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Stability of a Class of Linear Switching Systems with Applications to Two Consensus Problems

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***Abstract*— In this paper, we first establish a stability result for a class of linear switching systems involving Kronecker product. The problem is intriguing in that the system matrix does not have to be Hurwitz in any time instant. We have established the main result by a combination of the Lyapunov stability analysis and a generalized Barbalat's Lemma applicable to piecewise continuous linear systems. As applications of this stability result, we study both the leaderless consensus problem and the leader-following consensus problem for general marginally stable linear multi-agent systems under switching network topology. In contrast with many existing results, our result only assume that the dynamic graph is uniformly connected.**

I. INTRODUCTION

In this paper, we first consider the stability property of a class of linear switching systems:

$$\dot{x}(t) = \left(I_N \otimes A - F_{\sigma(t)} \otimes (BB^T P)\right) x(t), \quad \sigma(t) \in \mathcal{P} \tag{1}$$

where $\sigma(t): [0, +\infty) \to \mathcal{P} = \{1, 2, ..., \rho\}$ for some integer $\rho \geq 1$, is a right continuous piecewise constant switching signal whose switching instants $\{t_i : i = 0, 1, ...\}$ satisfies, for all $i \geq 1$, $t_i - t_{i-1} \geq \tau$ for a positive constant τ, $A \in R^{n\times n}$, $B \in R^{n\times m}$, and P is some positive definite matrix.

This class of linear switching systems arises in the consensus analysis of a general class of multi-agent systems

$$\dot{x}_i = Ax_i + Bu_i, \quad i = 1, 2, ..., N \tag{2}$$

where $x_i \in R^n$, $u_i \in R^m$ are the state and control of the agent i.

Two types of consensus problems can be associated with system (2). The first type is called leaderless consensus problem as studied in, e.g., [11], [19], [20], [21] and the second type is called leader-following consensus as studied in, e.g., [12]. As will be seen later, the solvability of both types of the consensus problems will rely on the asymptotic stability of a system of the form (1) where the matrix $F_{\sigma(t)}$ is determined by a dynamic graph that is dictated by a switching signal $\sigma(t)$. The fixed network can be viewed as a special case of the dynamic network with $\rho = 1$. In this case, system (1) reduces to a linear time-invariant system.

Denote $\bar{A}_{\sigma(t)} = I_N \otimes A - F_{\sigma(t)} \otimes (BB^T P)$. If there exists a positive definite matrix P such that

$$P\bar{A}_{\sigma(t)} + \bar{A}^T_{\sigma(t)} P < 0 \quad \forall\, t \geq 0,$$

then system (1) is asymptotically stable with $V(x) = x^T Px$ as the common Lyapunov function [13]. Other methods such as the multiple Lyapunov function [2], the generalized LaSalle's invariance principle [5] can only be used to determine the asymptotic stability of the linear switching systems for the case when $\bar{A}_{\sigma(t)}$ is Hurwitz. What challenges our problem is that we will not assume that the system matrix $\bar{A}_{\sigma(t)}$ is Hurwitz at any time instant. This case happens when dealing with the two consensus problems of system (2) without assuming the associated dynamic graph is connected at any time instant [8], [10], [12], [15].

Many efforts have been made on studying the stability of the linear switching systems of the form (1). For example, when $A = 0$, $B = 1$, $P = 1$, $F_{\sigma(t)} = \mathcal{L}_{\sigma(t)}$ where $\mathcal{L}_{\sigma(t)}$ is the Laplacian of the switching network graph, (1) represents the overall closed-loop system that consists of a distributed control protocol and a group of single integrators [8], [10], [14], [15].

More recently, when studying the the leader-following consensus for system (2) for the case where the graph is undirected and the leader is uniformly globally reachable by all followers in [12], the authors showed that the problem is solvable if a system of the form (1) is asymptotically stable and they further showed that (1) is asymptotically stable provided that there exists a positive definite matrix P satisfying the following two algebraic matrix inequalities:

$$PA + A^T P - 2\delta_{min} PBB^T P + \delta_{min} I < 0 \tag{3}$$

$$PA + A^T P \leq 0 \tag{4}$$

However, no discussion on the solvability of (3) and (4) was given in [12]. In fact, it can be verified that, for a simple forced harmonic system, i.e., a system with $A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the two inequalities (3) and (4) do not admit a solution.

Nevertheless, as the first main result of this paper, we will show that, under the same assumptions on $F_{\sigma(t)}$ as in [12], and the following assumption

Assumption 1: There exists a positive definite matrix P satisfying the linear matrix inequality

$$PA + A^T P \leq 0 \tag{5}$$

the linear switching system (1) is asymptotically stable. Thus, the constraint imposed by the inequality (3) can be removed. Clearly, Assumption 1 is weaker than the two inequalities (3) and (4). In fact, Assumption 1 always holds when A is Hurwitz or marginally stable, i.e., all the eigenvalues of A have negative real part or are semi-simple with zero real

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part. On the other hand, as will be pointed out in Remark 5, when A is neutrally stable, i.e., all the eigenvalues of A are semi-simple with zero real part, the two algebraic matrix inequalities usually do not admit a common positive definite solution.

