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Dwell-time approach to input-output stability properties for discrete-time linear hybrid systems†

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***Abstract*— Motivated by studying a sampled-data (or discrete-time) version of switched linear systems, we consider a class of time-varying dynamical systems that consist of switching of a number of discrete-time linear time-invariant (sub)systems. For these systems, we propose LMI (Linear Matrix Inequality) formulations of analyzing their input-output stability properties, including both ℓ-2 stability and passivity, by constraining switching signals via the concept of dwell time. As a natural byproduct, we provide a numerical procedure of optimally computing ℓ-2 bounds with respect to dwell time for switched linear systems in the discrete-time setting. Finally, the LMI formulation of ℓ-2 stability is applied to evaluating control system performance for an industrial refrigeration process that is regulated by several switched proportional-integral (PI) controllers.**

I. INTRODUCTION

Switched control strategy has been widely used in the design of automatic control systems to deal with plant dynamics, handle constraints, and improve performance; for example, see [11], [16], [19], [4], [18], [13], [3]. Naturally, some simple and practical low-cost analytical/numerical tools rather than exhaustive simulations or expensive experiments are highly desired for verifying stability and system performance. Unfortunately, both stability analysis and performance evaluation are often technically challenging as the resulting closed-loop control systems become hybrid dynamical systems, which typically consist of complicated interactions between continuous and discrete dynamics. The continuous dynamics primarily describes the closed-loop system behavior when a fixed controller is active during some time span. The discrete dynamics typically captures state reset when different controllers are switched at some time instants, where the state often includes the plant variables, the controller variables, and/or the controller modes; a practical example of state reset encountered in switched proportional-integral (PI) controllers is integrator reset during mode switching, often adopted to guarantee bumpless transfer for the actuator (see [2, Section IV] or [3, Section 4]).

In this paper, following the stability analysis of *linear hybrid systems* (i.e. switched linear systems additionally associated with linear state reset) recently developed in [2], [3], we focus on their sampled-data (or discrete-time) analogues, *discrete-time linear hybrid systems* (DLHS), where the continuous dynamics of each mode is a uniformly-discretized linear system and the discrete dynamics during mode switching is a linear map. We are interested in constructive tools of analyzing input-output stability, including both ℓ-2 stability and passivity, for DLHS.

The ℓ-2 stability analysis for DLHS is in part motivated by an open problem posed in [7] on understanding quantitative relationship between $\mathcal{L}$-2 induced gains and dwell time[1] (i.e. the computation of $\mathcal{L}$-2 induced gains versus the dwell time, see Figures 1 and 2 in Section VI for example) for switched linear systems. To the best of our knowledge, the open problem in [7] has not been completely solved, though some important aspects of the problem have been reported in [8], [20], [14], [6], [9], and [10]; to mention a few, in [9] the authors showed that $\mathcal{L}$-2 stability can be characterized by the existence of a convex homogeneous (of degree two) Lyapunov function, though the construction of such a storage function remains a theoretical challenge; in [6] the authors proposed a suboptimal LMI (Linear Matrix Inequality) formulation of estimating $\mathcal{L}$-2 induced gains versus dwell time, though the estimated gains may be conservative.

In this paper, we study an analogue problem to that in [7] and we propose an LMI formulation of optimally estimating ℓ-2 induced gains versus dwell time for DLHS. As a natural byproduct, we also provide a numerical solution to a discrete-time version of the open problem in [7], since DLHS simply reduce to switched discrete-time linear systems when state reset is ignored. Moreover, our ideas of deriving LMI formulation of ℓ-2 stability with respect to dwell time here can be easily extended to studying passivity with respect to dwell time; such an extension is quite natural as both ℓ-stability and passivity share similar LMI formulations for either continuous-time or discrete-time LTI (Linear Time Invariant) systems (see Section II).

This paper is organized as follows. Section II contains some preliminaries on LMI formulations of analyzing ℓ-2 stability and passivity for discrete-time LTI systems. Section III introduces DLHS and stability notions. Sections IV and V present LMI formulations of ℓ-2 stability and passivity, respectively, for DLHS. Section VI illustrates the application of the proposed LMI formulation of ℓ-2 stability to evaluating system performance for an industrial refrigeration process. The last section draws the conclusions.

We start by listing some basic notation.

- Denote by $\mathbb{Z}$ the set of integers; given any set $S \subset \mathbb{R}$, denote $\mathbb{Z}_S := \mathbb{Z} \cap S$.

