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Offline NMPC for continuous-time systems using sum of squares

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Abstract— **An offline nonlinear model predictive control (NMPC) approach for continuous time nonlinear systems subject to input and state constraints is presented. The approach deals with nonlinear systems which can be represented by polynomial parameter-varying systems. Since the applicability of NMPC is often limited by the speed at which an optimization problem can be solved online, we propose an NMPC scheme with drastically reduced online computational burden. The basic idea involves the offline computation of nested invariant sets and associated feedback laws by solving a convex optimization problem subject to sum of squares (SOS) constraints via semidefinite programming (SDP). Online, a search algorithm is executed to determine the feedback law suitable for the current state. The resulting offline NMPC controller guarantees stability and constraint satisfaction. Its applicability and effectiveness is shown by means of simulation of an example system.**

I. INTRODUCTION

Nonlinear model predictive control (NMPC) has been a prominent subject of research in the last two decades. The standard NMPC scheme is as follows: at each sampling instant, an optimal control problem is solved online, based on the current measurement of the system states, to determine an optimal input trajectory. The first part of this trajectory is applied to the system until the next sampling instant when the procedure is repeated using an update of the state measurement. Several finite horizon NMPC schemes with guaranteed stability have been developed [1]–[3]. However, first, these approaches may lead to hard to solve, non-convex optimization problems and second, the solution is usually an open-loop trajectory and not a state-feedback. Thus, rather short sampling intervals are required to counteract disturbances and/or model plant mismatch. This may lead to computational issues when the optimization problem cannot be solved fast enough. To overcome these problems, an idea to perform a repeated online calculation of a state feedback has been developed for linear systems [4] which has been extended to Lur'e systems [5] and polynomial control systems [6]. In [7]–[9], the basic idea of [4] is modified in order to solve the optimization problem offline to reduce online computational effort. The method is based on the offline computation of nested invariant sets and corresponding explicit control laws that are stored in a lookup table. The online computation is then limited to finding the smallest possible invariant set which contains the state at the current sampling instant, and to determine the corresponding feedback matrix which defines the control input. In [5], such an offline NMPC scheme has been derived for a continuous-time setting. For polynomial control systems as treated in [6], an offline NMPC approach is described in [10], however in a discrete-time setting and resulting in a non-convex optimization problem.

Polynomial control systems have received increasing attention since the development of the SOS relaxation [11]. SOS techniques take advantage of the fact that SOS problems can be formulated as a semidefinite program which allows for a computationally efficient solution of the control problem. Semidefinite Programs (SDPs) can be solved efficiently by interior-point methods [12] with solvers such as SeDuMi [13] after preprocessing with a parser such as YALMIP [14]. For a recent overview on the SOS method and its applications in control, see [15].

A particular application are polynomial approaches using linear-like system representations which are employed (e.g. [16], [17]) because they allow to prescribe a specific form of Lyapunov functions and thus, to formulate a convex optimization problem which would not be possible with the direct system representation. In the presented work, we extend these approaches by using polynomial parameter-varying (PPV) systems.

PPV systems present an interesting extension of the well-known concept of linear parameter-varying (LPV) systems. LPV systems can result from modeling using Linear Differential Inclusion (LDI) techniques [18], or Takagi-Sugeno (T-S) fuzzy models [19]. The technique using a linear model consequence has been used extensively to develop stability criteria, as well as for controller synthesis and observer design (e.g. [20], [21]).

Recently, a new approach to represent general input-affine nonlinear systems was developed [22] that uses polynomial model consequences instead of linear model consequences, and then employs the SOS method. This approach leads to polynomial parameter-varying systems. Usually, for a given nonlinear system, LPV systems require more parameters to represent the system dynamics in comparison to PPV systems since the class of LPV models is embedded in the class of PPV systems.

Systems with parameter-dependency have been investigated thoroughly, especially in the context of robust control where the parameter is regarded as an uncertainty, e.g. [23]–[25]. In some applications it is assumed that a bound on the time derivative on the parameter is available [26] or that the parameters are time-invariant [27]. The approach in

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this paper however differs from these results, because the parameters are not *really* unknown but used to model non-polynomial nonlinearities. This makes the parameters state-dependent and thus time-variant. Although their variation rates are known, these are not used since they may contain non-polynomial nonlinearities and their incorporation would increase the computational effort significantly. Using PPV systems, the class of nonlinear systems to which the NMPC approach [6] can be applied is enlarged. Additionally, a computationally attractive offline formulation of the NMPC approach is developed using the idea of nested invariant sets. The remainder of the paper is organized as follows. Section II introduces the notation, some definitions and useful lemmas that are important for the subsequent discussion. In Section III, the system description and the problem formulation is presented. Section IV shows the main results with the derivation of a stabilizing feedback control law which is then extended to offline NMPC. A numerical example is given in Section V, and the paper concludes with a summary in Section VI.

