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Invariance Kernels of Single-Input Planar Nonlinear Systems

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***Abstract*— The problem of determining invariance kernels for planar single-input nonlinear systems is considered. If K is a closed set, its invariance kernel is the largest subset of K with the property of being positively invariant for arbitrary measurable input signals. It is shown that the boundary of the invariance kernel is a concatenation of solutions of two so-called extremal vector fields. Moreover, only the solutions through a finite number of special points are of interest. This result makes it possible to devise an algorithm which determines the invariance kernel of a simply connected set in a finite number of steps.**

I. Introduction

In this paper we consider the planar system

$$\Sigma : \; \dot{x} = \lambda(t) f_1(x) + [1 - \lambda(t)] f_2(x), \tag{1}$$

where $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ are two C^1 planar vector fields and $\lambda(t)$ is a signal in the class $\mathcal{U}$ of measurable functions $\mathbb{R} \to [0, 1]$. We make a number of generic assumptions which are listed in Section III. Viewing $\lambda(t)$ as a control signal, Σ is a control-affine system. Vice versa, any control-affine system $\dot{x} = f(x) + g(x)u$ with scalar compact-valued controls $u \in [u_{\min}, u_{\max}] \subset \mathbb{R}$ can be expressed in the form (1) by letting $f_1(x) = f(x) + g(x)u_{\min}$, $f_2(x) = f(x) + g(x)u_{\max}$, and expressing $u(t) = \lambda(t)u_{\min} + (1 - \lambda(t))u_{\max}$.

The objective of this paper is the characterization of the invariance kernel of a closed set K, defined next.

Definition 1.1: Let $K \subset \mathbb{R}^2$ be a closed set. K is **positively invariant** (or strongly invariant) for Σ if for all $\lambda(t) \in \mathcal{U}$ and all $x_0 \in K$, the solution of Σ with initial condition $x(0) = x_0$ remains in K for all $t \geq 0$. The **invariance kernel** $K^\star$ of K for system Σ is the maximal positively invariant subset of K.

The invariance kernel of a closed set is closed, for if a set is positively invariant its closure is positively invariant as well. The notion of positive invariance (or strong invariance) of K defined above requires *all* solutions of Σ originating in K to remain in K for all positive time. In contrast, K is said to be *weakly invariant*, or *viable*, for Σ if for all $x_0 \in K$, at least one solution of Σ through x_0 remains in K for all $t \geq 0$. Accordingly, the *viability kernel* of K for system Σ is the maximal subset of K with the property of being viable for Σ.

The theory of viability and invariance kernels was developed by J.P. Aubin and co-workers in the general setting of differential inclusions. The reader is referred to Aubin's book [1] for an overview of the subject, and to the work of Frankowska and Quincampoix [2] and Saint-Pierre [3] for an algorithm to compute viability kernels. Rieger [4] gave convergence estimates for this algorithm. To relate our problem statement to Aubin's general theory, we remark that to system (1) one can associate the differential inclusion

$$\Sigma_I : \; \dot{x} \in F(x) := \mathrm{co}\{f_1(x), f_2(x)\} \text{ a.e.} \tag{2}$$

where $\mathrm{co}\{f_1(x), f_2(x)\}$ denotes the convex hull of $f_1(x)$ and $f_2(x)$. By Filippov's Selection Lemma (see [5]), trajectories of Σ_I in (2) are solutions of Σ corresponding to suitable selections $\lambda(t) \in \mathcal{U}$. Vice versa, it is obvious that solutions of Σ with $\lambda(t) \in \mathcal{U}$ are trajectories of Σ_I. Therefore, there is a one-to-one correspondence between solutions of Σ and those of Σ_I. Owing to this correspondence, determining the invariance kernel of K for Σ or for Σ_I is the same, and all results concerning invariance kernels of differential inclusions apply directly to Σ in (1).

Recently, invariance kernels were used in [6] to define and quantify a notion of stability margin for wind turbines. In control theory, viability kernels often appear in the form of maximal controlled invariant sets, while invariance kernels are associated with notions of robustness. Recently, Broucke and Turriff [7] gave an explicit characterization of the viability kernel for a class of control affine systems when K is the sublevel set of a smooth function. They showed that under certain conditions the viability kernel is a sublevel set of a hitting time function.

Letting $\mathcal{U}^\pm \subset \mathcal{U}$ be the class of measurable functions $\mathbb{R} \to \{0, 1\}$ and taking $\lambda(t) \in \mathcal{U}^\pm$, Σ becomes a switched system. In this context, the invariance kernel $K^\star$ of K is the maximal subset of K with the property that for any switching signal $\lambda(t) \in \mathcal{U}^\pm$, solutions of Σ originating in $K^\star$ remain in $K^\star$ in positive time. It turns out that the invariance kernels one obtains by letting either $\lambda(t) \in \mathcal{U}$ or $\lambda(t) \in \mathcal{U}^\pm$ in Definition 1.1 coincide. The results in this paper are therefore relevant to the literature on switched systems. See, for instance, the work in [8], [9], [10].