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[1] Dwell time is a well-known concept of guaranteeing stability by simply constraining switching rates; more precisely, it is a sufficiently large time interval between any two successive switching instants to guarantee exponential stability for switched linear systems [15, Lemma 2].

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- Given a vector $v \in \mathbb{R}^n$, v' denotes the transpose of v, and $|v|$ denotes Euclidean norm of v.
- Let $M^{'}$ denote the transpose of $M \in \mathbb{R}^{n\times m}$. In the context of a symmetric matrix $\begin{bmatrix} M_{11} & \bigstar \\ M_{21} & M_{22} \end{bmatrix}$, the symbol $\bigstar$ means the transpose of M_{21}.

II. Preliminaries: LMI Formulations of ℓ-2 Stability and Passivity for Discrete-Time LTI Systems

Consider a linear time-invariant (LTI) system as follows,

$$\begin{cases} x(t+1) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is state, $u(t) \in \mathbb{R}^p$ is input, and $y(t) \in \mathbb{R}^q$ is output.

The following results describe the LMI formulations of analyzing ℓ-2 stability and passivity for (1).

Lemma 2.1 ([20, Lemma 2]): Let $\gamma > 0$. The following statements are equivalent for the LTI system (1):

1) it is exponentially stable and has *ℓ-2 gain* γ, i.e. for each solution x to (1) with zero initial condition,

$$0 < -\frac{1}{\gamma}\sum_{\tau=0}^{t} y'(\tau)y(\tau) + \gamma \sum_{\tau=0}^{t} u'(\tau)u(\tau)$$

for all $t \in \mathbb{Z}_{[0,\infty)}$;

2) there exist $\gamma > 0$ and a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} A^{'}PA - P + \frac{1}{\gamma}C^{'}C & \bigstar \\ B^{'}PA + \frac{1}{\gamma}D^{'}C & B^{'}PB + \frac{1}{\gamma}D^{'}D - \gamma I \end{bmatrix} < 0,$$

or equivalently

$$\begin{bmatrix} A^{'}PA - P & \bigstar & \bigstar \\ B^{'}PA & B^{'}PB - \gamma I & \bigstar \\ C & D & -\gamma I \end{bmatrix} < 0.$$

Lemma 2.2 ([1, Lemma 2]): For the LTI system (1), if there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} A^{'}PA - P & \bigstar \\ B^{'}PA - \frac{1}{2}C & B^{'}PB - \frac{D}{2} - \frac{D^{'}}{2} \end{bmatrix} \le 0,$$

then the LTI system (1) is *passive*, i.e. for each solution x to (1) with zero initial condition,

$$0 \le \sum_{\tau=0}^{t} u'(\tau)y(\tau) \qquad \forall t \in \mathbb{Z}_{[0,\infty)}.$$

III. DLHS and Stability Notion

In [2], [3], the authors study stability analysis for a class of time-varying dynamical systems by simply incorporating state reset (during mode switches) into the standard switched linear systems, and they are called *linear hybrid systems*. For a detailed motivation of studying linear hybrid systems, we refer the reader to an industrial example reported in [2, Section IV] or [3, Section 4], where a dynamical plant is regulated by several switched PI controllers to meet the state constraints and the bumpless transfer technique is added to protect the actuator device.

In this paper, we consider a class of time-varying dynamical systems that represents a discrete-time version of linear hybrid systems.

Let $\mathcal{I} := \{1, 2, \cdots, N\}$, $\mathcal{I}_s \subset \mathcal{I}\times\mathcal{I}$, and

$$\begin{aligned} \mathcal{M}_\mathcal{A} &:= \left\{A_i \in \mathbb{R}^{n\times n} \ : \ i \in \mathcal{I}\right\}, \\ \mathcal{M}_\mathcal{B} &:= \left\{B_i \in \mathbb{R}^{n\times p} \ : \ i \in \mathcal{I}\right\}, \\ \mathcal{M}_\mathcal{C} &:= \left\{C_i \in \mathbb{R}^{q\times n} \ : \ i \in \mathcal{I}\right\}, \\ \mathcal{M}_\mathcal{D} &:= \left\{D_i \in \mathbb{R}^{q\times p} \ : \ i \in \mathcal{I}\right\}, \\ \mathcal{M}_s &:= \left\{A_{i,j} \in \mathbb{R}^{n\times n} \ : \ (i,j) \in \mathcal{I}_s\right\}, \end{aligned}$$

For the set $\mathcal{M}_s$ containing "state reset" matrices, we assume that, for each $i \in \mathcal{I}$, $A_{i,i}$ is an identity matrix. Denote

$$\mathcal{I}_s^{\neq} := \{(i,j) \in \mathcal{I}_s \ : \ i \neq j \quad \& \quad A_{i,j} \in \mathcal{M}_s\}.$$

