPS. Using Lyapunov stability arguments, a stable control law is provided for each of the mobile agents while taking account of all the aforementioned factors. To demonstrate the effectiveness of such a mobile sensor network in improving the system performance, numerical simulations for a 1D diffusion equation are included.

II. Problem formulation

Consider a linear infinite-dimensional dynamical system defined on a simply-connected spatial domain $\Omega \subset \mathbb{R}^m$ ($m \le 3$), which can be modeled by the following abstract differential equation with the output equation parameterized by the sensor positions [26]–[28]:

$$\begin{cases} \dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \quad x(0) = x_0 \in \mathcal{X}, \\ y(t;\theta) = \mathcal{C}(\theta)x(t), \end{cases} \tag{1}$$

where $x(t)$ is in a state space $\mathcal{X}$, $u(t)$ is in an input signal space $\mathcal{U}$, and $y(t)$ is in an output space $\mathcal{Y}$ ($\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are Hilbert spaces). Here t is time, $\mathcal{A} : \mathcal{X} \to \mathcal{X}$ is a possibly unbounded state operator, $\mathcal{B} : \mathcal{U} \to \mathcal{X}$ is an input opetator, and $\mathcal{C} : \mathcal{X} \to \mathcal{Y} = \mathbb{R}^n$ is an output operator.

We assume that the state $x(t)$ is observed by n mobile sensors whose configurations are $\theta_i(t)$, $i = 1, \dots, n$. In other words, setting $\theta(t) = (\theta_1(t), \dots, \theta_n(t))$, in (1) we have

$$\mathcal{C}(\theta(t))x(t) = \begin{bmatrix} \mathcal{C}_1(\theta_1(t))x(t) \\ \vdots \\ \mathcal{C}_n(\theta_n(t))x(t) \end{bmatrix}. \tag{2}$$

The θ_i's are understood here simply as, e.g., the centers of gravity of the mobile robots carrying the measurement equipment, but this description can be easily extended to cover, e.g., the robot heading angles. The operator $\mathcal{C}$ describes how the sensing devices operate on and interact with the spatial process. We consider two main types of sensing [29]:

Case 1. *Internal zone sensors:* At each instant t, observations represent a spatial average of $x(t)$ over some sensing region:

$$\mathcal{C}_i(\theta_i(t))x(t) = \int_\Omega c_i(\xi, \theta_i(t))x(t,\xi)\,\mathrm{d}\xi, \tag{3}$$

where $c_i(\,\cdot\,, \theta_i(t)) \in L^2(\Omega)$ with support $S_i(t)$ in Ω, $i = 1, \dots, n$. We assume that $S_i(t) \cap S_j(t) = \emptyset$ whenever $i \ne j$.

Case 2. *Internal pointwise sensors:* The supports $S_i(t)$ reduce to points:

$$\mathcal{C}_i(\theta_i(t))x(t) = \int_\Omega \delta(\xi - \theta_i(t))x(t,\xi)\,\mathrm{d}\xi, \tag{4}$$

where $\delta(\,\cdot\,)$ is the Dirac mass concentrated at the origin.

Representative systems that can be modeled by the above evolution equation are the advection-diffusion PDEs in one and two spatial dimensions. For the 1D diffusion equation on the interval $\Omega = [0, \ell]$ with n mobile pointwise sensors at locations $\theta_i(t) = \xi_{si}(t) \in \Omega \subset \mathbb{R}^1$

$$\begin{gathered} x_t(t,\xi) = a x_{\xi\xi}(t,\xi) + b(\xi)w(t), \\ x(t,0) = 0 = x(t,\ell), \quad x(0,\xi) = x_0(\xi), \end{gathered} \tag{5}$$

a being a diffusion coefficient and $b \in L^2(\Omega)$, the i^{th} output operator takes the specific form of a spatial delta function

$$\mathcal{C}_i(\theta_i(t))x(t) = x(t, \theta_i(t)) = \int_0^\ell \delta(\xi - \theta_i(t))x(t,\xi)\,\mathrm{d}\xi, \tag{6}$$

for $i = 1, \dots, n$. The choice of a delta function as the sensor model is commonly made and in the problem under consideration it provides an analytical expression for the sensor guidance. It should be noted that since the sensing devices are to be moving throughout the spatial domain, the vector of sensor locations θ is time varying.