This paper makes two main contributions. The first one, in Theorem 7.1, is a characterization of the boundary of the invariance kernel for the planar system Σ in terms of integral curves of extremal vector fields through a finite number of

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special points. The concatenation of such integral curves must obey precise rules in order to form a feasible boundary of $K^\star$. The second main contribution of this paper is an algorithm which exploits the finiteness of special points and the concatenation rules to determine the invariance kernel in a finite number of steps. In this paper we assume, among other things, that K is a simply connected set, but our algorithm can be adapted to the situation when K is not simply connected. Due to space limitationa, a number of proofs have been omitted in this paper.

Throughout this paper we use the following notation. If $S \subset \mathbb{R}^2$, S^c denotes the complement of S, $S^c = \mathbb{R}^2 \backslash S$. The notation $\langle \cdot, \cdot \rangle$ is used to denote the Euclidean inner product. Finally, $\operatorname{int} S$ denotes the interior of the set S.

II. Preliminary definitions

We now present the basic notions used in the characterization of invariance kernels. Let $\mathcal{R}^+ = \{x \in \mathbb{R}^2 : \det[f_1(x)\ f_2(x)] > 0\}$, $\mathcal{R}^- = \{x \in \mathbb{R}^2 : \det[f_1(x)\ f_2(x)] < 0\}$. $\mathcal{R}^+$ and $\mathcal{R}^-$ are open sets. In $\mathcal{R}^+$, f_2 points to the left-hand side of f_1, while in $\mathcal{R}^-$ f_2 points to the right-hand side of f_1.

Definition 2.1: The **extremal vector fields** $f_R(x)$ and $f_L(x)$ are defined as

$$f_L(x) = \begin{cases} f_1(x) & x \in \mathcal{R}^+ \\ f_2(x) & x \in \mathcal{R}^- \end{cases}, \quad f_R(x) = \begin{cases} f_2(x) & x \in \mathcal{R}^+ \\ f_1(x) & x \in \mathcal{R}^-. \end{cases}$$

The solutions[1] at time t of the extremal vector fields f_L and f_R are called **extremal solutions** and are denoted by $\phi_L(t, x_0)$ and $\phi_R(t, x_0)$, respectively. The images of extremal solutions on the plane are called **extremal arcs.** In particular, the **L-arc (resp. R-arc) through** x_0 is the image of the map $t \mapsto \phi_L(t, x_0)$ (resp., $t \mapsto \phi_R(t, x_0)$) for t ranging over some interval over which the map is defined. Extremal fields have been independently studied in relation to attainable sets by Baitman [11], Butenina [12], and Davydov [13].

Definition 2.2: A connected subset of ∂K along which both f_1 and f_2 point inside of K or are tangent to ∂K is said to be an **invariant arc of ∂K**. Each endpoint of an invariant arc of ∂K is called a **t^∂ point**.

We give an orientation to extremal arcs and invariant arcs of ∂K as follows. We give ∂K a positive orientation so that a point moving along ∂K finds the interior of K to its left-hand side. The orientation of extremal arcs is induced by the time parametrization of the corresponding extremal solutions, so that the orientation indicates the direction of increasing time. The notion of orientation of arcs allows us to say, for instance, that arc A crosses arc B leftward.

Definition 2.3: Suppose that $\bar{x}$ is an equilibrium of f_1 (resp., f_2). An extremal arc through $\bar{x}$ is said to be an **equilibrium extremal arc through** $\bar{x}$ if on a neighborhood of $\bar{x}$ it coincides with an f_1 arc (resp., f_2 arc). If, instead, the extremal arc coincides with an f_2 arc (resp., f_1 arc) in a neighborhood of $\bar{x}$, then it is said to be a **non-equilibrium extremal arc through** $\bar{x}$.

Definition 2.4: We define the **collinearity set** $\mathcal{L}$ as

$$\mathcal{L} = \{x \in \mathbb{R}^2 : \det[f_1(x)\ f_2(x)] = 0\}.$$

and the sets $\mathcal{L}^+ = \{x \in \mathcal{L} : \langle f_1(x), f_2(x) \rangle > 0\}$, $\mathcal{L}^- = \{x \in \mathcal{L} : \langle f_1(x), f_2(x) \rangle < 0\}$.