Given a *switching signal* $\sigma : \mathbb{Z}_{[0,\infty)} \mapsto \mathcal{I}$, consider a class of discrete-time dynamical systems denoted as follows

$$\begin{cases} x(t+1) = A_{\sigma(t+1),\sigma(t)}\left(A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)\right) \\ y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{cases} \tag{2}$$

where $t \in \mathbb{Z}_{[0,\infty)}$ is time, $x(t) \in \mathbb{R}^n$ is state, $u(t) \in \mathbb{R}^p$ is input and $y(t) \in \mathbb{R}^q$ is output[2]. The system (2) is called *discrete-time linear hybrid systems* (DLHS).

When $\mathcal{M}_s$ consists of identity matrices only (i.e. no state resets occur during mode switchings), the system (2) covers a discrete-time switched linear system

$$\begin{cases} x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{cases} \tag{3}$$

When both input and output are removed, the system (2) simply reduces to

$$x(t+1) = A_{\sigma(t+1),\sigma(t)}A_{\sigma(t)}x(t). \tag{4}$$

Stability properties will be studied by imposing certain switching signals specified by the dwell time concept to the system (2). Given any switching signal $\sigma : \mathbb{Z}_{[0,\infty)} \mapsto \mathcal{I}$, without loss of generality, assume there exists a sequence of monotonically increasing integers $\{t_k\}_{k=0}^K$ (we allow $K = \infty$) such that $\bigcup_{k=0}^K \mathbb{Z}_{[t_k,t_{k+1})} = \mathbb{Z}_{[0,\infty)}$, σ is constant on $\mathbb{Z}_{[t_k,t_{k+1})}$ for all $k \in \mathbb{Z}_{[0,K]}$, and $\sigma(t_{k+1}-1) \neq \sigma(t_{k+1})$ for all $k \in \mathbb{Z}_{[1,K]}$. In what follows, the switching signal σ is always associated with the sequence $\{t_k\}_{k=0}^K$ unless specifically stated. Let $T_* \ge T > 0$ (we allow $T_* = \infty$ and for such a case $[T, T_*]$ means $[T,\infty)$) be given, denote by $\mathcal{S}_{[T,T_*]}$ the set of all switching signals σ such that $T \le t_{k+1} - t_k \le T_*$; here T corresponds to the *dwell time* concept in the literature [15].

The system (2) under switching signals $\mathcal{S}_{[T,T_*]}$ is said to be *ℓ-2 stable with finite gain* $\gamma > 0$ if each solution x with zero initial condition to (2) under switching signals $\mathcal{S}_{[T,T_*]}$ satisfies

$$0 < -\frac{1}{\gamma}\sum_{\tau=0}^{t} y'(\tau)y(\tau) + \gamma\sum_{\tau=0}^{t} u'(\tau)u(\tau) \qquad \forall t \in \mathbb{Z}_{[0,\infty)}.$$

[2]We assume that the vectors u and y have the same dimension (i.e. $p = q$) for any relevant statement of passivity throughout the paper.

The system (2) is said to be *passive* if each solution x with zero initial condition to (2) under switching signals $\mathcal{S}_{[T,T_*]}$ satisfies

$$0 \leq \sum_{\tau=0}^{t} u'(\tau)y(\tau) \qquad \forall t \in \mathbb{Z}_{[0,\infty)}.$$

The origin of (4) under switching signals $\mathcal{S}_{[T,T_*]}$ is *uniformly exponentially stable* if there exist $r \geq 1$ and $\theta \in [0,1)$ such that each solution x to (4) under $\sigma \in \mathcal{S}_{[T,T_*]}$ satisfies $|x(t)| \leq r|x(0)|\theta^t$ for all $t \in \mathbb{Z}_{[0,\infty)}$.