As mentioned above, the stability analysis of the system of the form (1) is motivated by studying two types of consensus problems associated with the system (2) under the switching network topology. The first type consensus problem, i.e., the leaderless consensus problem was first studied for the fixed network in [11], [14], [17], [19], [20], and for the switching network in [8], [10], [14], [15], [17], [21]. For the switching network case, the investigations are mainly focused on the single integrator system [8], [10], [14], [15] or the harmonic system [17]. Even for the harmonic system, the result relies on a somehow restrictive assumption that the network graph is connected at any time instant. Recently, the authors in paper [21] studied the leaderless consensus of system (2) under the condition that the network is connected at any time instant or is frequently connected with time period T.[1] The second type consensus problem, i.e., the leader-following consensus problem has also been widely studied for both the fixed network in [12], [16], and the switching network in [6], [7], [8], [12]. Perhaps, the most general case was studied recently in [12]. However, as mentioned above, the result is hinged on the existence of a positive definite matrix P satisfying the two inequalities (3) and (4).

Nevertheless, our main result on the stability of system (1) has offered an opportunity to tackle both the leaderless and the leader-following consensus problems for the class of linear system (2) where the matrix A is marginally stable without assuming the associated switching network is connected at any time instant. As a matter of fact, for the leaderless case, we will solve the problem via the distributed state feedback protocol under the weaker assumption that the dynamic graph is uniformly connected, which means that the network can be disconnected at all time instants as opposed to the existing result in, say, [21]. On the other hand, for the leader-following case, under the same assumptions on the dynamic graph topology as those in [12], and without relying on the common solution of two inequalities (3) and (4), we will provide a solution for the problem via the distributed state feedback protocol.

The rest of this paper is organized as follows. In Section II, we study the stability of the linear switching system (1). In Section III, we present the problem formulation of consensus of multi-agent systems. In Section IV and V we study the leaderless consensus and the leader-following consensus via state feedback for the linear system (2), respectively.

Throughout this paper, $\otimes$ denotes the Kronecker product of matrices. $\mathbb{1}_N$ denotes a N dimensional column vector with all elements 1, $\lceil a \rceil$ denotes the largest integer less than the real number a, and $A^{\frac{1}{2}}$ denotes the square root of a positive semi-definite matrix A.

[1]The network topology is said to be frequently connected with time period T, if there exists a $T > 0$, such that for any $t > 0$, there exists a $t^* \in [t, t+T)$ such that the graph $\mathcal{G}(t^*)$ is connected.

II. Stability Analysis of the Linear Switching System (1)

In this section, we first present a generalized version of Babalat's Lemma which is applicable to piecewise continuous systems. Using this Lemma, we study the stability of linear switching system (1).

Lemma 1: [Generalized Babalat's Lemma] Let $t_i \in [0, +\infty)$, $i = 0, 1, 2, \cdots$, satisfying $t_0 = 0$, $t_{i+1} - t_i \geq \tau > 0$. Suppose $V(t)$: $[0, +\infty) \to R$ satisfies

1) $\lim_{t\to\infty} V(t)$ exists;
2) $V(t)$ is twice differentiable in each interval $[t_i, t_{i+1})$;
3) $\ddot{V}(t)$ is bounded over $[0, +\infty)$ in the sense that

$$\sup_{t_i \leq t < t_{i+1} \ i=0,1,2,\ldots} |\ddot{V}(t)| < +\infty.$$

Then $\dot{V}(t) \to 0$ as $t \to \infty$.

Due to the space limit, the proof of Lemma 1 is omitted here.

Remark 1: Since $V(t)$ can be piecewise continuous, Lemma 1 can be seen as an extension of Babalat's Lemma [18] to the class of piecewise continuous functions by allowing $\dot{V}(t)$ to be piecewise continuous and requiring $\ddot{V}(t)$ to be bounded.

Remark 2: Another version of the generalized Babalat's Lemma was given as Lemma 1 of [9]. However that version requires $V(t)$ to be continuous.

Applying Lemma 1 to non-increasing and lower bounded continuous function $V(t)$ gives the following result.

Corollary 1: Given the sequence $\{t_i\}$ with $t_0 = 0$, $t_{i+1} - t_i \geq \tau$, for some $\tau > 0$, $i = 0, 1, 2, \ldots$, suppose that a scalar continuous function $V(t)$, $t \in [0, \infty)$ satisfies

1) $V(t)$ is lower bounded;
2) $\dot{V}(t)$ is non-positive and differentiable in each interval $[t_i, t_{i+1})$;
3) $\ddot{V}(t)$ is bounded over $[0, +\infty)$ in the sense that

$$\sup_{t_i \leq t < t_{i+1} \ i=0,1,2,\ldots} |\ddot{V}(t)| < +\infty.$$

Then $\dot{V}(t) \to 0$ as $t \to \infty$.