II. Preliminaries and Notation

We call the set of all strictly positive numbers $\mathbb{R}_{++}$. The set of symmetric matrices is denoted by $\mathcal{S}^n$, symmetric matrices which are positive (semi-)definite belong to the set $\mathcal{S}^n_{++}$ ($\mathcal{S}^n_{+}$). Furthermore, symmetric block matrices $\left[\begin{smallmatrix} A & C \\ C^T & D \end{smallmatrix}\right]$ are abbreviated by $\left[\begin{smallmatrix} A & C \\ * & D \end{smallmatrix}\right]$.

The set of polynomial $n_1 \times n_2$-matrices is denoted by $\mathbb{R}^{n_1 \times n_2}[x]$. We call a symmetric polynomial matrix $P(x) \in \mathcal{S}^m[x] \subset \mathbb{R}^{m\times m}[x]$ a sum of squares (briefly $P(x) \in \Sigma^m[x]$), if there exists a polynomial matrix $Q(x) \in \mathbb{R}^{m\times m_1}[x]$ such that $P(x) = Q^T(x)Q(x)$.

Lemma 1: (Putinar's Positivstellensatz [28]) Let the domain $\mathcal{D}$, defined by

$$\mathcal{D} := \{x \in \mathbb{R}^n \mid h_j(x) \geq 0,\, h_j \in \mathbb{R}[x],\, j \in \{1, \ldots, m\}\},$$

be a compact subset of $\mathbb{R}^n$ and $P(x) \in \mathcal{S}^n[x]$. Then the following statements are equivalent:

(i) $P(x) \in \mathcal{S}^n_{++}$ for all $x \in \mathcal{D}$.
(ii) $P(x) = S_0(x) + \sum_{j=1}^m S_j(x)h_j(x)$ for some SOS matrices $S_j(x) \in \Sigma^n[x]$, $j = 0, 1, \ldots, m$.

(ii)⇒(i) is still true if the degree of $S_j(x)$ is restricted.

Lemma 2: (cf. [29, Theorem 4.10]) Consider the nonlinear control system $\dot{x} = f(x)+g(x)u$ with the state feedback $u = u(x)$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

$$k_1||x||_2^2 \leq V(x) \leq k_2||x||_2^2,$$
$$\frac{\partial V(x)}{\partial x}(f(x) + g(x)u(x)) \leq -k_3||x||_2^2$$

for all $x \in D$, with $k_1, k_2, k_3 \in \mathbb{R}_{++}$. Then, the controller $u(x)$ exponentially stabilizes the equilibrium $x = 0\,\forall\, x \in D$.

Proposition 1: For $\alpha \in \mathbb{R}_{++}$, $\mathcal{V}(\alpha) := \{x \in D \mid V(x) \leq \alpha\}$ is an invariant set with respect to the dynamics of the nonlinear control system with $u = u(x)$ if Lemma 2 holds and $\mathcal{V}(\alpha) \in D$, i.e. $x(t_0) \in \mathcal{V}(\alpha) \Rightarrow x(t) \in \mathcal{V}(\alpha)$ for all $t \geq t_0$.

III. Problem Formulation

Consider a general input-affine nonlinear system

$$\dot{x} = f(x) + g(x)u, \tag{1}$$

where the state x evolves on a compact domain $\mathcal{D} \subset \mathbb{R}^n$ and the input is denoted by $u \in \mathbb{R}^{n_u}$. The functions $f : \mathcal{D} \to \mathbb{R}^n$ and $g : \mathcal{D} \to \mathbb{R}^{n\times n_u}$ are nonlinear, continuous and satisfy $f(0) = 0$. Nonlinear system (1) can be exactly represented [30] by a parameter-varying polynomial system of the form

$$\dot{x} = A(x, \theta(x))Z(x) + B(x, \theta(x))u \tag{2a}$$

with a vector of polynomials $Z(x) \in \mathbb{R}^{n_Z}[x]$ ($n_Z \geq n$) and the state- and parameter-dependent system and input matrices $A(x, \theta) \in \mathbb{R}^{n\times n_Z}[x, \theta]$ and $B(x, \theta) \in \mathbb{R}^{n\times n_u}[x, \theta]$. $Z(x)$ satisfies $Z(0) = 0$ iff $x = 0$. The function $\theta : \mathbb{R}^n \to \Gamma \subset \mathbb{R}^{n_\theta}$ is bounded and evolves continuously over time. Since $\theta(x)$ is used to model non-polynomial nonlinearities, its value can be safely assumed to be known online. However, when formulating and solving SOS conditions, it is necessary to regard $\theta(x)$ as an unknown parameter and only use polynomial information about the domain Γ. In these instances, the parameter is denoted by θ, otherwise by $\theta(x)$. Using the parameters allows the application of SOS techniques to non-polynomial nonlinear systems. The bounded domain Γ is assumed to be a semi-algebraic set described by