Similarly, for the 2D diffusion PDE on the rectangle $\Omega = [0, L_\xi] \times [0, L_\zeta]$ with n mobile sensors at locations $\theta_i(t) = (\xi_{si}(t), \zeta_{si}(t)) \in \Omega \subset \mathbb{R}^2$

$$\begin{gathered} x_t(t,\xi,\zeta) = a\left(x_{\xi\xi}(t,\xi,\zeta) + x_{\zeta\zeta}(t,\xi,\zeta)\right) + b(\xi,\zeta)u(t), \\ x(t,\xi,0) = x(t,\xi,L_\zeta) = 0, \; x(t,0,\zeta) = x(t,L_\xi,\zeta) = 0, \\ x(0,\xi,\zeta) = x_0(\xi,\zeta), \end{gathered} \tag{7}$$

the i^{th} output operator is represented by the 2D delta function and it takes the form

$$\begin{aligned} &\mathcal{C}_i(\theta(t))x(t) = x(t, \xi_{si}(t), \zeta_{si}(t)) \\ &\quad = \int_0^{L_\xi}\int_0^{L_\zeta} \delta(\xi - \xi_{si}(t))\delta(\zeta - \zeta_{si})x(t,\xi,\zeta)\,\mathrm{d}\zeta\,\mathrm{d}\xi \end{aligned} \tag{8}$$

for $i = 1, \dots, n$.

It is assumed that all sensing devices have the same sensing characteristics, which then leads to a *homogeneous sensor network*. This means that, by abuse of notation, for the vector of sensor locations θ we have $\mathcal{C}_i(\theta) = \mathcal{C}(\theta_i)$ and

$$y(t;\theta) = \begin{bmatrix} \mathcal{C}(\theta_1)x(t) \\ \vdots \\ \mathcal{C}(\theta_n)x(t) \end{bmatrix}.$$

The above can be interpreted as having one single output operator that represents the sensor model and it is evaluated at different spatial locations within the spatial domain Ω.

Problem statement: The problem under consideration is to propose an integrated state estimation and mobile sensor guidance scheme in order to estimate the distributed process state faster than with the use of an estimator with a static sensor network.

Associated with the above estimation problem are the following questions:

1) How to choose the estimator structure (Kalman or Luenberger observer)?
2) How to navigate the sensors within the spatial domain Ω via the sensor spatial relocation, that would help the estimation problem while ensuring that agent collision and agent clustering is avoided?
3) How to incorporate the agent dynamics directly into the estimation problem without having to propose a lower level control scheme for tracking of the agents with the reference trajectory derived by the estimation scheme?

III. State estimator with mobile sensing agents

The above three questions are answered using Lyapunov stability arguments for the associated state estimation error system. For simplicity and without restriction of generality, in what follows we assume that our spatiotemporal system is one-dimensional, i.e., $m = 1$.

A. Simplified spatially distributed filter architecture with mobile agents

To directly answer the first question is to use a Luenberger observer in which the filter gain attains a specific form of a "collocated" sensor, in the sense of $\mathcal{L}(\theta) = -\mathcal{C}^*(\theta)\Gamma$, where Γ is an $n \times n$ positive definite matrix and the asterisk denotes the adjoint operator.

The proposed state estimator takes the form

$$\begin{aligned} \dot{\widehat{x}}(t) &= (\mathcal{A} + \mathcal{L}(\theta)\mathcal{C}(\theta))\,\widehat{x}(t) - \mathcal{L}(\theta)y(t;\theta), \\ \widehat{x}(0) &= \widehat{x}_0 \neq x(0). \end{aligned} \tag{9}$$

To examine the stability and extract the guidance laws based on Lyapunov stability arguments, one considers the associated error dynamics $e(t) = x(t) - \widehat{x}(t)$, governed by the following evolution equation in $\mathcal{X}$

$$\dot{e}(t) = \mathcal{A}_{\text{cl}}(\theta)e(t), \quad e(0) \neq 0, \tag{10}$$

where $\mathcal{A}_{\text{cl}}(\theta) = \mathcal{A} - \mathcal{C}^*(\theta)\Gamma\mathcal{C}(\theta)$.

The output estimation error is then given by

$$\varepsilon(t) = \begin{bmatrix} \varepsilon_1(t) \\ \vdots \\ \varepsilon_n(t) \end{bmatrix} = \begin{bmatrix} \mathcal{C}(\theta_1)e(t) \\ \vdots \\ \mathcal{C}(\theta_n)e(t) \end{bmatrix}. \tag{11}$$

B. Dynamics of mobile agents

It is assumed that each sensing device is affixed on a mobile agent (a terrain vehicle or a mobile robot) with dynamics governed by

$$m_i\ddot{\theta}_i(t) + d_i\dot{\theta}_i(t) + k_i\theta_i(t) = f_i(t), \quad i = 1, \ldots, n, \tag{12}$$

where $f_i(t)$ are agent controls and m_i, d_i and k_i are given parameters. With the above expression for the vehicle dynamics, we can now answer the last two questions associated with the proposed estimation problem.