Proof: Since $V(t)$ is lower bounded and continuous and $\dot{V}(t)$ is non-positive, $\lim_{t\to\infty} V(t)$ exist. Applying Lemma 1, the conclusion of Corollary 1 holds. □

Lemma 2: Suppose an integrable function $f(t)$: $[0, +\infty) \to R$ satisfies $\lim_{t\to\infty} f(t) = 0$. Then for any arbitrary positive number T_0, $\lim_{t\to\infty} \int_t^{t+T_0} f(s)ds = 0$.

The proof of Lemma 2 is obvious and is omitted here.

We now consider the stability property of the class of linear switching systems described by (1). For this purpose, let us first introduce the following definition:

Definition 1 (see in [6], [8]): A time signal $\sigma : [0, \infty) \to \mathcal{P}$ where $\mathcal{P} = \{1, 2, \cdots, \rho\}$ for some integer $\rho > 0$ is said to be *a piecewise constant switching signal with dwell time* τ if there exists a sequence $\{t_i : i = 0, 1, \ldots\}$ satisfying, for all $i \geq 1$, $t_i - t_{i-1} \geq \tau$ for some positive constant τ such that, over each interval $[t_i, t_{i+1})$, $\sigma(t) = p$ for some integer $1 \leq p \leq \rho$. $t_0, t_1, t_2, \ldots$ are called switching instants.

Now we present our main result:

Theorem 1: Consider the linear switching system (1), where $\sigma(t)$ is a piecewise constant switching signal with dwell time τ, $F_{\sigma(t)}$ is symmetric and positive semi-definite for any $t \geq 0$, and the pair (A, B) is controllable with A satisfying Assumption 1. Then

(i) If some solution $x(t)$ of (1) has the property that there exists a subsequence $\{i_k\}$ of $\{i : i = 0, 1, ...\}$ with $t_{i_{k+1}} - t_{i_k} < \nu$ for some positive ν such that $x(t_{i_k})$ is orthogonal to the null space of the matrix $(\sum_{q=i_k}^{i_{k+1}-1} F_{\sigma(t_q)}) \otimes I_n$, then

$$\lim_{t\to\infty} x(t) = 0. \tag{6}$$

(ii) If there exists a subsequence $\{i_k\}$ of $\{i : i = 0, 1, ...\}$ with $t_{i_{k+1}} - t_{i_k} < \nu$ for some positive ν such that the matrix $(\sum_{q=i_k}^{i_{k+1}-1} F_{\sigma(t_q)})$ is nonsingular, then the origin of the system (1) is asymptotically stable.

Proof: Part (i): Let $\xi(t) = (I_N \otimes P^{\frac{1}{2}})x(t)$, $\bar{A} = P^{\frac{1}{2}} A P^{-\frac{1}{2}}$, $\bar{B} = P^{\frac{1}{2}} B$. Then system (1) is equivalent to

$$\dot{\xi}(t) = \left(I_N \otimes \bar{A} - F_{\sigma(t)} \otimes (\bar{B}\bar{B}^T)\right)\xi(t), \quad \sigma(t) \in \mathcal{P} \tag{7}$$

Let

$$V(\xi(t)) = \frac{1}{2}\xi^T(t)\xi(t). \tag{8}$$

Then, the derivative of $V(\xi(t))$ along system (7) exists on every interval $[t_i, t_{i+1})$, $i = 0, 1, 2, \cdots$, and is given by, noting $\bar{A}^T + \bar{A} \leq 0$,

$$\begin{aligned} \dot{V}(\xi(t))\mid_{(7)} &= \frac{1}{2}\xi^T(t)\left((I_N \otimes \bar{A} - F_{\sigma(t)} \otimes (\bar{B}\bar{B}^T))^T \right. \\ &\quad \left. + (I_N \otimes \bar{A} - F_{\sigma(t)} \otimes (\bar{B}\bar{B}^T)\right)\xi(t) \\ &\leq -\xi^T(t)(F_{\sigma(t)} \otimes (\bar{B}\bar{B}^T))\xi(t) \leq 0 \end{aligned} \tag{9}$$

So $V(\xi(t)) \leq V(\xi(0))$, i.e. $||\xi(t)|| \leq ||\xi(0)||$ for any $t \geq 0$. Since $\sigma(t) \in \mathcal{P}$ and $\mathcal{P}$ is a finite set, $||F_{\sigma(t)} \otimes I_n||$, $||F^{\frac{1}{2}}_{\sigma(t)} \otimes I_n||$, $|F_{\sigma(t)} \otimes (\bar{B}\bar{B}^T)||$ and $||I_N \otimes \bar{A} - F_{\sigma(t)} \otimes (\bar{B}\bar{B}^T)||$ are all bounded. Then by (7), $||\dot{\xi}(t)||$ is bounded, so is $\ddot{V}(\xi(t))$. Therefore, by Corollary 1, $\lim_{t\to\infty} \dot{V}(\xi(t)) = 0$, i.e., $\lim_{t\to\infty} \xi^T(t)(F_{\sigma(t)} \otimes (\bar{B}\bar{B}^T))\xi(t) = 0$ which in turn implies $\lim_{t\to\infty}(F^{\frac{1}{2}}_{\sigma(t)} \otimes \bar{B}^T)\xi(t) = 0$. Since $||(F^{\frac{1}{2}}_{\sigma(t)} \otimes I_n)||$ is bounded,

$$\lim_{t\to\infty}(F_{\sigma(t)} \otimes \bar{B}^T)\xi(t) = 0. \tag{10}$$

We will show that (10) implies (6) by the following two steps:

Step-1: We first show that (10) implies

$$\lim_{t\to\infty}(F_{\sigma(t)} \otimes I_n)\xi(t) = 0. \tag{11}$$

In fact, (10) is equivalent to

$$\lim_{t\to\infty}(I_N \otimes (-\bar{B}^T))(F_{\sigma(t)} \otimes I_n)\xi(t) = 0. \tag{12}$$

Note that $\xi(t)$ is bounded. Let $\eta(t) = (F_{\sigma(t)} \otimes I_n)\xi(t)$. Then $\eta(t)$ is also bounded, and the derivative of $\eta(t)$ that exists on every interval $[t_i, t_{i+1})$, $i = 0, 1, 2, \cdots$, is given by

$$\begin{aligned} \dot{\eta}(t) &= (F_{\sigma(t)} \otimes I_n)(I_N \otimes \bar{A} - F_{\sigma(t)} \otimes (\bar{B}\bar{B}^T))\xi(t) \\ &= (I_N \otimes \bar{A} - F_{\sigma(t)} \otimes (\bar{B}\bar{B}^T))(F_{\sigma(t)} \otimes I_n)\xi(t) \\ &= (I_N \otimes \bar{A} - F_{\sigma(t)} \otimes (\bar{B}\bar{B}^T))\eta(t) \end{aligned} \tag{13}$$

By (12),

$$\lim_{t\to\infty}(I_N \otimes (-\bar{B}^T))\eta(t) = 0. \tag{14}$$

Applying Lemma 1 to each component of the vector $(I_N \otimes (-\bar{B}^T))\eta(t)$ leads to $\lim_{t\to\infty}(I_N \otimes (-\bar{B}^T))\dot{\eta}(t) = 0$. Combining (13), (14) and the fact that $||F_{\sigma(t)} \otimes I_n||$ is bounded, we get

$$\begin{aligned} &\lim_{t\to\infty}(I_N \otimes (-\bar{B}^T))(I_N \otimes \bar{A})\eta(t) \\ &= \lim_{t\to\infty}(I_N \otimes (-\bar{B}^T))\left[\dot{\eta}(t) + (F_{\sigma(t)} \otimes (\bar{B}\bar{B}^T))\eta(t)\right] \\ &= 0 + 0 = 0. \end{aligned} \tag{15}$$

Repeating this process, we have

$$\lim_{t\to\infty}(I_N \otimes (-\bar{B}^T))(I_N \otimes \bar{A})^k \eta(t) = 0, \;\; k = 2, ..., nN - 1 \tag{16}$$

Since the pair (A, B) is controllable, so is $(\bar{A}, \bar{B})$. Thus the pair $(-\bar{B}^T, \bar{A})$ is observable, so is the pair $(I_N \otimes (-\bar{B}^T), I_N \otimes \bar{A})$. Therefore by (14), (15) and (16), we can obtain $\lim_{t\to\infty} \eta(t) = 0$, i.e., (11) holds.

Step-2: We now show that (11) implies (6). For any $t \in [t_{i_k}, t_{i_{k+1}})$. Let t_{i_k}, $t_{i_k+1}, ..., t_{i_{k+1}-1}$ be the switching instants in $[t_{i_k}, t_{i_{k+1}})$. Then (11) implies

$$\lim_{k\to\infty}(F_{\sigma(t_{i_k+j})} \otimes I_n)\xi(t_{i_k+j}) = 0. \quad j = 0, 1, ..., p_k \tag{17}$$

where $p_k \triangleq i_{k+1} - i_k - 1 \leq \lceil \frac{\nu}{\tau} \rceil + 1$. Since, for $j = 1, ..., p_k$,

$$\xi(t_{i_k+j}) = e^{(I_N \otimes \bar{A})(t_{i_k+j} - t_{i_k+j-1})}\xi(t_{i_k+j-1}) + \Delta_j(k) \tag{18}$$

where

$$\Delta_j(k) = \int_{t_{i_k+j-1}}^{t_{i_k+j}} e^{(I_N \otimes \bar{A})(t_{i_k+j} - s)}(-F_{\sigma(s)} \otimes (\bar{B}\bar{B}^T))\xi(s)ds.$$