$$\Gamma := \{\theta \in \mathbb{R}^{n_\theta} \,|p_k(\theta) \geq 0,\, k \in \mathcal{K}\}, \tag{2b}$$

where $p_k(\theta) \in \mathbb{R}[\theta]$. Furthermore, we define $M(x) \in \mathbb{R}^{n_Z\times n}[x]$ as $M(x) = \frac{\partial Z(x)}{\partial x}$, and $\mathcal{J} = \{j_1, \ldots, j_m\}$ ($m < n_u$) as the row indices of $B(x, \theta)$ whose corresponding rows are equal to zero, which then also defines $A_j(x, \theta)$ as the j-th row of $A(x, \theta)$ and $\tilde{x} = (x_{j_1}, \ldots, x_{j_m})$.

The control task is to stabilize the origin of system (1) such that a set of combined state and input constraints are satisfied while minimizing the following infinite horizon cost functional with weighting matrices $Q \in \mathcal{S}^{n_Z}_{++}$ and $R \in \mathcal{S}^{n_u}_{++}$:

$$J(x(\cdot), u(\cdot)) = \int_{t_0}^{\infty} Z^T(x(\tau))QZ(x(\tau)) + u^T(\tau)Ru(\tau)\mathrm{d}\tau. \tag{3}$$

The constraints are described by the sets $\mathcal{C}_i$ defined as

$$\mathcal{C}_i = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n+n_u} \middle| |c_i(x)Z(x) + d_i(x)u| \leq 1 \right\}, \tag{4}$$

where $c_i(x) \in \mathbb{R}^{1\times n_Z}[x]$ and $d_i(x) \in \mathbb{R}^{1\times n_u}[x]$ for $i \in \mathcal{I} = \{1, \ldots, r\}$. The overall set of all combined input and state constraints is then given by the intersection of all constraint sets, i.e. $\mathcal{C} = \cap_{i\in\mathcal{I}}\mathcal{C}_i$. We can define a set of pure state constraints which is given by $\mathcal{C}_s = \cap_{i\in\mathcal{I}_s}\mathcal{C}_i$ with $\mathcal{I}_s := \{i \in \mathcal{I} \mid d_i(x) = 0\}$. We assume that the compact set $\mathcal{D}$ is a semi-algebraic set containing the set of pure state constraints $\mathcal{C}_s$, i.e. $\mathcal{C}_s \subseteq \mathcal{D}$, described by $\mathcal{D} = \{x \in \mathbb{R}^n | h_\mathcal{D}(x) \geq 0\}$, with $h_\mathcal{D}(x) = 1 - Z^T(x)P_0^{-1}Z(x)$ and $P_0 \in \mathcal{S}^n_{++}$.

To satisfy the control task, we apply a parameter-dependent state feedback control law $u(x) = K(x, \theta(x))Z(x)$ which is obtained via the solution of a convex optimization problem.

IV. Main Results

In the following, we first present the main theoretical result, then regard a special case and lastly describe an offline NMPC algorithm for PPV systems.

A. Time-invariant feedback control

We exploit the idea of invariant sets for the derivation of a convex optimization problem for the computation of the stabilizing feedback controller which guarantees constraint satisfaction and minimizes an upper bound on the cost functional (3).

Theorem 1: Consider system (1) in representation (2a) subject to state and input constraints (4). For given ϵ and state $x(t_0)$ at time t_0, suppose the optimization problem

$$\begin{gathered}\underset{\substack{\alpha\in\mathbb{R}_{++},X(\tilde{x})\in\mathcal{S}^{n_Z}[\tilde{x}],Y(x,\theta)\in\mathbb{R}^{n_u\times n_Z}[x,\theta],\\ S_{h_{-2}}(x,\theta),S_{h_{-1}}(x,\theta)\in\Sigma^{n_Z}[x,\theta],\ S_{h_0}(x,\theta)\in\Sigma^{2n_Z+n_u}[x,\theta],\\ S_{h_i}(x,\theta)\in\Sigma^{n_Z+1}[x,\theta]\,\forall\, i\in\mathcal{I},\ S_{c_{i,k}}(x,\theta)\in\Sigma^{n_Z+1}[x,\theta]\,\forall\, k\in\mathcal{K},\, i\in\mathcal{I},\\ S_{p_k}(x,\theta)\in\Sigma^{2n_Z+n_u}[x,\theta]\,\forall\, k\in\mathcal{K},}}{\text{minimize}}\quad \alpha\end{gathered} \tag{5a}$$