C. Lyapunov-based guidance scheme

The choice of the Lyapunov function will help with the derivation of the guidance scheme. However, additional modifications to the nominal function consisting of the estimation error norm and the vehicle energy (kinetic and potential) should be incorporated in order to account for collision avoidance and clusterization [6]. The Lyapunov function thus consists of three expressions

$$V(t) = V_{\text{error}}(t) + V_{\text{collision}}(t) + V_{\text{vehicle}}(t). \tag{13}$$

The first Lyapunov function is given by

$$V_{\text{error}}(t) = -\frac{1}{2}\langle \mathcal{A}_{\text{cl}}(\theta)e(t), e(t)\rangle \tag{14}$$

and describes the "energy" of the error system (10). This function is the same as the one used in [20] and essentially represents the negative of the derivative of the state estimation error norm along the trajectories of (10). The negative sign ensures the positive definiteness of $V_{\text{error}}(t)$. Such a choice, even in the absence of vehicle kinematics, ensures that the "control" variable $\theta(t)$ will appear in the time derivative of $V_{\text{error}}(t)$. The second Lyapunov function is given by

$$V_{\text{collision}}(t) = wP(t), \tag{15}$$

where $P(t) = \frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} P_{ij}(\theta_i, \theta_j)$, the individual penalty terms are

$$P_{ij}(\theta_i, \theta_j) = \left\{\max\left(0, \frac{1}{(\theta_i - \theta_j)^2 - r^2}\right)\right\}^2 \tag{16}$$

and r denotes the minimum safe distance between the agents. The parameter $w > 0$ is a user-defined weight. It provides a penalty term for collision and clusterization avoidance. It is a simplified version of the one used in [30].

The third component Lyapunov function describes the vehicle energy which is simply taken as the sum of kinetic and potential energies of the vehicles

$$V_{\text{vehicle}}(t) = \frac{1}{2}\sum_{i=1}^{n}\Big(m_i\dot{\theta}_i^2(t) + k_i\theta_i^2(t)\Big). \tag{17}$$

The time derivative of the Lyapunov function is found via the time derivatives of the three component Lyapunov functions. The following technical lemmas provide the details of the derivatives of these Lyapunov functions and allow for the stability analysis and guidance of the integrated estimation system and guidance policies. Details of their proofs can be found in Appendices A, B and C, respectively.

Lemma 1: The time derivative of $V_{\text{error}}(t)$ along the trajectory of the error system (10) is

$$\dot{V}_{\text{error}}(t) = -\|A_{\text{cl}}(\theta)e(t)\|^2 + \varepsilon^T(t)\Gamma\frac{\partial\varepsilon(t)}{\partial\theta}\dot{\theta}(t). \tag{18}$$

It should be noted that this choice of the Lyapunov function yields an explicit expression of the gradient of the output estimation error with respect to the sensor location and of the sensor velocities.

Lemma 2: The time derivative of $V_{\text{collision}}(t)$ is

$$\dot{V}_{\text{collision}}(t) = w\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\partial P_{ij}}{\partial\theta_i}\,\dot{\theta}_i(t). \tag{19}$$

Lemma 3: The time derivative of $V_{\text{vehicle}}(t)$ along the vehicle trajectories (12) is

$$\dot{V}_{\text{vehicle}}(t) = \sum_{i=1}^{n}\left(\dot{\theta}_i(t)f_i(t) - d_i\dot{\theta}_i^2(t)\right) \tag{20}$$

Since the sensing devices are assumed identical, one must make the explicit assumption of homogeneous network.

Assumption 1 (Homogeneous network): All sensing devices have the same characteristics, in the sense that for the vector of sensor locations $\theta\in\Omega^n$ we have $\mathcal{C}_i(\theta)=\mathcal{C}(\theta_i)$ for $i=1,\ldots,n$.

Equipped with the above results, one may be able to provide for a guidance policy based on Lyapunov stability arguments. This leads to the following result which provides the vehicle guidance with collision-free conditions. The next result extends to the case of communication constraints.