IV. LMI Formulation of ℓ-2 Stability for DLHS

We start with some observations and notation for (2). For each $T \in \mathbb{Z}_{[1,\infty)}$, all solutions $x(\cdot)$ to an LTI system $[A_i, B_i, C_i, D_i]$ satisfy

$$\begin{cases} x(t+T) = \mathbb{A}_i(T)x(t) + \mathbb{B}_i(T)\mathbf{u}(t,T), \\ \mathbf{y}(t,T) = \mathbb{C}_i(T)x(t) + \mathbb{D}_i(T)\mathbf{u}(t,T), \end{cases} \tag{5}$$

where $t \in \mathbb{Z}_{[0,\infty)}$ is time and the matrices $\mathbb{A}_i(T)$, $\mathbb{B}_i(T)$, $\mathbb{C}_i(T)$, $\mathbb{D}_i(T)$ are defined as follows

$$\mathbb{A}_i(T) := A_i^T,$$
$$\mathbb{B}_i(T) := \begin{bmatrix} B_i & A_iB_i & \cdots & A_i^{T-1}B_i \end{bmatrix},$$
$$\mathbb{C}_i(T) := \begin{bmatrix} C_iA_i^{T-1} \\ \vdots \\ C_iA_i \\ C_i \end{bmatrix},$$
$$\mathbb{D}_i(T) :=$$
$$\begin{bmatrix} D_i & C_iB_i & C_iA_iB_i & \cdots & C_iA_i^{T-3}B_i & C_iA_i^{T-2}B_i \\ 0 & D_i & C_iB_i & \cdots & C_iA_i^{T-4}B_i & C_iA_i^{T-3}B_i \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & C_iB_i & C_iA_iB_i \\ 0 & 0 & 0 & \cdots & D_i & C_iB_i \\ 0 & 0 & 0 & \cdots & 0 & D_i \end{bmatrix},$$

(note that $\mathbb{A}_i(1) = A_i$, $\mathbb{B}_i(1) = B_i$, $\mathbb{C}_i(1) = C_i$, $\mathbb{D}_i(1) = D_i$) and the vectors $\mathbf{u}(t,T)$, $\mathbf{y}(t,T)$ are defined as follows

$$\mathbf{u}(t,T) := \begin{bmatrix} u(t+T-1) \\ u(t+T-2) \\ \vdots \\ u(t+1) \\ u(t) \end{bmatrix}$$

and

$$\mathbf{y}(t,T) := \begin{bmatrix} y(t+T-1) \\ y(t+T-2) \\ \vdots \\ y(t+1) \\ y(t) \end{bmatrix}.$$

The above representation for each subsystem with varying T is similar to the standard approach of solving linear model predictive control problem via quadratic programming.

With the notation of (5), we now state the LMI formulation of ℓ-2 stability for (2) as follows.

Theorem 4.1: Let $T \in \mathbb{Z}_{[1,\infty)}$ and $\gamma > 0$ be given. If there exists a collection of symmetric positive definite matrices $\{P_1, P_2, \cdots, P_N\}$ of compatible dimensions such that

$$\begin{bmatrix} A_i'P_iA_i - P_i & \star & \star \\ B_i'P_iA_i & B_i'P_iB_i - \gamma I & \star \\ C_i & D_i & -\gamma I \end{bmatrix} < 0 \qquad \forall i \in \mathcal{I}, \tag{6}$$

and

$$\begin{bmatrix} \Xi_{11} & \star & \star \\ \Xi_{21} & \Xi_{22} & \star \\ \mathbb{C}_i(T) & \mathbb{D}_i(T) & -\gamma I \end{bmatrix} < 0 \qquad \forall (i,j) \in \mathcal{I}_s^{\neq} \tag{7}$$

where

$$\Xi_{11} = \mathbb{A}_i(T)'A_{i,j}'P_jA_{i,j}\mathbb{A}_i(T) - P_i,$$
$$\Xi_{21} = \mathbb{B}_i(T)'A_{i,j}'P_jA_{i,j}\mathbb{A}_i(T),$$
$$\Xi_{22} = \mathbb{B}_i(T)'A_{i,j}'P_jA_{i,j}\mathbb{B}_i(T) - \gamma I,$$

then the system (2) under switching signals $\mathcal{S}_{[T,\infty)}$ is ℓ-2 stable with finite gain γ.

Specializing Theorem 4.1 to switched linear systems gives the following corollary, which is a numerical solution to a discrete-time analogue of the open problem in [7].

Corollary 4.1: Let $T \in \mathbb{Z}_{[1,\infty)}$ and $\gamma > 0$ be given. If there exists a collection of symmetric positive definite matrices $\{P_1, P_2, \cdots, P_N\}$ of compatible dimensions such that (6) and the following LMI condition hold

$$\begin{bmatrix} \mathbb{A}_i(T)'P_j\mathbb{A}_i(T) - P_i & \star & \star \\ \mathbb{B}_i(T)'P_j\mathbb{A}_i(T) & \mathbb{B}_i(T)'P_j\mathbb{B}_i(T) - \gamma I & \star \\ \mathbb{C}_i(T) & \mathbb{D}_i(T) & -\gamma I \end{bmatrix} < 0$$
$$\forall (i,j) \in \mathcal{I}_s^{\neq}$$

then the system (3) under switching signals $\mathcal{S}_{[T,\infty)}$ is ℓ-2 stable with finite gain γ.