Let $\Upsilon = \max_{t\in[\tau,\nu]} ||e^{(I_N \otimes \bar{A})t}||$. Then Υ is finite since $||e^{(I_N \otimes \bar{A})t}||$ is continuous on $[\tau, \nu]$. Since $\tau \leq t_{i_k+j} - t_{i_k+j-1} \leq t_{i_{k+1}} - t_{i_k} < \nu$, we have

$$||\Delta_j(k)|| \leq \Upsilon \int_{t_{i_k+j-1}}^{t_{i_k+j}} ||(I_N \otimes \bar{B})(F_{\sigma(s)} \otimes \bar{B}^T)\xi(s)||ds$$

By (10), and Lemma 2,

$$\lim_{k\to\infty} \Delta_j(k) = 0 \tag{19}$$

Since the product of $(I_N \otimes \bar{A})$ and $(F_{\sigma(t_{i_k+1})} \otimes I_n)$ is commutative, so is the product of $e^{(I_N \otimes \bar{A})(t_{i_k+1} - t_{i_k})}$ and $(F_{\sigma(t_{i_k+1})} \otimes I_n)$. Then by (17), (18), (19), and the fact that $||F_{\sigma(t_{i_k+1})} \otimes I_n||$ is bounded, we have

$$\begin{aligned} &\lim_{k\to\infty} e^{(I_N \otimes \bar{A})(t_{i_k+1} - t_{i_k})}(F_{\sigma(t_{i_k+1})} \otimes I_n)\xi(t_{i_k}) \\ &= \lim_{k\to\infty}(F_{\sigma(t_{i_k+1})} \otimes I_n)e^{(I_N \otimes \bar{A})(t_{i_k+1} - t_{i_k})}\xi(t_{i_k}) \\ &= \lim_{k\to\infty}(F_{\sigma(t_{i_k+1})} \otimes I_n)(\xi(t_{i_k+1}) - \Delta_1(k)) \\ &= 0 - 0 = 0 \end{aligned}$$

Since $\max_{t\in[\tau,\nu]} ||e^{-(I_N \otimes \bar{A})t}||$ is finite as $||e^{-(I_N \otimes \bar{A})t}||$ is continuous on $[\tau, \nu]$, we have

$$\lim_{k\to\infty}(F_{\sigma(t_{i_k+1})} \otimes I_n)\xi(t_{i_k}) = 0 \tag{20}$$

Repeating this process we can obtain

$$\lim_{k\to\infty}(F_{\sigma(t_{i_k+j})}\otimes I_n)\xi(t_{i_k})=0,\quad j=0,...,p_k. \tag{21}$$

Let $J_k=(\sum_{q=i_k}^{i_{k+1}-1}F_{\sigma(t_q)})\otimes I_n$ and $\zeta_k=J_k\xi(t_{i_k})$. Then (21) leads to

$$\lim_{k\to\infty}\zeta_k=0 \tag{22}$$

Since $x(t_{i_k})$ and hence $\xi(t_{i_k})$ is orthogonal to the null space of J_k, we have $\xi(t_{i_k})=J_k^{\dagger}\zeta_k$ where $(J_k)^{\dagger}$ is the Moore-Penrose inverse of J_k [1]. Thus, $||\xi(t_{i_k})||\leq||J_k^{\dagger}||||\zeta_k||$. Since $\sum_{q=i_k}^{i_{k+1}-1}F_{\sigma(t_q)}=\sum_{q=i_k}^{i_k+p_k}F_{\sigma(t_q)}$, $p_k\leq\lceil\frac{\nu}{\tau}\rceil+1$, and $\mathcal{P}$ is a finite set, the set $\{J_k^{\dagger},k=1,2,\cdots,\}$ contains only finitely many distinct real numbers. Thus, there exists a finite real number J such that, for all $k=1,2,\cdots$, $||J_k^{\dagger}||\leq J$. Then $||\xi(t_{i_k})||\leq J||\zeta_k||$ Therefore, (22) implies

$$\lim_{k\to\infty}\xi(t_{i_k})=0. \tag{23}$$

By (9), $||\xi(t)||$ is non-increasing as $t\to\infty$. This fact together with (23) concludes $\lim\limits_{t\to\infty}\xi(t)=0$. Thus,

$$\lim_{t\to\infty}x(t)=\lim_{t\to\infty}(I_N\otimes P^{-\frac{1}{2}})\xi(t)=0.$$

Part (ii): Since J_k is nonsingular, the Moore-Penrose inverse of J_k becomes the inverse of J_k. We still have (23), and hence, any solution of (1) will approach the origin asymptotically. Therefore, the origin of system (1) is asymptotically stable. □

Remark 3: Our result is in contrast with the recent result in [3]where the stability property of a linear switching system of the form $\dot{x}=A_{\sigma(t)}x$, where $\sigma(t)\in\mathcal{P}$, with $\mathcal{P}$ finite, and $A_{\sigma(t)}$ may not be Hurwitz is thoroughly studied. The result in [3] relies on the existence of the so-called common joint quadratic Lyapunov function[2][3]. However, by (9) and some direct calculations, we can conclude that the quadratic function (8) cannot be a common joint quadratic Lyapunov function for system (1) unless B is of full row rank.