On $\mathcal{L}$, f_1 and f_2 are collinear. On $\mathcal{L}^-$, f_1 and f_2 are antiparallel. Points in $\mathcal{L}$ that are neither in $\mathcal{L}^+$ nor in $\mathcal{L}^-$ are equilibria of f_1 or f_2. The set $\mathcal{L}$ is closed and in this paper we will assume (see Section III) that it is a one dimensional embedded submanifold. The extremal vector fields f_L, f_R are discontinuous on $\mathcal{L}$. The existence and uniqueness of extremal solutions is discussed in Section IV.

Definition 2.5: A point p in $\mathcal{L}^-$ is called a **t^- point** if the trajectories of f_1 and f_2 through p remain on one side of $\mathcal{L}^-$ (i.e., $\langle f_1, f_2 \rangle$ has constant sign along the trajectories) for some interval of time containing $t = 0$.

If ∂K is differentiable in a neighborhood of a t^∂ point, then at least one of the vector fields f_1, f_2 must be tangent to ∂K at the t^∂ point.

Definition 2.6: The **attainable set** $\mathcal{A}(x_0, t)$ of Σ from x_0 at time t is $\mathcal{A}(x_0, t) = \{x(t) : x(\cdot)$ is a solution of Σ with $x(0) = x_0$ for some $\lambda(\cdot) \in \mathcal{U}\}$.

By Theorem 1 in Section 2.8 of [14], the set-valued map $(x_0, t) \mapsto \mathcal{A}(x_0, t)$ is upper semicontinuous. Moreover, $\mathcal{A}(x_0, t)$ is compact and non-empty [15].

Remark 2.7: By definition of f_L and f_R, for each $x \in \mathbb{R}^2 \backslash \mathcal{L}$ and all $\lambda \in (0, 1)$, the vector $\lambda f_1(x) + (1 - \lambda) f_2(x)$ points to the left-hand side of $f_L(x)$ and to the right-hand side of $f_R(x)$. Moreover, when λ is 0 or 1, the vector $\lambda f_1(x) + (1 - \lambda) f_2(x)$ is tangent to either $f_L(x)$ or $f_R(x)$. Therefore, all solutions of Σ in $\mathbb{R}^2 \backslash \mathcal{L}$ are either tangent to or cross R-arcs rightward and L-arcs leftward. In particular, R-arcs (resp., L-arcs) in $\mathbb{R}^2 \backslash \mathcal{L}$ are either tangent to or cross L-arcs (resp., R-arcs) leftward (resp., rightward). Actually, it can be shown that the above statement is true not just for arcs in $\mathbb{R}^2 \backslash \mathcal{L}$, but in the entire $\mathbb{R}^2$ except at equilibria of f_1 or f_2. Extremal arcs cannot self-intersect at points other than equilibria of f_1 and f_2.

III. Standing Assumptions

Throughout this paper we assume that K is simply connected and its boundary is a C^1 Jordan curve. Additionally, we make these assumptions:

(i) The set $\mathcal{L}$ is a one dimensional embedded submanifold, i.e., it is the union of a countable number of disjoint regular curves.

(ii) There is a finite number of t^- points in K, and there is at most a finite number of points on ∂K at which either f_1 or f_2 are tangent to ∂K.

(iii) The equilibria of f_1 and f_2 in K are hyperbolic (implying that all equilibria are isolated) and the linearization at each equilibrium has distinct eigenvalues. Moreover, none of the equilibria of f_1 is an equilibrium of f_2.

(iv) No equilibria of f_1 and f_2 lie on ∂K.

[1] By a solution of an extremal vector field we mean an absolutely continuous function $x(t) : (a, b) \to \mathbb{R}^2$ which satisfies the differential equation associated to the extremal vector field for almost all $t \in (a, b)$.

(v) The slow manifolds of nodes (stable or unstable) of f_1 and f_2 are not tangent to $\mathcal{L}$.
(vi) No t^∂ points lie on $\mathcal{L}$.
(vii) There is a finite number of points on $\mathcal{L}^+ \cap K$ where f_1 and f_2 are tangent to $\mathcal{L}^+$.
(viii) There is at most a finite number of closed extremal arcs in K.

Assumptions (i)-(vii) are C^1-generic. Assumptions (iii)-(vii) could be relaxed, but are made to avoid the need for special cases and to simplify the presentation.

IV. Properties of extremal solutions

The extremal vector fields f_L and f_R are discontinuous on $\mathcal{L}$. Issues of existence and uniqueness of solutions of vector fields of this kind have been extensively investigated by Filippov [14]. Solutions of f_L and f_R exist everywhere on the plane. The next two lemmas discuss issues of non-uniqueness and continuity of the solution maps ϕ_L and ϕ_R.