In order to develop a numerical procedure from Theorem 4.1, we start with some observations.

- For each fixed $T \in \mathbb{Z}_{[1,\infty)}$, the search of positive definite matrices $\{P_1, \cdots, P_N\}$ and minimal $\gamma > 0$ such that the inequalities (6) and (7) hold becomes a convex optimization problem, that is,
$$\min_{P_1>0,\ \cdots,\ P_N>0} \{\gamma > 0 \ : \ \text{(6) and (7) hold}\}. \tag{8}$$
- For any $T \in \mathbb{Z}_{[1,\infty)}$ and any $T_* \in \mathbb{Z}_{[1,\infty)}$ with $T < T_*$, we know that $\mathcal{S}_{[T_*,\infty)} \subseteq \mathcal{S}_{[T,\infty)}$ and hence we infer that the minimal γ for (8) is nonincreasing as T increases.

So a natural question to ask is how to efficiently find a minimal $T \in \mathbb{Z}_{[1,\infty)}$ such that the optimization problem (8) becomes numerically solvable. It turns out the answer to this question lies in finding a "minimal" dwell time to guarantee exponential stability of the system (4), as stated in the following results.

Theorem 4.2: Let $T \in \mathbb{Z}_{[1,\infty)}$ be given. If there exists a collection of symmetric positive definite matrices $\{P_1, P_2, \cdots, P_N\}$ of compatible dimensions such that

$$A_i'P_iA_i < P_i \qquad \forall i \in \mathcal{I}, \tag{9}$$

and

$$(A_i')^T A_{i,j}' P_j A_{i,j} (A_i)^T < P_i \qquad \forall (i,j) \in \mathcal{I}_s^{\neq}, \tag{10}$$

then the origin of the system (4) under switching signals $\mathcal{S}_{[T,\infty)}$ is uniformly exponentially stable.

Proposition 4.1: Let $T \in \mathbb{Z}_{[1,\infty)}$ and a collection of symmetric positive definite matrices $\{P_1, P_2, \cdots, P_N\}$ of compatible dimensions satisfy (9) and (10) for the system (4). Then there exists $\gamma > 0$ such that $\{P_1, P_2, \cdots, P_N\}$ also satisfy (6) and (7) for the system (2).

Theorem 4.2 deserves several remarks. It is not hard to see that the condition that exponential stability of all subsystems guarantees the existence of (sufficiently large) $T > 0$ such that both (9) and (10) are feasible. We also remark that, when the system (4) becomes a discrete-time switched linear system, Theorem 4.2 covers [5, Theorem 1] and also gives a stronger stability result (rather than global asymptotic stability as claimed in [5, Theorem 1]). Note that, for linear time-varying systems, global asymptotic stability is not necessarily equivalent to uniform asymptotic stability or uniform exponential stability (see [17, Example 22.12]), though the latter two stability properties are indeed equivalent (see [17, Theorem 22.14] or [12, Theorem 4.11]).

In practice, given a system (2), one can start with Thm 4.2 to solve the following optimization problem for (4)

$$\min_{T>0,\ P_1>0,\ \cdots,\ P_N>0} \{T \ : \ (9) \text{ and } (10) \text{ hold}\}; \tag{11}$$

once a minimal dwell time $T_{\min}$ is computed from (11), one can rely on Proposition 4.1 and solve the optimization problem (8) for any $T > T_{\min}$ (and hence the computed ℓ-2 gains form a function of T). In summary, the above procedure has fairly straightforward coding and easy implementation for practical problems.

V. LMI Formulation of Passivity for DLHS

In continuous-time/discrete-time LTI systems, both ℓ-stability analysis and passivity analysis share similar LMI formulations. The ideas of deriving Theorem 4.1 is readily extended to studying passivity for the system (2).