III. Preliminaries of Consensus Problem

In this section, we first summarize some terminologies on a graph, and then present the problem formulation of both leaderless consensus and leader-following consensus of linear multi-agent systems.

A. Graph

We first introduce some graph notations which can be found in [4]. A graph $\mathcal{G}=(V,E)$ consists of a finite node set $V=\{1,...,N\}$ and an edge set $E=\{(i,j):\ i,j\in V\}\subseteq V\times V$. If edge $(i,j)\in E$, node i is called a neighbor of node j. N_i denotes the set of labels of those nodes that are neighbors of the node i. If the graph $\mathcal{G}$ contains a sequence of edges of the form $(i_1,i_2),(i_2,i_3),\cdots,(i_k,i_{k+1})$, then the set $\{(i_1,i_2),(i_2,i_3),\cdots,(i_k,i_{k+1})\}$ is called a path of $\mathcal{G}$ from i_1 to i_{k+1} and node i_1 is said to be reachable from node i_{k+1}.

[2]The function $V(x)=x^TPx$ with P a positive definite matrix is called a common joint quadratic Lyapunov function if P has the property that $PA_i+A_i^TP=-Q_i\leq 0,\quad\forall\, i\in\mathcal{P}$, and $Q\triangleq\sum_{i=1}^{N}Q_i>0$.

The edge (i,j) is called undirected if $(i,j)\in E$ implies $(j,i)\in E$. The graph is called undirected if every edge in E is undirected. A graph $\mathcal{G}$ is called strongly connected if there exists a path between any two distinct nodes. A undirected and strongly connected graph $\mathcal{G}$ is called connected. A graph $\mathcal{G}_s=(V_s,E_s)$ is a subgraph of $\mathcal{G}=(V,E)$ if $V_s\subseteq V$ and $E_s\subseteq V_s\times V_s$.

The weighted adjacency matrix of a graph $\mathcal{G}$ is a nonnegative matrix $\mathcal{A}=[a_{ij}]\in R^{N\times N}$, where $a_{ii}=0$ and $a_{ij}>0$ $\Leftrightarrow$ $(j,i)\in E$. The Laplacian of a graph $\mathcal{G}$ is denoted by $\mathcal{L}=[l_{ij}]\in R^{N\times N}$, where $l_{ii}=\sum_{j=1}^{N}a_{ij}$ and $l_{ij}=-a_{ij}$ if $i\neq j$.

Remark 4: Clearly, $\mathcal{L}\mathbb{1}_N=0$ where $\mathbb{1}_N$ is an $N\times 1$ column vector whose elements are all 1. Moreover, $\mathcal{L}$ is symmetric and positive semi-definite if and only if the graph $\mathcal{G}$ is undirected [4], and $\mathcal{L}$ has exactly one zero eigenvalue and its null space is $span\{\mathbb{1}_N\}$ if $\mathcal{G}$ is strongly connected [14].

Given a set of r graphs $\mathcal{G}_k=(V,E_k),\ k=1,...,r$ with the same node set, the graph $\mathcal{G}=(V,E)$ where $E=\bigcup_{k=1}^{r}E_k$ is called the union of $\mathcal{G}_k$, denoted by $\bigcup_{k=1}^{r}\mathcal{G}_k$. Given a node set $V=\{1,...,N\}$ and a piecewise constant switching signal σ, we can define a time-varying graph $\mathcal{G}_{\sigma(t)}=(V,E_{\sigma(t)})$ where $E_{\sigma(t)}\subseteq V\times V$ for all $t\geq 0$. $\mathcal{G}_{\sigma(t)}$ will be called a dynamic graph. Let $\mathcal{G}([t,t+s))$ be the union graph of all the graphs $\mathcal{G}_{\sigma(t)}$ over the time interval $[t,t+s)$, $s>0$, i.e.,

$$\mathcal{G}([t,t+s))=\bigcup_{t_i\in[t,t+s)}\mathcal{G}_{\sigma(t)}.$$

B. Two Consensus Problems

We now introduce both the leaderless consensus problem and the leader-following consensus problem of multi-agent systems of the form (2) under the switching network. We assume that (A,B) is controllable and Assumption 1 holds. Associated with system (2), we can define a dynamic graph $\mathcal{G}_{\sigma(t)}=(V,E_{\sigma(t)})$ where $V=\{1,\ldots,N\}$ and $(j,i)\in E_{\sigma(t)}$ if and only if the control u_i can make use of x_j for feedback at time t.

Denote the weighted adjacency matrix of the dynamic graph $\mathcal{G}_{\sigma(t)}$ by $\mathcal{A}_{\sigma(t)}=[a_{ij}(t)]_{N\times N}$. Then we can define the distributed state feedback protocol as follows:

$$u_i(t)=K\sum_{j=1}^{N}a_{ij}(t)(x_j(t)-x_i(t))\quad i=1,...,N \tag{24}$$

where K is some gain matrix to be defined later.

Definition 2 (Leaderless Consensus Problem): Given the system (2) and a dynamic graph $\mathcal{G}_{\sigma(t)}$, find the distributed state feedback protocol (24) such that, for $i,j=1,...,N$, $x_i(t)-x_j(t)\to 0$ as $t\to\infty$.