Lemma 4.1: Extremal solutions of Σ exist through each $x_0 \in \mathbb{R}^2$. Locally near each point $x_0 \in \mathbb{R}^2$, there is only one L-arc and one R-arc through x_0, except in the following cases:

(i) If $x_0 \in \mathcal{L}^-$ and x_0 is not a t^- point, then through x_0 there are either two L-arcs which converge to and two R-arcs which diverge from x_0, or two L-arcs which diverge from and two R-arcs that converge to x_0. In a neighborhood of x_0, the two L-arcs (resp., R-arcs) coincide with an arc of f_1 in $\mathcal{R}^+$ (resp., in $\mathcal{R}^-$) and and an arc of f_2 in $\mathcal{R}^-$ (resp., in $\mathcal{R}^+$).
(ii) If x_0 is an equilibrium of f_1 or f_2 then there is one non-equilibrium extremal arc through x_0 and several, possibly infinite, equilibrium extremal arcs through x_0.

Lemma 4.2: Suppose that $x_0 \notin \mathcal{L}^-$ and x_0 is not an equilibrium of f_1 or f_2. Suppose that the unique solution $x(t)$ of f_L (resp., f_R) through x_0 is defined on $[0,T] \subset \mathbb{R}$ and such that, for all $t \in [0,T]$, $x(t) \notin \mathcal{L}^-$ and $x(t)$ is not an equilibrium of f_1 or f_2. Then, there exists a neighborhood U of x_0 such that the map $\phi_L(t, x_0)$ (resp., $\phi_R(t, x_0)$) is continuous on $[0,T] \times U$.

Finally, we characterize equilibrium extremal arcs in a neighborhood of a node (stable or unstable). Before stating the next result, we recall that if the linearization of a planar vector field at a node has two distinct eigenvalues, then the fast manifold of the node is the invariant manifold of the vector field associated with the eigenvalue which has the largest absolute value, while the slow manifold is associated with the eigenvalue with smallest absolute value.

Lemma 4.3: Suppose that an L-arc (resp., R-arc) γ is an equilibrium extremal arc through a node $\bar{x}$, and that, in a neighborhood of $\bar{x}$, γ does not coincide with the fast manifold of $\bar{x}$. Then, there exists a ball $\mathcal{B}$ centred at $\bar{x}$ and a circle segment $\mathcal{S} \subset \partial\mathcal{B}$ with a unique intersection point $p = \mathcal{S} \cap \gamma$ such that all L-arcs (resp., R-arcs) through $\mathcal{S}$ remain in $\mathcal{B}$ in positive or negative time and are equilibrium extremal arcs.

V. Extremal arcs and boundary of the invariance kernel

The significance of extremal arcs, as pertains to the determination of invariance kernels, is that they form the boundary of attainable sets of Σ, as shown in the next lemma. Thus, extremal arcs delimit bundles of arcs of Σ through points in $\mathbb{R}^2$ resulting from arbitrary choices of $\lambda(t) \in \mathcal{U}$. This feature of extremal arcs, together with the so-called barrier property presented in Proposition 5.2 below, will be used in Proposition 5.3 to establish a relationship between extremal arcs and boundaries of invariance kernels. Before stating the lemma, we recall that Σ is said to be small-time locally controllable (STLC) from x_0 if, for all $T > 0$, x_0 lies in the interior of $\mathcal{A}(x_0, [0,T])$.

Lemma 5.1: Let $x_0 \in \mathbb{R}^2$ be such that Σ is **not** STLC from x_0. Suppose that, for some $T > 0$, a solution $x(t)$ of Σ with initial condition x_0 has the property that $x(t) \in \partial\mathcal{A}(x_0, t)$ for all $t \in [0,T]$. Then, $x(t)$ is a concatenation of extremal solutions.

The boundary of invariance kernels enjoys the so-called barrier property.

Proposition 5.2 (Barrier property [16]): Let $K^\star$ be the invariance kernel of K for (1), and assume it is not empty. Then, for any x_0 in $\partial K^\star$ there exists $\lambda(t) \in \mathcal{U}$ such that the solution to (1) with initial condition $x(0) = x_0$ remains in $\partial K^\star$ for all $t \geq 0$, or until it reaches ∂K.

The proof is completely analogous to the proof of Theorem 4.18 in [17], and is therefore omitted.

Lemma 5.1 and Proposition 5.2 yield the following.

Proposition 5.3: If $K^\star$ is non-empty, then each connected component of $\partial K^\star$ is a concatenation including extremal arcs and invariant arcs of ∂K.

We conclude this section with a result clarifying which equilibria of f_1 and f_2 are feasible on $\partial K^\star$.

Lemma 5.4: The only equilibria of f_1 and f_2 that may belong to $\partial K^\star$ are nodes (stable or unstable) and saddle points, and the only points in $\partial K^\star \cap \mathcal{L}^-$ are t^- points.