Theorem 1: Consider the infinite dimensional system (1) where the sensors satisfy the homogeneous network condition in Assumption 1. The guidance policy

$$\begin{aligned} f_i(t) = &-\frac{\partial\varepsilon_i(t)}{\partial\theta_i}\sum_{j=1}^{n}\gamma_{ij}\varepsilon_j(t) - l_i\dot{\theta}_i(t)\\ &- w\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\partial P_{ij}}{\partial\theta_i}, \quad i=1,\ldots,n, \end{aligned} \tag{21}$$

for the mobile sensors with dynamics given by (12), with $\gamma_{ij}>0$ user-defined communication gains, with $w>0$ user-defined collision avoidance gain, and with l_i user-defined velocity feedback gains such that $d_i+l_i>0$, results in a stable system.

Proof: Consider the Lyapunov function (13). Using Lemmas 1, 2 and 3 we have

$$\begin{aligned} \dot{V}(t) = &-\|A_{\text{cl}}(\theta)e(t)\|^2 + \varepsilon^T(t)\Gamma\frac{\partial\varepsilon(t)}{\partial\theta}\dot{\theta}(t)\\ &+ w\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\partial P_{ij}}{\partial\theta_i}\,\dot{\theta}_i(t)\\ &+\sum_{i=1}^{n}\left(\dot{\theta}_i(t)f_i(t) - d_i\dot{\theta}_i^2(t)\right). \end{aligned} \tag{22}$$

The Jacobian matrix $\frac{\partial\varepsilon(t)}{\partial\theta}$ is given by

$$\frac{\partial\varepsilon(t)}{\partial\theta} = \begin{bmatrix} \frac{\partial\varepsilon_1}{\partial\theta_1} & \frac{\partial\varepsilon_1}{\partial\theta_2} & \cdots & \frac{\partial\varepsilon_1}{\partial\theta_n}\\ \frac{\partial\varepsilon_2}{\partial\theta_1} & \frac{\partial\varepsilon_2}{\partial\theta_2} & \cdots & \frac{\partial\varepsilon_2}{\partial\theta_n}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial\varepsilon_n}{\partial\theta_1} & \frac{\partial\varepsilon_n}{\partial\theta_2} & \cdots & \frac{\partial\varepsilon_n}{\partial\theta_n} \end{bmatrix}.$$

However, since each agent uses its own sensor to estimate its own output estimation error, the above Jacobian matrix is diagonal:

$$\frac{\partial\varepsilon(t)}{\partial\theta} = \text{Diag}\left[\frac{\partial\varepsilon_1}{\partial\theta_1},\ldots,\frac{\partial\varepsilon_n}{\partial\theta_n}\right].$$

and therefore the second term in (22) simplifies to

$$\varepsilon^T(t)\Gamma(\theta)\frac{\partial\varepsilon(t)}{\partial\theta}\dot{\theta}(t) = \varepsilon^T\Gamma\begin{bmatrix}\dot{\theta}_1\frac{\partial\varepsilon_1}{\partial\theta_1}\\ \vdots\\ \dot{\theta}_n\frac{\partial\varepsilon_n}{\partial\theta_n}\end{bmatrix}.$$

The last term in (22) can be compactly written as

$$\sum_{i=1}^{n}\left(\dot{\theta}_i(t)f_i(t) - d_i\dot{\theta}_i^2(t)\right) = f^T(t)\dot{\theta}(t) - \dot{\theta}^T(t)D\dot{\theta}(t),$$

where

$$f(t) = \begin{bmatrix}f_1(t)\\ \vdots\\ f_n(t)\end{bmatrix}, \quad \dot{\theta}(t) = \begin{bmatrix}\dot{\theta}_1(t)\\ \vdots\\ \dot{\theta}_n(t)\end{bmatrix},$$
$$D = \text{Diag}\left[d_1,\ldots,d_n\right].$$

The third term in (22) is written as

$$w\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\partial P_{ij}}{\partial\theta_i}\,\dot{\theta}_i = w\mathbf{1}^T\left(\frac{\partial P}{\partial\theta}\right)\dot{\theta},$$

where $\mathbf{1}$ is the n-dimensional vector of 1's. The above allow one to rewrite (22) as

$$\begin{aligned} \dot{V}(t) &= -\|A_{\text{cl}}(\theta)e(t)\|^2 + \varepsilon^T(t)\Gamma\frac{\partial\varepsilon(t)}{\partial\theta}\dot{\theta}\\ &\quad + w\mathbf{1}^T\left(\frac{\partial P}{\partial\theta}\right)\dot{\theta} + f^T(t)\dot{\theta}(t) - \dot{\theta}^T(t)D\dot{\theta}(t)\\ &= -\|A_{\text{cl}}(\theta)e(t)\|^2 + \Big(\varepsilon^T(t)\Gamma\frac{\partial\varepsilon(t)}{\partial\theta} + w\mathbf{1}^T\left(\frac{\partial P}{\partial\theta}\right)\\ &\quad + f^T(t) - \dot{\theta}^T(t)D\Big)\dot{\theta}(t) \end{aligned}$$