In the leaderless consensus problem described above, the steady-state behavior of the solution of each subsystem is immaterial. There is another type of consensus problem called leader-following consensus problem where the solution of each subsystem is required to approach some signal $x_0(t)$ asymptotically. In this paper, we assume that the signal $x_0(t)$

is generated by a linear system of the form:

$$\dot{x}_0 = Ax_0 \tag{25}$$

where $x_0 \in R^n$ is the state of the system.

In what follows, we call system (2) and system (25) follower system and leader system, respectively. Associated with system (2) and system (25), we define another dynamic graph $\bar{\mathcal{G}}_{\sigma(t)} = \{\bar{V}, \bar{E}_{\sigma(t)}\}$ where $\bar{V} = \{0, 1, ..., N\}$ and $(j,i) \in \bar{E}_{\sigma(t)}$ if and only if the control u_i can make use of x_j for feedback at time t. Clearly, $\mathcal{G}_{\sigma(t)}$ is a subgraph of $\bar{\mathcal{G}}_{\sigma(t)}$ and can be obtained from $\bar{\mathcal{G}}_{\sigma(t)}$ by removing the node 0 from $\bar{V}$ and all edges incident on node 0 at time t from $\bar{E}_{\sigma(t)}$. Let $\Delta_{\sigma(t)}$ be an $N \times N$ nonnegative diagonal matrix whose ith diagonal element is $b_i(t)$, where $b_i(t) > 0$ if $(0,i) \in \bar{E}_{\sigma(t)}$ and $b_i(t) = 0$ if otherwise. We define the matrix $H_{\sigma(t)} = \mathcal{L}_{\sigma(t)} + \Delta_{\sigma(t)}$. Then $H_{\sigma(t)}$ is symmetric and positive semi-definite for any $t \geq 0$ if $\mathcal{L}_{\sigma(t)}$ is. Since $\mathcal{L}_{\sigma(t)} \mathbb{1}_N = 0$, we have $H_{\sigma(t)} \mathbb{1}_N = \Delta_{\sigma(t)} \mathbb{1}_N$ for any $t \geq 0$.

We consider the distributed state feedback protocol as follows:

$$\begin{aligned} u_i = K \left(\textstyle\sum_{j=1}^{N} a_{ij}(t)(x_j - x_i) + b_i(t)(x_0 - x_i) \right) \\ i = 1, ..., N \end{aligned} \tag{26}$$

where K is some gain matrix to be defined later.

Definition 3 (Leader-Following Consensus Problem): Given the leader system (25), the follower system (2) and a dynamic graph $\bar{\mathcal{G}}_{\sigma(t)}$, find the distributed state feedback protocol (26) such that, for $i = 1, ..., N$, $x_i(t) - x_0(t) \to 0$ as $t \to \infty$.

Remark 5: In paper [12], the authors considered the distributed state feedback protocol (26) with $K = B^T P$, where P is the common positive definite solution of inequalities (3) and (4). However, when A is neutrally stable, the inequalities (3) and (4) usually do not have a common positive definite solution. In fact, if a positive definite matrix P satisfies (4), it must satisfy $PA + A^T P = 0$. Thus, the inequality (3) reduces to $-2PBB^T P + I < 0$, which is possible only if B is of full row rank. We will show later that the two inequality constraints (3) and (4) can be reduced to one simple inequality $PA + A^T P \leq 0$ which always has a positive definite solution P when A is neutrally stable.

IV. Solvability of Leaderless Consensus

We need the following assumptions.

Assumption 2: The dynamic graph $\mathcal{G}_{\sigma(t)}$ is undirected for any $t \geq 0$.

Assumption 3: There exists a subsequence $\{i_k\}$ of $\{i : i = 0, 1, ...\}$ with $t_{i_{k+1}} - t_{i_k} < \nu$ for some positive ν such that the union graph $\mathcal{G}([t_{i_k}, t_{i_{k+1}}))$ is connected.

Remark 6: (i) Under Assumption 2, the Laplacian $\mathcal{L}_{\sigma(t)}$ of $\mathcal{G}_{\sigma(t)}$ is symmetric and positive semi-definite for any $t \geq 0$.

(ii) If a dynamic graph satisfies Assumption 3, it is called uniformly connected over $[0, +\infty)$ [10], or jointly connected over $[t_{i_k}, t_{i_{k+1}})$ in [8].

Remark 7: Let $\mathcal{L}^k = \sum_{q=i_k}^{i_{k+1}-1} \mathcal{L}_{\sigma(t_q)} = [l_{ij}^k]$. Then, for all $i, j = 1, \cdots, N$, $l_{ii}^k \leq 0$, $l_{ij}^k \geq 0$ when $i \neq j$, and $\sum_{j=1}^{N} l_{ij}^k = 0$. Thus, the matrix $\mathcal{L}^k$ can be viewed as a Laplacian of the union graph $\bigcup_{q=i_k}^{i_{k+1}-1} \mathcal{G}_{\sigma(t_q)} = \mathcal{G}([t_{i_k}, t_{i_{k+1}}))$. By Remark 4, Assumption 2 implies $\mathcal{L}^k$ is positive semi-definite, and Assumption 3 implies $\mathcal{L}^k$ has exactly one zero eigenvalue and its null space is $span\{\mathbb{1}_N\}$.