VI. Concatenation of extremal arcs and invariant arcs of ∂K

Proposition 5.3 indicates that the boundary of the invariance kernel $K^\star$ is formed by concatenations of extremal arcs and invariant segments of ∂K. The result below identifies all feasible concatenations on $\partial K^\star$. Before stating the proposition, we introduce some notation. We will use the shorthands HH, HT, TT to signify "head-to-head," "head-to-tail," and "tail-to-tail," respectively. The notation A$\overset{p}{\rightarrow\leftarrow}$B will be used to indicate an HH concatenation at point p between arcs A and B, where the symbols A, B belong to the list { L, R, ∂K } (∂K stands for invariant arc of ∂K). Similarly, A$\overset{p}{\rightarrow\rightarrow}B, A\overset{p}{\leftarrow\rightarrow}$B will be used to indicate HT and TT concatenations, respectively. To state that a concatenation occurs at a saddle or node (stable or unstable) of f_1 or f_2 (recall that foci are ruled out by Lemma 5.4) we will set $p = \circ$, while to state that the concatenation occurs anywhere on a set S we will set $p = S$. If p is omitted then the location of the concatenation is unspecified. To illustrate, $\mathrm{L}\overset{t^\partial}{\to\!\to}\partial K$ denotes an HT concatenation of an L-arc with an invariant arc of ∂K at a t^∂ point, and $\partial K \overset{\partial K}{\to\!\to} \mathrm{L}$ denotes an HT concatenation of an invariant arc of ∂K and an L-arc occurring anywhere on ∂K.

Proposition 6.1: On $\partial K^\star$, the only feasible concatenations involving extremal arcs and invariant arcs of ∂K are:

(HH) $\mathrm{L} \overset{\circ}{\to\!\leftarrow} \mathrm{R}$, $\mathrm{L} \overset{t^-}{\to\!\leftarrow} \mathrm{R}$, $\partial K \overset{t^\partial}{\to\!\leftarrow} \mathrm{R}$,

(HT) $\mathrm{L} \overset{t^\partial}{\to\!\to} \partial K$, $\partial K \overset{\partial K}{\to\!\to} \mathrm{L}$, $\mathrm{L} \to\!\to \mathrm{L}$, $\mathrm{R} \to\!\to \mathrm{R}$.

(TT) $\partial K \overset{\partial K}{\leftarrow\!\to} \mathrm{R}$, $\mathrm{L} \overset{\bar{x}}{\leftarrow\!\to} \mathrm{R}$, where $\bar{x}$ is either a t^- point or any point in $(\mathcal{L}^-)^c$.

VII. Main Result

In this section we present the main theoretical result of this paper characterizing the boundary of the invariance kernel. This result relies on Proposition 6.1 and other properties proved earlier.

Theorem 7.1: Each connected component of $\partial K^\star$ is either a closed extremal arc, a closed invariant arc of ∂K, or it is the concatenation of extremal arcs and invariant arcs of $\partial K^\star$ according to the rules listed in Proposition 6.1. An extremal arc which is not closed can only be part of $\partial K^\star$ if one of its endpoints is a t^∂ point, a t^- point, or an equilibrium (saddle or node) of f_1 or f_2. γ is a permissible equilibrium extremal arc through a node on $\partial K^\star$ only if at least one of the following holds:

(i) γ coincides with the fast manifold of $\bar{x}$ locally around $\bar{x}$.

(ii) γ is the non-equilibrium extremal arc of another equilibrium (saddle or node of f_1 or f_2), or an extremal arc through a t^-, t^∂ point.

(iii) γ is simultaneously an equilibrium extremal arc for $\bar{x}$ and for another equilibrium $\bar{y} \neq \bar{x}$. In this case, either γ is of type (i), or locally around $\bar{y}$, γ coincides with the stable/unstable manifold of $\bar{y}$, if $\bar{y}$ is a saddle, or the fast manifold of $\bar{y}$ if $\bar{y}$ is a node.

Proof: Suppose, by way of contradiction, that $\partial K^\star$ contains an extremal arc γ which is not closed and whose endpoints violate the conditions of the theorem. In light of this contradiction assumption and Proposition 6.1, the head of γ must be a node $\bar{x}$, and γ must be an equilibrium extremal arc which does not belong to any of the types (i)-(iii) in the theorem statement. Suppose, without loss of generality, that γ is an L-arc. Since in any neighborhood of $\bar{x}$ γ does not coincide with the fast manifold of $\bar{x}$, by Lemma 4.3 there exists a ball $\mathcal{B}$ centred at $\bar{x}$ and a circle segment $\mathcal{S} \subset \partial\mathcal{B}$ with a unique intersection point $p = \mathcal{S} \cap \gamma$ such that all L-arcs through points in $\mathcal{S}$ remain in $\mathcal{B}$ in positive time, and are all equilibrium extremal arcs. Since $\bar{x} \in \partial K^\star \backslash \partial K$, the ball $\mathcal{B}$ can be taken small enough that $p \in \partial K^\star \backslash \partial K$ as well. Let $\bar{p} \in \gamma$ be a point in the interior of $\mathcal{B}$ and denote by q the tail of γ. Then, there exist $T_2 > T_1 > 0$ such that $\phi_L(T_1, q) = p$ and $\phi_L(T_2, q) = \bar{p}$. By the contradiction assumption, γ does not contain t^- points and so by Lemma 5.4 it follows that $\gamma \cap \mathcal{L}^- = \varnothing$. Consequently, by Lemma 4.2 there exists a neighborhood U of q such that the map $\phi_L : [0, T_2] \times U \to \mathbb{R}^2$ is continuous. By continuity, there exists a neighborhood $V \subset U$ of q such that the following two properties hold:

(a) $\phi_L([0, T_2], V) \cap \partial\mathcal{B} \subset \mathcal{S}$,

(b) $\phi_L(T_2, V) \subset \mathcal{B}$.

The two properties above imply that all L-arcs through points in V intersect S and, by Lemma 4.3, they are equilibrium extremal arcs, i.e., their head is at $\bar{x}$. Next, we investigate the available concatenations at the tail q of γ. There are three cases.

Case 1: $q \in (\mathcal{L}^-)^c \backslash \partial K$ is not an equilibrium. Since, by the contradiction assumption, γ does not contain t^- and t^∂ points, we also have that $\gamma \in (\mathcal{L}^-)^c \backslash \partial K$. Moreover, we can assume that V is small enough that $V \subset \operatorname{int} K$. By Proposition 6.1, at q there must be a TT concatenation between γ and an R-arc η. Extend η in negative time from q, and denote by η' the extended arc. If $q \notin \mathcal{L}^+$, then $f_1(q)$ and $f_2(q)$ are linearly independent. Therefore, in a neighborhood of q, without loss of generality V, the arc η' is transversal to all L-arcs. If, on the other hand, $q \in \mathcal{L}^+$, then by assumption (vii) in Section III, η' is transversal to L-arcs in a punctured neighborhood of q, without loss of generality in $V \backslash \{q\}$. In both cases, in any neighborhood of q contained in V there exists $q' \in \eta' \cap V$ with the property that $q' \notin K^\star$, and therefore such that the L-arc γ' through q' is not contained in $K^\star$. Since $q' \in V$, γ' has its head at $\bar{x}$. Since $\gamma \subset (\mathcal{L}^-)^c \backslash \partial K$, an open set, q' can be chosen such that $\gamma' \subset (\mathcal{L}^-)^c \backslash \partial K$ as well. The set obtained from $K^\star$ by replacing the concatenation $\gamma \overset{q}{\leftarrow\!\to} \eta$ with $\gamma' \overset{q'}{\leftarrow\!\to} \eta'$ is contained in K, is positively invariant, and contains $K^\star$, contradicting the assumption that $K^\star$ is the invariance kernel of K.

Case 2: $q \in \partial K$. Since, by the contradiction assumption, q is not a t^∂ point, it follows that q is not the endpoint of an invariant arc of ∂K. If the vectors $f_1(q)$, $f_2(q)$ point to the interior of K, then the invariant arc of ∂K containing q is transversal to L-arcs in a neighborhood of q, without loss of generality in V. If, on the other hand, $f_1(q)$ or $f_2(q)$ are tangent to ∂K, then by assumption (ii) in Section III the invariant arc of ∂K containing q is transversal to L-arcs in a punctured neighborhood of q, without loss of generality in $V \backslash \{q\}$. In both cases, in any neighborhood of q contained in V there exists $q' \in \partial K$ such that the L-arc γ' through q' is contained in K but is not contained in $K^\star$, and has head at $\bar{x}$. As before, replacing the concatenation $\gamma \overset{q}{\leftarrow\!\to} \eta$ with $\gamma' \overset{q'}{\leftarrow\!\to} \partial K$ we obtain a positively invariant set contained in K which is contains $K^\star$, a contradiction.

Case 3: $q \in (\mathcal{L}^-)^c \cap \partial K$ is a node and γ is an equilibrium extremal arc through q which, near q, does not coincide with the fast manifold of q. By Lemma 4.3, there exists a ball $\mathcal{B}'$ centred at q and a circle segment $\mathcal{S}' \subset \partial\mathcal{B}'$ with a unique intersection point $p' = \mathcal{S}' \cap \gamma$ such that all L-arcs through $\mathcal{S}'$ remain in $\mathcal{B}'$ in negative time and are equilibrium extremal arcs through q. We can assume that $\mathcal{S}' \subset V$ (for, if that isn't the case, we can make $\mathcal{B}'$ smaller). Thus, all L-arcs through S' have tail at q and head at $\bar{x}$. In particular, one can choose a point on $\mathcal{S}'$ outside of $\mathcal{K}^\star$ through which there is an L-