Theorem 5.1: Let $T \in \mathbb{Z}_{[1,\infty)}$. If there exists a collection of symmetric positive definite matrices $\{P_1, P_2, \cdots, P_N\}$ of compatible dimensions such that

$$\begin{bmatrix} A_i' P_i A_i - P_i & \star \\ B_i' P_i A_i - \frac{1}{2} C_i & B_i' P_i B_i - \frac{1}{2} D_i - \frac{1}{2} D_i' \end{bmatrix} < 0 \qquad \forall i \in \mathcal{I}, \tag{12}$$

and

$$\begin{bmatrix} \Xi_{11} & \star \\ \Xi_{21} & \Xi_{22} \end{bmatrix} < 0 \qquad \forall (i,j) \in \mathcal{I}_s^{\neq}, \tag{13}$$

where

$$\Xi_{11} = \mathbb{A}_i(T)' A_{i,j}' P_j A_{i,j} \mathbb{A}_i(T) - P_i,$$
$$\Xi_{21} = \mathbb{B}_i(T)' A_{i,j}' P_j A_{i,j} \mathbb{A}_i(T) - \frac{1}{2}\mathbb{C}_i,$$
$$\Xi_{22} = \mathbb{B}_i(T)' A_{i,j}' P_j A_{i,j} \mathbb{B}_i(T) - \frac{1}{2}\mathbb{D}_i - \frac{1}{2}(\mathbb{D}_i)',$$

then the system (2) under switching signals $\mathcal{S}_{[T,\infty)}$ is passive.

Specializing Theorem 5.1 to the setting of switched linear systems becomes the following corollary.

Corollary 5.1: Let $T \in \mathbb{Z}_{[1,\infty)}$. If there exists a collection of symmetric positive definite matrices $\{P_1, P_2, \cdots, P_N\}$ of compatible dimensions such that (12) and the following LMI condition hold

$$\begin{bmatrix} \mathbb{A}_i(T)' P_j \mathbb{A}_i(T) - P_i & \star \\ \mathbb{B}_i(T)' P_j \mathbb{A}_i(T) - \frac{1}{2}\mathbb{C}_i & \begin{matrix} \mathbb{B}_i(T)' P_j \mathbb{B}_i(T) \\ -\frac{1}{2}\mathbb{D}_i - \frac{1}{2}(\mathbb{D}_i)' \end{matrix} \end{bmatrix} < 0$$
$$\forall (i,j) \in \mathcal{I}_s^{\neq},$$

then the system (3) under switching signals $\mathcal{S}_{[T,\infty)}$ is passive.

Compared with the passivity analysis for discrete-time dynamical systems in piecewise affine form [1], the LMI formulation in Corollary 5.1 may become useful when the polyhedral partition of the state space becomes too complicated to handle through S-procedure.

VI. An Industrial Switched Control System

In this section we show how the ℓ-2 stability analysis in Section IV can be applied to a practical example. The model data of the example are derived from an industrial process in [3, Section 4], which is regularized by four switched PI controllers, by discretizing its closed-loop model with sample time equal to one. Consequently, we consider the system (2) with $x \in \mathbb{R}^5$, $y \in \mathbb{R}$, $u \in \mathbb{R}$, $\mathcal{I} = \{1, 2, 3, 4\}$ and

$$A_1 = \begin{bmatrix} 0.9962 & 0 & 0 & 0 & -0.0002 \\ 0.0676 & 0.9940 & 0 & 0 & 0.0034 \\ 0.2246 & 0 & 0.8251 & 0 & 0.0112 \\ -0.0041 & 0 & 0 & 0.9835 & -0.0002 \\ 0.0160 & 0 & 0 & 0 & 1.0000 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.9994 & 0.0014 & 0 & 0 & -0.0002 \\ 0 & 0.9640 & 0 & 0 & 0.0033 \\ 0 & -0.0997 & 0.8251 & 0 & 0.0111 \\ 0 & 0.0018 & 0 & 0.9835 & -0.0002 \\ 0 & -0.0664 & 0 & 0 & 0.9999 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0.9994 & 0 & 0.0028 & 0 & -0.0001 \\ 0 & 0.9940 & -0.0584 & 0 & 0.0030 \\ 0 & 0 & 0.6318 & 0 & 0.0100 \\ 0 & 0 & 0.0035 & 0.9835 & -0.0002 \\ 0 & 0 & -3.0060 & 0 & 0.9800 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0.9994 & 0 & 0 & -0.0160 & -0.0002 \\ 0 & 0.9940 & 0 & 0.3384 & 0.0034 \\ 0 & 0 & 0.8251 & 1.1240 & 0.0111 \\ 0 & 0 & 0 & 0.9631 & -0.0002 \\ 0 & 0 & 0 & 1.6466 & 0.9998 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.0000 & 0.0000 & 0.0001 & -0.0000 & 0.0160 \end{bmatrix}',$$
$$B_2 = B_3 = B_4 = 0_{5\times 1},$$
$$C_1 = C_2 = C_3 = C_4 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$
$$D_1 = -1,$$
$$D_2 = D_3 = D_4 = 0,$$
$$A_{i,j} = \begin{bmatrix} I_{4\times 4} & 0_{4\times 1} \\ \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \Theta_{i,j} & 1 \end{bmatrix}, \qquad i \neq j,$$