The choice $\varepsilon^T(t)\Gamma\frac{\partial\varepsilon(t)}{\partial\theta} + w\mathbf{1}^T\left(\frac{\partial P}{\partial\theta}\right) + f^T(t) = -\dot{\theta}^T(t)L$, where L is a diagonal $n\times n$ matrix such that $D+L$ is positive definite, yields

$$f(t) = -\left(\frac{\partial\varepsilon(t)}{\partial\theta}\right)^T\Gamma\varepsilon(t) - w\left(\frac{\partial P}{\partial\theta}\right)^T\mathbf{1} - L\dot{\theta}(t). \tag{23}$$

In terms of the individual agents, the guidance policy is

$$f_i(t) = -\frac{\partial\varepsilon_i(t)}{\partial\theta_i}\left(\sum_{j=1}^{n}\gamma_{ij}\varepsilon_j(t)\right) - l_i\dot{\theta}_i(t) - w\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{\partial P_{ij}}{\partial\theta_i}$$

for $i = 1, \ldots, n$. The Lyapunov function then has the derivative given by

$$\dot{V}(t) = -\|A_{\text{cl}}(\theta)e(t)\|^2 - \dot{\theta}^T(t)(L+D)\dot{\theta}(t) \leq 0.$$

■

Remark 1 (decoupled guidance): When the collision-avoidance modification is not implemented, i.e., when $w = 0$, we have

$$f_i(t) = -\frac{\partial \varepsilon_i(t)}{\partial \theta_i} \sum_{j=1}^{n} \gamma_{ij}\varepsilon_j(t) - l_i\dot{\theta}_i(t), \quad i = 1, \ldots, n.$$

One may choose Γ to be a diagonal matrix and in this case the guidance

$$f_i(t) = -\gamma_{ii}\frac{\partial \varepsilon_i(t)}{\partial \theta_i}\varepsilon_i(t) - l_i\dot{\theta}_i(t), \quad i = 1, \ldots, n$$

becomes decoupled, with each agent being aware of only its own output estimation error, i.e., we are faced with severe inter-agent disconnectivity.

Remark 2 (inter-agent communication): A limited inter-agent communication can be represented by the dependence of the gain matrix Γ on the vehicle positions θ. Some of its elements may be zero designating loss of communication between two agents. Note, however, that $\Gamma(\theta)$ must then be positive definite, which restricts possible parameterizations.

Logically, the gain γ_{ij} must depend on the distance between Agent i and Agent j. For a longer distance, this gain may be close to zero (or just be zero). This may warrant a study on its own, but for now, we propose the gain matrix of the form

$$\Gamma(\theta) = \Big[\gamma_{ij}(\theta_i, \theta_j)\Big], \tag{24}$$

where $\gamma_{ij}(\theta_i, \theta_j) = \alpha\exp(-\beta|\theta_i - \theta_j|)$. Here α and β are fixed coefficients. What is more, β steers the strength of the coupling between the two agents. (The larger β, the looser the coupling; for large β the gains γ_{ij} will be practically zero even for agents which are close to each other.)

The above form (drawn from geostatistics [31], where it corresponds to a typical covariance kernel of a random field) guarantees that $\Gamma(\theta)$ will be positive definite whenever all agents have different locations.

Introduction of (24) implies the appearance of an additional term when determining $\dot{V}_{\text{error}}(t)$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle e(t), A_{\text{cl}}(\theta)e(t)\rangle = \|A_{\text{cl}}(\theta)e(t)\|^2 - \langle\frac{\partial \varepsilon(t)}{\partial \theta}\dot{\theta}, \Gamma\varepsilon(t)\rangle - \underbrace{\langle\varepsilon(t), \frac{\mathrm{d}\Gamma(\theta)}{\mathrm{d}t}\varepsilon(t)\rangle}_{\text{new term}}.$$