Initial condition		extremal arc	integration direction
t^∂ point, tail of inv. arc	f_L is tangent	L	rev.
		R	fwd.
	f_R is tangent	do nothing	
t^∂ point, head of inv. arc	f_L is tangent	do nothing	
	f_R is tangent	L	fwd.
		R	rev.
t^- point		L	fwd.
		L	rev.
		R	fwd.
		R	rev.
node stable or (unstable)		non-eq	fwd.
			rev.
		eq., fast manifold	rev. (fwd.)
			rev. (fwd.)
saddle		non-eq	fwd.
			rev.
		eq., stable manifold	rev.
			rev.
		eq.,unstable manifold	fwd.
			fwd.

TABLE I

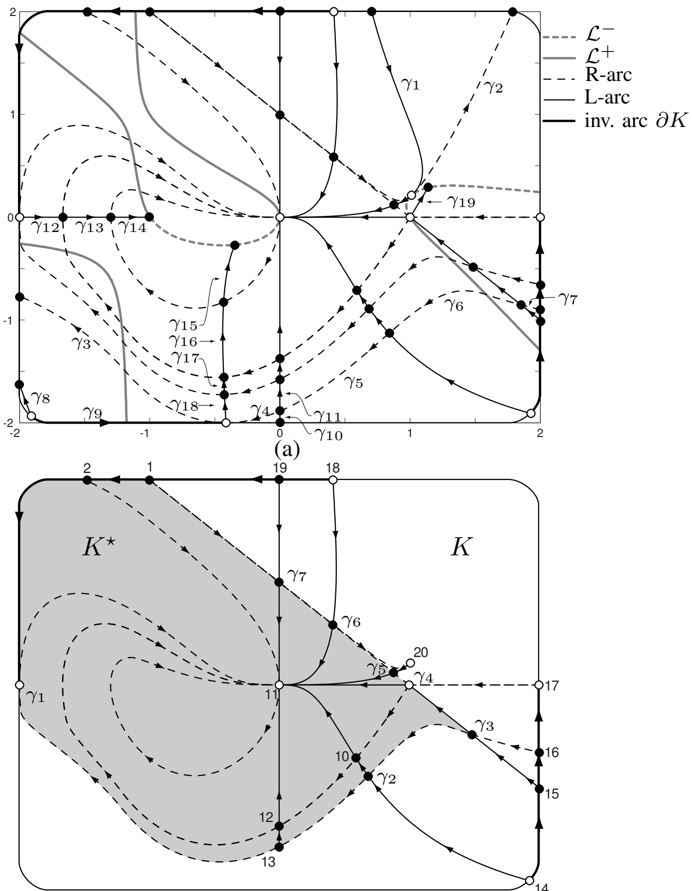
RULES OF INTEGRATION THROUGH SPECIAL POINTS.

arc γ' with tail at q and head at $\bar{x}$ such that $\gamma' \not\subset K^\star$ but $\gamma' \subset K$. By replacing γ with γ' we enlarge $K^\star$ and get a contradiction. ■

VIII. INVARIANCE KERNEL ALGORITHM

In the exposition of this algorithm, it is assumed that any closed extremal arcs are known. Moreover, it is assumed that K is not positively invariant, for in this case trivially $K^\star = K$. The following algorithm determines the invariance kernel of a simply connected and compact set K in a finite number of steps. The algorithm has a rigorous justification based on Proposition 6.1 and Theorem 7.1. The justification is omitted due to space limitations.