subject to

$$X(\tilde{x})-\epsilon I_{n_Z}-S_{h_{-2}}(x)h_{\mathcal{D}}(x)\in\Sigma^{n_Z}[x], \tag{5b}$$

$$P_0-X(\tilde{x})-S_{h_{-1}}(x)h_{\mathcal{D}}(x)\in\Sigma^{n_Z}[x], \tag{5c}$$

$$\begin{bmatrix}1 & Z^T(x(t_0))\\ * & X(\tilde{x}(t_0))\end{bmatrix}\in\mathcal{S}_+^{n_Z+1}, \tag{5d}$$

$$\Phi(x,\theta)-\Upsilon(x,\theta) -S_{h_0}(x,\theta)h_{\mathcal{D}}(x)\in\Sigma^{2n_Z+n_u}[x,\theta], \tag{5e}$$

$$\Psi_i(x,\theta)-\Upsilon_{c,i}(x,\theta) -S_{h_i}(x,\theta)h_{\mathcal{D}}(x)\in\Sigma^{n_Z+1}[x,\theta]\ \forall i\in\mathcal{I} \tag{5f}$$

is feasible with the substitutions $\Phi(x,\theta)$, $\Psi_i(x,\theta)$, $\Upsilon(x,\theta)$ and $\Upsilon_{c,i}(x,\theta)$ given in (9) below. Then, with

$$K(x,\theta):=Y(x,\theta)X^{-1}(\tilde{x})\ \text{ and }\ P(\tilde{x}):=\alpha X^{-1}(\tilde{x}) \tag{6}$$

the following properties hold:

(a) The feedback law

$$u(t)=K(x(t),\theta(x(t)))Z(x(t)),\ t\geq t_0, \tag{7}$$

guarantees exponential stability of (2a) by means of the Lyapunov function $V(x)=Z^T(x)P(\tilde{x})Z(x)$. Further, the domain $\mathcal{V}(\alpha)$ is an invariant set.

(b) The solution of the optimization problem (5) minimizes the upper bound $V(x(t_0))$ on the cost functional (3) at time t_0.

(c) The satisfaction of the combined state and input constraints (4) is guaranteed for all $t\geq t_0$.

Proof: The proof is divided into three parts establishing the properties (a)-(c).

Part (a): The important steps are first applying Positivstellensatz (Lemma 1) to (5e) ensuring local positive definiteness on the domain $\mathcal{D}\times\Gamma$ (represented by $h_{\mathcal{D}}$ and $p_k(\theta)$) [1], and in a second step to exploit Schur complement of $\Phi(x,\theta)$ (cf. [6]). Then, we use the fact that if a matrix $\Lambda\in\mathcal{S}_+^n\Rightarrow\tilde{Z}^T\Lambda\tilde{Z}\geq 0$,

[1]Using the slack variables $S_{h_0}(x,\theta)$ and $S_{p_k}(x,\theta)$.

$$\Phi(x,\theta):=\begin{bmatrix}-\Delta(x,\theta)-\Delta^T(x,\theta)+\Omega(x,\theta) & * & *\\ Q^{\frac{1}{2}}X(\tilde{x}) & \alpha I_{n_Z} & *\\ R^{\frac{1}{2}}Y(x,\theta) & 0 & \alpha I_{n_u}\end{bmatrix} \tag{9a}$$

with $\Delta(x,\theta)=M(x)(A(x,\theta)X(\tilde{x})+B(x,\theta)Y(x,\theta))$, and $\Omega(x,\theta)=\sum_{j\in J}\frac{\partial X(\tilde{x})}{\partial x_j}(A_j(x,\theta)Z(x))$.

$$\Psi_i(x,\theta):=\begin{bmatrix}1 & c_i(x)X(\tilde{x})+d_i(x)Y(x,\theta)\\ * & X(\tilde{x})\end{bmatrix}, \tag{9b}$$

$$\Upsilon(x,\theta):=\sum_{k=1}^{n_\theta}p_k(\theta)S_{p_k}(x,\theta), \tag{9c}$$

$$\Upsilon_{c,i}(x,\theta):=\sum_{k=1}^{n_\theta}p_k(\theta)S_{c_{i,k}}(x,\theta). \tag{9d}$$

$$\tilde{\Phi}(x):=\begin{bmatrix}\tilde{\Phi}_{11}(x) & \dots & \tilde{\Phi}_{1n_\theta}(x)\\ \vdots & \ddots & \vdots\\ \tilde{\Phi}_{n_\theta 1}(x) & \dots & \tilde{\Phi}_{n_\theta n_\theta}(x)\end{bmatrix} \tag{10a}$$

where

$$\tilde{\Phi}_{kl}(x):=\begin{bmatrix}-\Delta_{kl}(x)-\Delta_{kl}^T(x)+\Omega_k(x) & X(\tilde{x})Q^{\frac{1}{2}} & Y_l^T(x)R^{\frac{1}{2}}\\ * & \alpha I_{n_Z} & 0\\ * & * & \alpha I_{n_u}\end{bmatrix}$$

with $\Delta_{kl}(x)=M(x)(A_k(x)X(\tilde{x})+B_k(x)Y_l(x))$, and $\Omega_k(x)=\sum_{j\in J}\frac{\partial X(\tilde{x})}{\partial x_j}(A_{k,j}(x)Z(x))$.