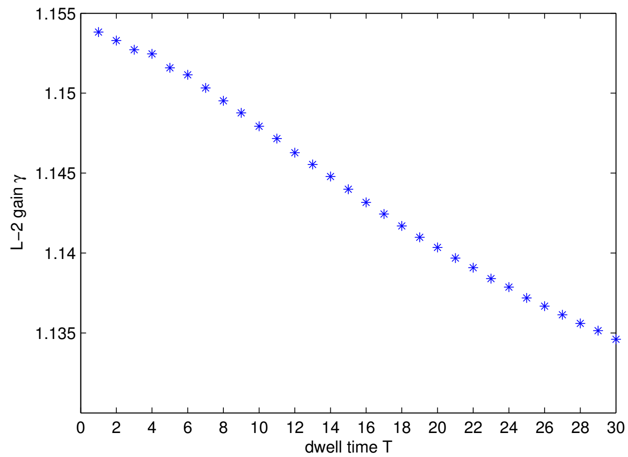

Fig. 1. An industrial example: decreasing of ℓ-2 gain γ with respect to dwell time T

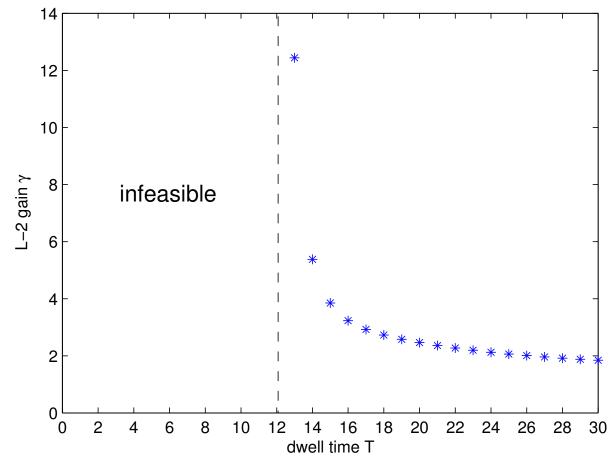

Fig. 2. When state reset is ignored: Nonexistence of γ for $T \leq 12$ and decreasing of γ for $T \geq 13$

where $\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} = \begin{bmatrix} -20 & 9.0 & 19.4 & -101 \end{bmatrix}$, $\Theta_{i,j} \in \mathbb{R}^{4\times 4}$ are with all zero entries except $\Theta_{i,j}(i,i) = -1$ and $\Theta_{i,j}(j,j) = 1$ (for example, if $i = 1$ and $j = 2$ then $\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix} \Theta_{1,2} = \begin{bmatrix} -c_1 & c_2 & 0 & 0 \end{bmatrix}$). Here x_1 is a controlled temperature, x_2 is a constraint temperature, x_3 and x_4 are constraint pressures, and x_5 is the integrator state from the switched controllers. The state reset matrices $A_{i,j}$ mainly describe integrator reset due to the bumpless transfer requirement of protecting the actuator devices during controller switches.

Figure 1 shows a monotonic relationship between ℓ-2 gain γ and dwell time T for (2) by using the LMI formulation in Theorem 4.1. When state reset is ignored (i.e. $\mathcal{M}_s$ contains only identity matrices), a similar relationship between ℓ-2 gain γ and dwell time T by using the LMI formulation in Theorem 4.1 is plotted in Figure 2; note that the optimization problem (8) is infeasible when $T \leq 12$. As predicted by Proposition 4.1, the LMI formulation in Theorem 4.2 gives estimates of the upper bound of dwell time $T = 1$ when state reset is captured and $T = 13$ when state reset is ignored to guarantee exponential stability for the system (4). It is worth mentioning that the upper bound of dwell time for exponential stability of the original closed-loop system in [3, Section 4] was reported as 0.001 when state reset is captured and as 12.425 when state reset is ignore in a similar LMI formulation. Both the results here and those in [3, Section 4] agree to indicate that the state reset has significantly enhanced stability and improved system performance for the switched control system in the refrigeration process.