1. **Initialization**
   Determine:
   1.1. t^∂ points in K,
   1.2. t^- points in K,
   1.3. nodes and saddles of f_1 or f_2 in K,
   1.4. closed extremal arcs in K.
2. **Integration**
   Using the integration rules in Table I, generate extremal arcs from all points computed in Part 1. The stopping criteria for the integration are:
   2.1. The solution hits $\mathcal{L}^-$ at a point which is not a t^- point.
   2.2. The solution hits ∂K at a point which does not lie on an invariant arc of ∂K.
   2.3. The solution hits an invariant arc of ∂K coming from $\operatorname{int} K$.
   2.4. The solution is detected to reach (in finite or infinite time) an equilibrium of f_1 or f_2 or to spiral (in positive or negative time) around a limit set.
3. **Pruning**
   Label all points identified in Part 1 (steps 1.1-1.4) as *special points*. Label as *significant* all special points, all the integration endpoints, and all points of intersection between extremal arcs generated in Part 2 or between extremal and invariant arcs of ∂K. Thus, special points are significant, but not vice versa.
   3.1. Partition each extremal arc resulting from an integration performed in Part 2 and invariant arcs of ∂K into subarcs whose heads and tails are the significant points. The subarcs inherit the orientation of the parent arc. In the rest of the algorithm below, these subarcs will be simply referred to as extremal arcs.
   3.2. Prune one L-arc γ and one R-arc η if γ and η have the same endpoints, and if neither endpoint is special.
   3.3. Prune any L-arc (resp. R-arc) with head at a point p which is not special if there is no L-arc (resp., R-arc) with tail at p.
   3.4. Prune any extremal arc whose head or tail is at a point where no other arc is connected.
   3.5. Repeat steps 3.3-3.4 until there is not more arc to prune.
   3.6. Prune extremal arcs that spiral around limit sets in positive or negative time.
   3.7. Eliminate from the list of significant points all points with no arcs attached, and points connecting only two arcs of the same type (L or R).
4. **Graph construction**
   Construct a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V}$ the set of vertices of $\mathcal{G}$ and $\mathcal{E}$ the set of edges of $\mathcal{G}$ as follows.
   Vertices of $\mathcal{G}$. Let $\mathcal{P}$ denote the set of significant points in K that remain after the pruning in Part 3.
   4.1. For every point $p \in \mathcal{P}$ which is special, create a vertex v_p.
   4.2. For every $p \in \mathcal{P}$ which is not special, create two vertices, denoted v_p^L and v_p^R.
   Edges of $\mathcal{G}$. Create directed edges between vertices associated with extremal arcs and invariant arcs of ∂K as follows:
   4.3. If p is the tail of an L-arc or an invariant arc of ∂K with head at q, create a directed edge from v_p, or v_p^L, to v_q, or v_q^L.
   4.4. If p is the tail of an R-arc with head at q, create a directed edge from v_q, or v_q^R, to v_p, or v_p^R.
   4.5. For every (v_p^L, v_p^R) pair, create a directed edge from v_p^R to v_p^L.
5. **Cycle Analysis**
   5.1. Find all simple cycles (i.e., closed paths that do not visit any vertex more than once) in the graph $\mathcal{G}$.
   5.2. Discard any cycles containing two vertices v_p^R, v_p^L that are not consecutive (when travelling in the direction of the edges of the graph).
   5.3. For each remaining cycle in $\mathcal{G}$, check whether the region in the plane delimited by the path associated to the cycle is positively or negatively invariant. If it is negatively invariant, discard the cycle.



(a)

(b)

Fig. 1. Outcome of parts 2 (integration) and part 3 (pruning) of the invariance kernel algorithm.

5.4. $K^\star$ is the union of all regions enclosed by closed paths associated to graph cycles and by closed extremal trajectories in K.

Remark 8.1: The simple cycles of $\mathcal{G}$ can be efficiently found using Tarjan's algorithm in [18], which has polynomial complexity $O((V{\cdot}E)(C{+}1))$, where V, E, C are the number of vertices, edges, and simple cycles in $\mathcal{G}$. The test in step 5.3 can be done simply by picking any non-special point p in the closed path and discarding the cycle if $f_1(p)$ points outside the region delimited by the path.

IX. Example

Consider the planar system $\dot{x} = \lambda(t)f_1(x) + [1 - \lambda(t)]f_2(x)$, where

$$f_1(x) = \begin{bmatrix} x_2 \\ x_1^2 + x_1x_2 - 1 \end{bmatrix}, \quad f_2(x) = \begin{bmatrix} -x_1 + 2x_1^2x_2 \\ -3x_2 \end{bmatrix}.$$

Let K be the box $\{(x_1, x_2) : |x_1| \leq 2, |x_2| \leq 2\}$ with rounded corners displayed in Figure 1. The corners are rounded to meet the standing assumptions in Section III, but the invariance kernel algorithm can be applied with no modification even when K has C^0 boundary.

The vector field f_1 has two equilibria, a stable focus at $(-1, 0)$ and a saddle at $(1, 0)$, while f_2 has only one equilibrium at $(0, 0)$, a stable node. The stable focus is not considered to be a special point.

The outcome of the integration part of the algorithm is displayed in Figure 1(a). Solid dots in the figure indicate all significant points arising from endpoints of integration and intersections of various arcs. The outcome of the pruning part of the algorithm is displayed in Figure 1(b), where the arcs $\gamma_1, \ldots, \gamma_{19}$ have been pruned in five executions of steps 3.3, 3.4 of the algorithm. As a result of this pruning, in step 3.7 a number of significant points with no arcs attached or points connecting only two arcs of the same type are eliminated.

In part 4 of the algorithm we construct the invariance graph $\mathcal{G}$. It has 33 nodes and 49 edges. It is not displayed due to space limitations. There is only one closed extremal arc in K, namely the dashed curve containing point 11 in Figure 1(b). The resulting invariance kernel $K^\star$ is the shaded area in Figure 1(a).

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