$$\tilde{\Psi}_{i,k}(x):=\begin{bmatrix}1 & c_i(x)X(\tilde{x})+d_i(x)Y_k(x)\\ * & X(\tilde{x})\end{bmatrix}. \tag{10b}$$

and apply the coordinate transformation $\tilde{Z}=\sqrt{\alpha^{-1}}PZ$ with the substitutions from (6). This implies

$$\dot{V}(x)\leq -Z^T(x)(Q+K^T(x)RK(x))Z(x)\,\forall x\in\mathcal{D}, \tag{8}$$

and thus, guarantees exponential stability.

Part (b): Integrating (8) from $\tau=t_0$ to $\tau\to\infty$ with the stabilizing feedback $u(x,\theta)=K(x,\theta)Z(x)$ yields

$$V(x(t_0))\geq\int_{t_0}^{\infty}Z^T(x(\tau))QZ(x(\tau))+u^T(\tau)Ru(\tau)\mathrm{d}\tau.$$

Inequality (5d) implies $\alpha\geq V(x(t_0))$. Thus, α gives an upper bound on the cost functional. Furthermore, by proposition 1, $\mathcal{V}(\alpha)$ is an invariant set.

Part (c) is omitted due to space limitations. The basic idea is to use condition (5f) to show that the computed invariant set lies inside the set of state and input constraints. The techniques used in the derivation are again Positivstellensatz and Schur complement (cf. Part (a)). ■

For more insight on this proof and the omitted parts, see [6] where a similar proof is presented.

Notice that the state-dependency of $X(\tilde{x})$ and thus of $P(\tilde{x})$ is restricted to the states in $\tilde{x}$ in order to prevent non-convex conditions.

B. Exploiting a special case

Similarly to LPV systems [18] and T-S fuzzy systems [19], PPV systems can sometimes be designed (e.g. following [22] or [30]) in such a way that the system matrices vary inside a convex polytope Ξ

$$\begin{bmatrix} A(x,\theta) & B(x,\theta) \end{bmatrix} \in \Xi,$$

which is defined by the convex hull of the n_θ local vertex matrices $[A_k(x), B_k(x)]$, $k = 1, \dots n_\theta$, i.e.

$$\Xi := \text{conv}\left\{ \begin{bmatrix} A_1(x) & B_1(x) \end{bmatrix}, \dots, \begin{bmatrix} A_{n_\theta}(x) & B_{n_\theta}(x) \end{bmatrix} \right\}.$$

Thus, the system matrices in (2) take the form

$$A(x,\theta) = \sum_{k=1}^{n_\theta} \theta_k A_k(x), \tag{11a}$$

$$B(x,\theta) = \sum_{k=1}^{n_\theta} \theta_k B_k(x), \tag{11b}$$

where $A_k(x) \in \mathbb{R}^{n \times n_Z}[x]$ and $B_k(x) \in \mathbb{R}^{n \times n_u}[x]$. In this case, the set (2b) is realized by the standard $(n_\theta - 1)$-simplex

$$\Gamma = \left\{ \theta \in \mathbb{R}^{n_\theta} \,\middle|\, \sum_{k=1}^{n_\theta} \theta_i = 1 \;,\; 0 \le \theta_k \le 1 \right\}. \tag{11c}$$

Finally, using

$$Y(x,\theta) := \sum_{k=1}^{n_\theta} \theta_k Y_k(x) \tag{11d}$$

with $Y_k(x) \in \mathbb{R}^{n_u \times n_Z}[x]$ ($k \in \mathcal{K} = \{1, ..., n_\theta\}$) the expressions (9a) and (9b) take the specific form[2]

$$\Phi(x,\theta) = (\theta \otimes I_{2n_Z+n_u})^T \tilde{\Phi}(x) (\theta \otimes I_{2n_Z+n_u}), \tag{12a}$$

$$\Psi(x,\theta) = \sum_{k=1}^{n_\theta} \theta_k \tilde{\Psi}_{i,k}(x), \tag{12b}$$

where $\tilde{\Phi}(x)$ and $\tilde{\Psi}_{k,i}(x)$ are given in (10). Naturally, it is possible to use these substitutions in the conditions (5e) and (5f) in this form but this still requires the use of slack variables which depend not only on x but also on θ. However, since (12a) and (12b) are homogeneous forms in θ of second respectively first order, there are possibilities to formulate computationally more attractive conditions being sufficient conditions to guarantee (5e) and (5f). In the following, we want to briefly mention some of the sufficient conditions known in the literature.