But we have

$$\langle\varepsilon(t), \frac{\mathrm{d}\Gamma(\theta)}{\mathrm{d}t}\varepsilon(t)\rangle = \sum_{i=1}^{n}\langle\varepsilon(t), \frac{\partial\Gamma(\theta)}{\partial\theta_i}\varepsilon(t)\rangle\dot{\theta}_i = \eta^T(t,\theta)\dot{\theta}(t),$$

where

$$\eta(t,\theta) = \begin{bmatrix} \varepsilon^T(t)\dfrac{\partial\Gamma(\theta)}{\partial\theta_1}\varepsilon(t) \\ \vdots \\ \varepsilon^T(t)\dfrac{\partial\Gamma(\theta)}{\partial\theta_n}\varepsilon(t) \end{bmatrix}$$

and

$$\eta_i(t,\theta) = \varepsilon^T(t)\frac{\partial\Gamma(\theta)}{\partial\theta_i}\varepsilon(t) = 2\alpha\beta\varepsilon_i(t)\sum_{\substack{j=1\\j\neq i}}^{n}\operatorname{sgn}(\theta_j - \theta_i)\exp(-\beta|\theta_i - \theta_j|)\varepsilon_j(t).$$

Accordingly, the agent guidance law (23) has to be modified as follows:

$$f_i(t) = -\frac{\partial\varepsilon_i(t)}{\partial\theta_i}\sum_{j=1}^{n}\gamma_{ij}(\theta_i,\theta_j)\varepsilon_j(t) - l_i\dot{\theta}_i(t) - \eta_i(t,\theta) - w\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\partial P_{ij}}{\partial\theta_i}. \tag{25}$$

IV. Example

We consider the 1D PDE in (5), modified to include a "moving" disturbance

$$x_t(t,\xi) = ax_{\xi\xi}(t,\xi) + d(t,\xi)v(t),$$
$$x(t,0) = 0 = x(t,\ell), \quad x(0,\xi) = x_0(\xi),$$

where $v(t)$ is the "intensity" and $d(t,\xi)$ is the spatial distribution of the moving disturbance or source. Using the sensor model (6) we have that

$$\frac{\partial\varepsilon_i(t)}{\partial\theta_i} = \frac{\partial}{\partial\theta_i}\int_0^\ell \delta(\xi - \theta_i)e(t,\xi)\,\mathrm{d}\xi = \frac{\partial}{\partial\theta_i}e(t,\theta_i) = e_\xi(t,\theta_i).$$

This essentially requires the spatial gradient of the output estimation error at the current sensor location $\theta_i(t)$. This explicit form of the spatial gradient, owing to the sensor model assumed in (6), permits one to realize the proposed guidance policy (21). For this case, the guidance policy is

$$f_i(t) = -e_\xi(t,\theta_i)\sum_{j=1}^{n}\gamma_{ij}e(t,\theta_j)) - l_i\dot{\theta}_i(t) - w\sum_{\substack{j=1\\j\neq i}}^{n}\frac{\partial P_{ij}}{\partial\theta_i}$$

for $i = 1, \ldots, n$, where the last term is given by (B.1).

As for a specific implementation, the spatial gradient was approximated using a finite divided central difference formula, which amounts to replacing a pointwise sensor by a two-pronged probe [22]. (In other words, each sensing vehicle has two pointwise sensors attached to it.)

The 1D PDE was simulated with 80 linear elements and the proposed filter was implemented with $n = 3$ mobile sensors. The spatial domain was $\Omega = (0,1)$, i.e., $\ell = 1$, and the diffusivity constant was set to $a = 5 \times 10^{-3}$. The initial sensor locations were taken as $\theta_1(0) = 0.25\ell$, $\theta_2(0) = 0.75\ell$ and $\theta_3(0) = 0.50\ell$ and the initial velocities were set to zero. The plant initial condition was set to $x(0,\xi) = \sin(\pi\xi/\ell)e^{-7\xi^2}$ and for the estimator was set to

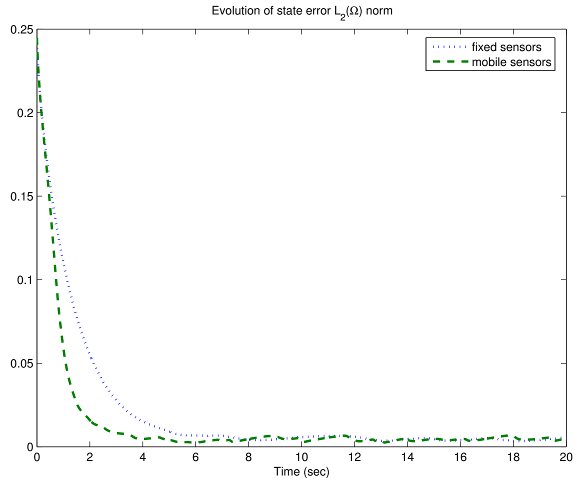

Fig. 1. Estimation of 1D diffusion: evolution of spatial L_2 norm of estimation error.