VII. Conclusions and Discussions

By constraining switching signals via dwell time, we establish both LMI formulation of ℓ-2 stability and that of passivity for DLHS (discrete-time linear hybrid systems). As a byproduct, we have provided a numerical solution to estimate nonconservative ℓ-gains versus dwell time for discrete-time switched linear systems. The results reported here are readily applicable to analyzing input-output stability properties for switched control of dynamical systems that have moderate sizes and have well approximation from sampled data. An example of such types is demonstrated by using the proposed LMI formulation of ℓ-stability to evaluate the control system performance for an industrial refrigeration process that is regulated by four switched PI controllers. Future work will focus on taking similar ideas to study a (possibly) complete solution to the open problem posed in [7].

References

[1] A. Bemporad, G. Bianchini, and F. Brogi. Passivity analysis and passification of discrete-time hybrid systems. *IEEE Transactions on Automatic Control*, 53(4):1004 – 1009, 2008.

[2] C. Cai and S. Mijanovic. Dwell-time approach to stability analysis for hybrid control systems: Application to an industrial refrigeration process. In *Proceedings of the 6th IEEE Conference on Automation Science and Engineering*, pages 728–733, 2010.

[3] C. Cai and S. Mijanovic. LMI-based stability analysis of linear hybrid systems with application to switched control of a refrigeration process. *Asian Journal of Control*, to appear, 2010.

[4] M.-L. Chiang and L.-C. Fu. Hybrid system based adaptive control for the nonlinear HVAC system. In *Proceedings of the 25th American Control Conference*, pages 5324–5329, 2006.

[5] J. Geromel and P. Colaneri. Stability and stabilization of discrete-time switched systems. *International Journal of Control*, 79(7):719 – 728, 2006.

[6] J. Geromel and P. Colaneri. $\mathcal{H}_\infty$ and dwell time specifications of switched linear systems. In *Proc. 47rd IEEE Conf. on Decision and Control*, pages 5318–5323, 2008.

[7] J. Hespanha. $\mathcal{L}_2$-induced gains of switched linear systems. In Vincent D. Blondel and Alexander Megretski, editors, *Unsolved Problems in Mathematical Systems & Control Theory*, pages 131–133. Princeton University Press, Princeton, NJ, 2003.

[8] J. Hespanha. Root-mean-square gains of switched linear systems. *IEEE Transactions on Automatic Control*, 48(11), Nov. 2003.

[9] K. Hirata and J. Hespanha. $\mathcal{L}_2$-induced gain analysis for a class of switched systems. In *Proc. 48th IEEE Conf. on Decision and Control*, pages 2138–2143, 2009.

[10] K. Hirata and J. Hespanha. $\mathcal{L}2$-induced gain analysis of switched linear systems via finitely parametrized storage functions. In *Proceedings of the 29th American Control Conference*, pages 4064–4069, 2010.

[11] H.A. Spang III and H. Brown. Control of jet engines. *Control Engineering Practice*, 7(9):1043–1059, 1999.

[12] H.K. Khalil. *Nonlinear Systems*. Prentice Hall, 3rd edition, 2002.
[13] J.-L. Lin and T.-J. Yeh. Mode switching control of dual-evaporator air-conditioning systems. *Energy Conversion and Management*, 50:1542–1555, 2009.
[14] M. Margaliot and J. Hespanha. Root-mean-square gains of switched linear systems: A variational approach. *Automatica*, 44(9):2398–2402, Sep. 2008.
[15] A.S. Morse. Supervisory control of families of linear set-point controllers Part I: Exact matching. *IEEE Transactions on Automatic Control*, 42(11):1500–1515, 1998.
[16] R. Oboe and M. Federico. Initial value compensation applied to disturbance observer-based servo control in HDD. In *Proceedings of the 7th International Workshop on Advanced Motion Control*, pages 34–39, 2002.
[17] W.J. Rugh. *Linear Systems Theory*. Prentice Hall, 2^{nd} edition, 1996.
[18] D. Sarabia, F. Capraro, L.F.S. Larsen, and C. de Prada. Hybrid control of a supermarket refrigeration system. In *Proceedings of the 26th American Control Conference*, pages 4178–4185, 2007.
[19] M.C. Turner, N. Aouf, D.G. Bates, I. Postlethwaite, and B. Boulet. Switched control of a vertical/short take-off land aircraft: an application of linear quadratic bumpless transfer. *Proceedings of the IMechE Part I: Journal of Systems & Control Engineering*, 220:157–170, 2006.
[20] G. Zhai, H. Lin, Y. Kim, J. Imae, and T. Kobayashi. L_2 gain analysis for switched systems with continuous-time and discrete-time subsystems. *International Journal of Control*, 78:1198–1205, 2005.