We start with the easier expression (12b) which appears in condition (5f). As (12b) is linear, and thus convex in θ which itself is constrained to lie in the convex polytope (11c), checking (5f) for all $\theta \in \Gamma$ can be reduced to checking the n_θ extreme points $\tilde{\Psi}_{i,k}(x)$ ($k \in \mathcal{K}$).

Exploiting the specific form (12a) to guarantee (5e) is a little bit more involved as (12a) is not necessarily convex in θ. This has led to various conditions differing in complexity on the one hand and loss of exactness on the other hand:

- Simple blockwise checking (cf. [31]):

  $$\tilde{\Phi}_{kl} + \tilde{\Phi}_{lk} - S_{h_{0,kl}}(x) h_{\mathcal{D}}(x) \in \Sigma^{2n_Z+n_u}[x]$$

  for all $k \in \mathcal{K}$ and all strictly positive $l \le k$.

- A series of relaxations using or based on the matrix version of Polya's theorem (see [32] for an overview) which can be extended from LMI to SOS conditions. As an example, we present [32, Theorem 3] (also in [21]) adapted to SOS conditions with the auxiliary matrix $\Pi(x) \in \Sigma^{n_\theta(2n_Z+n_u)}[x]$:

  $$\Phi_{kl} + \Phi_{lk} - \Pi_{kl} - \Pi_{lk} - S_{h_{0,kl}} h_{\mathcal{D}} \in \Sigma^{2n_Z+n_u}[x],$$

  for all $k \in \mathcal{K}$ and all strictly positive $l \le k$ (the block structure of $\tilde{\Phi}$ given in (12a) also applies to Π).

C. *Offline NMPC*

The computation of a feedback control law instead of an open-loop input trajectory in Theorem 1 allows to react to disturbances and/or model plant mismatch. However, by being forced to obey the parameterized feedback law (7) instead of any open-loop input trajectory, the input is strongly restrained. This may result in conservative control inputs.

To overcome this restriction, the results from Theorem 1 are employed in combination with an offline NMPC strategy to obtain the algorithm below. In the following, the index i describes the association of matrices, optimization variables, functions etc. with the i-th algorithm step. The user-selected number of nested invariant sets is denoted by S, with a higher number leading to better performance at the cost of a larger lookup table which needs to be stored.

Algorithm 1: (Offline NMPC)

Offline: Consider system (2a) subject to the state and input constraints (4). Given an initial feasible state vector x_1, generate a lookup table of α_i, $P_i(\tilde{x})$, $K_i(x,\theta)$ ($i = 1, \dots, S$) in the following way. Set $i := 1$.

1) Compute and store α_i, $P_i(\tilde{x})$, $K_i(x,\theta)$ using Theorem 1, and if $i > 1$ with the additional constraint

$$X_{i-1}(\tilde{x}) - X_i(\tilde{x}) \in \Sigma^{n_Z}[x]. \tag{13}$$

2) If $i < S$, choose a new state vector $x_{i+1} \in \text{int}(\mathcal{V}_i(\alpha_i))$, where $\text{int}(\mathcal{V}_i(\alpha_i))$ is the interior of the invariant set $\mathcal{V}_i(\alpha_i) = \{x \in \mathcal{D} | Z^T(x) P_i(\tilde{x}) Z(x) \le \alpha_i\}$. Then, set $i := i + 1$ and go back to step 1.

Online: Consider a state $x(t_0)$ at time t_0 which lies inside $\mathcal{V}_1(\alpha_1)$. Perform the following steps.

1) Maximize i_0 ($i_0 \in [1, S]$) such that

$$x(t_0) \in \mathcal{V}_{i_0}(\alpha_{i_0}).$$

Set $i := i_0$. If $i_0 = S$, go to step 4, else to step 3.
2) As soon as $Z(x(t))^T P_{i+1}(\tilde{x}(t)) Z(x(t)) \le \alpha_i$, set $i := i + 1$. If $i = S$, go to step 4, else continue with step 3.
3) Apply the control input $u(x) = K_i(x, \theta(x)) Z(x)$ and go back to step 2.
4) Apply the control input $u(x) = K_S(x, \theta(x)) Z(x)$ for all times.