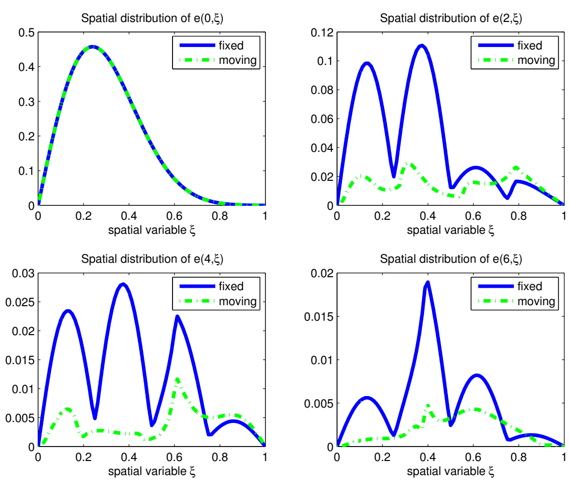

Fig. 2. Estimation of 1D diffusion: state estimation error vs spatial variable at different time instances.

$\widehat{x}(0, \xi) = 0$. For simplicity, the gain matrix Γ was taken as constant, thereby implementing an *all-to-all* connectivity

$$\Gamma = 10^{-4} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

The weight on the collision penalty was chosen as $w = 10^{-4}$ and the collision-free radius as $r = 0.01$. The vehicle parameters were chosen as $m_i = 1, d_i = \sqrt{2}$ and $k_i = 1$.

The system was simulated for 20 seconds. The moving source was taken as $d(t, \xi) = \delta(\xi - \xi_c(t))$, with the centroid of the source given by $\xi_c(t) = \ell(0.35 \cos(\pi t/20) + 0.5)$.

Figure 1 depicts the evolution of the state error norm for both fixed and mobile sensors, where it is observed that mobile sensors can lead to a faster convergence of $e(t, \xi)$ to zero. The same is observed in Figure 2 when the state error $e(t, \xi)$ is plotted versus the spatial variables for four arbitrarily selected time instances. Once again, when sensors are allowed to move, the convergence of $e(t, \xi)$ is faster. Finally, the trajectory of the three mobile sensors is depicted

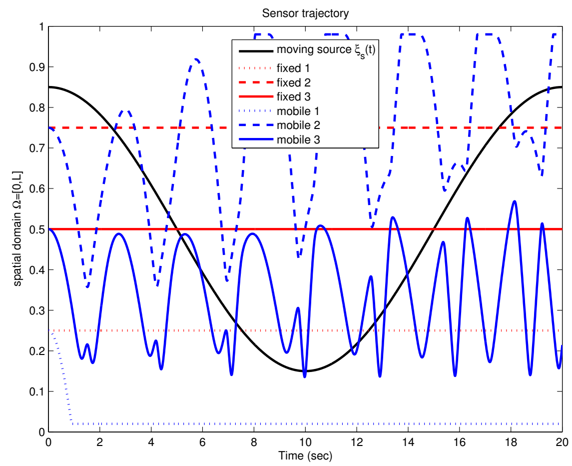

Fig. 3. Estimation of 1D diffusion: sensor trajectories.

in Figure 3. Observe that the mobile agents do not collide with one another.

V. Conclusions

The aim of this work was to propose a relatively simple guidance scheme for a network of (possibly) partially connected sensor-equipped vehicles to estimate a spatially distributed process described by a linear PDE. The novelty here is that the state observer structure was decided *a priori* for implementation simplicity and real-time realization, while taking into account dynamic models of sensing agent movements and collision avoidance conditions. Numerical results confirm the effectiveness of the proposed scheme.

Numerous important topics exceed the scope of this paper. They include the well-posedness of the observer equations, a thorough study of the influence of inter-agent communication or the numerical approximation to the spatial gradient, as well as extensions to more spatial dimensions or the case of measurement and process noise. This will appear in a forthcoming publication.