Theorem 2: Consider the system (1) in the representation (2a) subject to the state and input constraints (4). For a state $x(t_0)$ at time t_0 satisfying $x(t_0) \in \mathcal{V}(\alpha_1)$, the offline NMPC Algorithm 1 exponentially stabilizes the system and guarantees constraint satisfaction.

[2] $\otimes$ denotes the Kronecker product.

Proof: From the additional constraint (13), we get

$$1 - Z^T(x)\frac{P_{i-1}(\tilde{x})}{\alpha_{i-1}}Z(x) \geq 1 - Z(x)^T\frac{P_i(\tilde{x})}{\alpha_i}Z(x)$$

which means that $\mathcal{V}_i(\alpha_i) \subset \mathcal{V}_{i-1}(\alpha_{i-1})$. This ensures that there exists a unique i_0 such that $x(t_0) \in \mathcal{V}_{i_0}(\alpha_{i_0})$.
Convergence: With the additional constraint, the closed-loop system becomes

$$\dot{x} = \begin{cases} (A + BK_i)Z & \text{if} \quad x(t) \in \mathcal{V}_i(\alpha_i) \\ & \phantom{\text{if}} \quad x(t) \notin \mathcal{V}_{i+1}(\alpha_{i+1}), \\ & \phantom{\text{if}} \quad i \neq S, \\ (A + BK_S)Z & \text{if} \quad x(t) \in \mathcal{V}_S(\alpha_S). \end{cases}$$

Since $\dot{V}_i < -k_i||x||_2^2$ ($k_i > 0$), the control law K_i will drive the state $x(t)$ satisfying

$$x(t) \in \mathcal{V}_i(\alpha_i),\, x(t) \notin \mathcal{V}_{i+1}(\alpha_{i+1}),\, i < S$$

into the next invariant set $\mathcal{V}_{i+1}(\alpha_{i+1})$ in finite time. By induction, the state converges towards $\mathcal{V}_S(\alpha_S)$ where it is driven to the origin by K_S.
Stability: Consider the two innermost sets where $\dot{V}_S < 0$ and $\dot{V}_{S-1} < 0$ respectively. If

$$\exists\, \delta_{S-1} > 0 \text{ such that } \min_{x \in \rho_S}(\delta_{S-1}V_{S-1} - V_S) > 0$$

is true, where $\rho_S = \{x \in \mathcal{D} | \alpha_S - Z^T(x)P_S(\tilde{x})Z(x) = 0\}$, a decreasing Lyapunov function can be constructed. Since

$$\exists\, \bar{\alpha}_{S-1} > 0 \text{ such that } \mathcal{V}_{S-1}(\bar{\alpha}_{S-1}) \subseteq \mathcal{V}_S(\alpha_S)$$

where $\mathcal{V}_{S-1}(\bar{\alpha}_{S-1}) = \{x \in \mathcal{D} | Z^T(x)P_{S-1}(\tilde{x})Z(x) \leq \bar{\alpha}_{S-1}\}$, it follows that $\delta_{S-1} > \frac{\alpha_S}{\bar{\alpha}_{S-1}}$. Recursively, a δ_i can be found for every transition which results in the following Lyapunov function

$$V(x) = \begin{cases} \delta_i Z^T(x)P_i(\tilde{x})Z(x) & \text{if} \quad x(t) \in \mathcal{V}(\alpha_i) \\ & \phantom{\text{if}} \quad x(t) \notin \mathcal{V}(\alpha_{i+1}), \\ & \phantom{\text{if}} \quad i \neq S, \\ Z(x)^T P_S(\tilde{x})Z(x) & \text{if} \quad x(t) \in \mathcal{V}(\alpha_S). \end{cases}$$

which is decreasing because all the V_i are decreasing.
Constraint satisfaction: When $x(t) \in \mathcal{V}_i(\alpha_i)$, the associated active control law K_i guarantees constraint satisfaction. ■
Even though the V_i are all decreasing, stability is not obvious here since the simple combination of the V_i might increase at the transitions from one invariant set into the next. The existence of scaling factors at the transition make sure that there always exists a strictly decreasing Lyapunov function.

Remark 1: The algorithm has two important properties:

- The convex optimization problem (5) with additional constraint (13) is an SDP, if the degrees of all polynomial variables are predetermined. In addition, feasibility for $i = 1$ recursively guarantees feasibility for all $i > 1$.
- For every x_i, the solution to the optimization problem (5) minimizes the upper bound $V_i(x_i)$ on the cost functional (3).

Remark 2: Because of condition (5d) in Theorem 1, the invariant sets and feedback laws depend on the initial condition and thus on the set of points chosen in the offline portion of the algorithm. Hence, the selection of points has an influence on the performance of the approach.
Obviously, Theorem 2 can be applied to polynomial systems directly without using the parameter-varying representation. In that case $n_\theta = 1$ and the conditions in Theorem 1 do not depend on θ. Theorem 1 is then identical to Theorem 1 in [6], and Theorem 2 is the corresponding off-line formulation. Accordingly, using Theorem 1 and the online formulation of [6], we can easily achieve an online NMPC approach for PPV systems.

V. Numerical Example

As an illustration of the applicability and effectiveness of the presented approach, a numerical example is considered:

$$\dot{x}_1 = x_2^2 + u, \tag{14a}$$

$$\dot{x}_2 = -\tan(x_1) + x_2. \tag{14b}$$

When this continuous system is discretized using Euler method with step size 1, it results in the system presented in [33]. In [30] a systematic and powerful method is presented to find a polynomial parameter-varying representation which extends the well-known sector-nonlinearity approach. This way, the following exact representation of (15) in the domain $\mathcal{D} := \{x \in \mathbb{R}^2 \mid -\frac{\pi}{4} \leq x_1 \leq \frac{\pi}{4}\}$ can be found

$$\begin{aligned} A_1(x) &= \begin{bmatrix} 0 & x_2 \\ -1 - 0.443x_1^2 & 1 \end{bmatrix}, \\ A_2(x) &= \begin{bmatrix} 0 & x_2 \\ -1 - \frac{1}{3}x_1^2 & 1 \end{bmatrix}, \quad Z(x) = x \\ B_1(x) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2(x) = B_1(x), \end{aligned} \tag{15}$$

with the state-dependent parameters

$$\theta_1(x) = \frac{3\tan(x_1) - 3x_1 - x_1^3}{3x_1^3(0.443 - \frac{1}{3})},\ \theta_2(x) = 1 - \theta_1(x),$$

where $\theta(x)$ evolves in the standard 1-simplex for all $x \in \mathcal{D}$. The state constraints are $|x_1| \leq \frac{\pi}{4}$, $|x_2| \leq \frac{\pi}{4}$, and the input is bounded by $|u| \leq 2$. The lookup-table for the offline approach is constructed by using ten states along the x_1-axis between $x_1 = 0.5$ and $x_1 = 0.05$, where the weighting matrices are chosen to be $Q = 100I_2$ and $R = 1$. Further, the maximal degree of all polynomial variables (including slack variables) is set to two. Finally, the initial condition for the simulation run is $x(0) = \begin{bmatrix} -0.55 & -0.05 \end{bmatrix}^T$.
To show the effectiveness of the NMPC approach in contrast to just using Theorem 1, the system states and control inputs are compared in Figure 1. The NMPC approach leads to much faster convergence, and the value of the cost functional is reduced by roughly 30%.

VI. Conclusion

This paper presents an offline NMPC approach for continuous-time nonlinear systems. The class of systems to which the controller can be applied is enlarged from polynomial systems to general nonlinear systems which can be represented by PPV systems. Since no online optimization

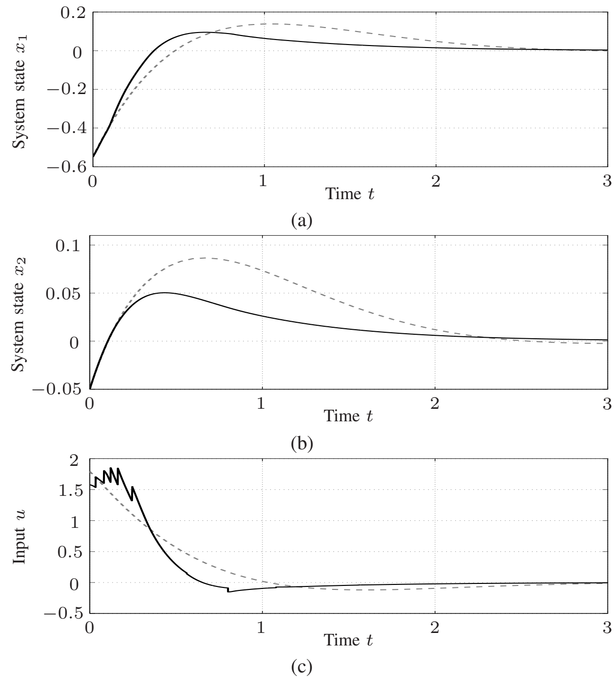

Fig. 1. (a) State trajectories x_1 for Theorem 2 (black, solid) and Theorem 1 (gray, dashed). (b) State trajectories x_2. (c) Control inputs u.

is necessary, the presented offline NMPC approach reduces online computational effort significantly compared to an online approach which was presented in [6], while sacrificing optimality. The derived control law guarantees stability and constraint satisfaction. Both applicability and effectiveness of the presented results have been shown through numerical simulation of an example system for which the NMPC approach reduced the cost significantly compared to the use of a time-invariant feedback matrix.

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