Appendix A

Proof of Lemma 1. The sought time derivative is

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle e(t), \mathcal{A}_{\text{cl}}(\theta)e(t)\rangle \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle e(t), \big(\mathcal{A} - \mathcal{C}^*(\theta)\Gamma\mathcal{C}(\theta)\big)e(t)\rangle \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\langle e(t), \mathcal{A}e(t)\rangle - \langle\underbrace{\mathcal{C}(\theta)e(t)}_{\varepsilon(t)}, \Gamma\underbrace{\mathcal{C}(\theta)e(t)}_{\varepsilon(t)}\rangle\Big\} \\
&= \langle\dot{e}(t), \mathcal{A}e(t)\rangle - \langle\dot{\varepsilon}(t), \Gamma\varepsilon(t)\rangle \\
&= \langle\dot{e}(t), \mathcal{A}e(t)\rangle - \langle\mathcal{C}(\theta)\dot{e}(t) + \frac{\partial\varepsilon(t)}{\partial\theta}\,\dot{\theta}, \Gamma\varepsilon(t)\rangle \\
&= \langle\dot{e}(t), \mathcal{A}_{\text{cl}}(\theta)e(t)\rangle - \langle\frac{\partial\varepsilon(t)}{\partial\theta}\,\dot{\theta}, \Gamma\varepsilon(t)\rangle \\
&= \|\mathcal{A}_{\text{cl}}(\theta)e(t)\|^2 - \varepsilon^T(t)\Gamma\frac{\partial\varepsilon(t)}{\partial\theta}\,\dot{\theta}.
\end{aligned} \tag{A.1}$$

Therefore, one has

$$\dot{V}_{\text{error}}(t) = -\|\mathcal{A}_{\text{cl}}(\theta)e(t)\|^2 + \varepsilon^T(t)\Gamma\frac{\partial\varepsilon(t)}{\partial\theta}\,\dot{\theta}.$$

For the 1D diffusion equation, the second term simplifies to

$$\begin{aligned}\frac{\partial\varepsilon_i}{\partial\theta_i} &= \frac{\partial}{\partial\theta_i}\int_0^\ell \delta(\xi-\theta_i)e(t,\xi)\,\mathrm{d}\xi \\ &= \frac{\partial}{\partial\theta_i}\,e(t,\theta_i) = \frac{\partial}{\partial\xi}\,e(t,\xi)\Big|_{\xi=\theta_i} = e_\xi(t,\theta_i).\end{aligned}$$

APPENDIX B

Proof of Lemma 2. The time derivative is

$$\frac{\partial P_{ij}}{\partial\theta_i} = \begin{cases}\dfrac{-4(\theta_i-\theta_j)}{\left((\theta_i-\theta_j)^2-r^2\right)^3} & \text{if } |\theta_i-\theta_j|>r,\\ \text{undefined} & \text{if } \theta_i=\theta_j,\\ 0 & \text{if } |\theta_i-\theta_j|<r.\end{cases} \tag{B.1}$$

Observing that

$$P_{ij} = P_{ji}, \quad \frac{\partial P_{ij}}{\partial\theta_i} = -\frac{\partial P_{ji}}{\partial\theta_i}, \tag{B.2}$$

then we have

$$\begin{aligned}\dot{V}_{\text{collision}}(t) &= \frac{1}{2}w\sum_{i=1}^n\sum_{\substack{j=1\\j\neq i}}^n\left(\frac{\partial P_{ij}}{\partial\theta_i}\,\dot{\theta}_i + \frac{\partial P_{ij}}{\partial\theta_j}\,\dot{\theta}_j\right)\\ &= \frac{1}{2}w\sum_{i=1}^n\sum_{\substack{j=1\\j\neq i}}^n\left(\frac{\partial P_{ij}}{\partial\theta_i}\,\dot{\theta}_i + \frac{\partial P_{ji}}{\partial\theta_j}\,\dot{\theta}_j\right)\\ &= w\sum_{i=1}^n\sum_{\substack{j=1\\j\neq i}}^n\frac{\partial P_{ij}}{\partial\theta_i}\,\dot{\theta}_i.\end{aligned} \tag{B.3}$$

APPENDIX C

Proof of Lemma 3. The time derivative is

$$\begin{aligned}\dot{V}_{\text{vehicle}}(t) &= \sum_{i=1}^n m_i\ddot{\theta}_i(t)\dot{\theta}_i(t) + k\dot{\theta}_i(t)\theta_i(t)\\ &= \sum_{i=1}^n\left(\dot{\theta}_i(t)f_i(t) - d_i\dot{\theta}_i^2(t)\right).\end{aligned}$$

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