Theorem 2: Under Assumptions 1, 2 and 3, the leaderless consensus problem is solvable by the distributed control protocol (24) with the gain matrix $K = B^T P$, where P is the positive definite solution of (5).

Proof: Under the distributed state feedback control protocol (24), the closed-loop system of agent i is

$$\dot{x}_i = Ax_i + BB^T P \sum_{j=1}^{N} a_{ij}(t)(x_j - x_i) \tag{27}$$

Denote by $x_c(t)$ the center of all agents at time t. which is

$$x_c(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t).$$

Since the network graph is undirected, by direct calculation, we have

$$\dot{x}_c(t) = \frac{\sum_{i=1}^{N} \dot{x}_i(t)}{N} = A \left(\frac{\sum_{i=1}^{N} x_i(t)}{N} \right) = Ax_c(t) \tag{28}$$

Like [21], we define a decomposition of $x_i(t)$ as

$$x_i(t) = x_c(t) + w_i(t) \quad i = 1, ..., N \tag{29}$$

where $x_i(t)$, $i = 1, ..., N$ is the state of agent i. (29) is equivalent to the following compact form

$$x(t) = \mathbb{1}_N \otimes x_c(t) + w(t) \tag{30}$$

where $x(t) = [x_1(t)^T, x_2(t)^T, ..., x_N(t)^T]^T$ and $w(t) = [w_1(t)^T, w_2(t)^T, ..., w_N(t)^T]^T$. The vector $w(t)$ is called the (group) disagreement vector.

By (27), (28) and (30), the overall closed-loop system with respect to the disagreement vector $w(t)$ can be written in the form

$$\dot{w}(t) = \left(I_N \otimes A - \mathcal{L}_{\sigma(t)} \otimes (BB^T P) \right) w(t) \tag{31}$$

Since $\sum_{i=1}^{N} w_i(t) = \sum_{i=1}^{N} x_i(t) - Nx_c(t) = 0$, $w(t)$ is orthogonal to $span\{\mathbb{1}_N \otimes I_n\}$ for any $t \geq 0$. By Remark 7, the null space of $\sum_{q=i_k}^{i_{k+1}-1} \mathcal{L}_{\sigma(t_q)}$ is $span\{\mathbb{1}_N\}$. So the null space of $(\sum_{q=i_k}^{i_{k+1}-1} \mathcal{L}_{\sigma(t_q)}) \otimes I_n$ is $span\{\mathbb{1}_N \otimes I_n\}$. Thus, $w(t_{i_k})$ is orthogonal to the null space of $(\sum_{q=i_k}^{i_{k+1}-1} \mathcal{L}_{\sigma(t_q)}) \otimes I_n$. Then by Theorem 1, $\lim_{t \to \infty} w(t) = 0$. Therefore, all the state $x_i(t)$ asymptotically converge to $x_c(t)$. The leaderless consensus is thus achieved. □

Remark 8: When $A = 0$ and $B = 1$, the plant (2) reduces to the single-integrator system. Obviously, this system satisfies Assumption 1. Thus, the leaderless consensus problem described in Definition 1 is solvable provided that the dynamic graph $\mathcal{G}_{\sigma(t)}$ is undirected and uniformly connected. In fact, by Theorem 2, with $K = 1$, the state feedback protocol is given by $u_i = \sum_{j=1}^{N} a_{ij}(t)(x_j(t) - x_i(t))$ and the closed-loop system is $\dot{x} = -\mathcal{L}_{\sigma(t)} x$ This problem has been well studied in [8], [10], [14], [15].

When $A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the plant (2) reduces to multiple harmonic oscillators. This system also satisfies Assumption 1. Thus, the leaderless consensus problem is also solvable provided that the dynamic graph $\mathcal{G}_{\sigma(t)}$ is undirected and uniformly connected. In fact, with $P = \begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix}$ and the control gain $K = B^T P = \begin{pmatrix} 0 & 1 \end{pmatrix}$, the state feedback protocol (24) solves the problem. This result extends the result in [17] where it is assumed that the network graph is directed and strongly connected at any time instant.

Remark 9: Theorem 2 also extends the result in [21] for the case A is marginal stable in the sense that the uniform connectivity is strictly weaker than connectivity or frequent connectivity.

V. Solvability of Leader-following Consensus

We need the following assumption and lemma:

Assumption 4: There exists a subsequence $\{i_k\}$ of $\{i : i = 0, 1, ...\}$ with $t_{i_{k+1}} - t_{i_k} < \nu$ for some positive ν such that every node is reachable form the node 0 in the union graph $\bar{\mathcal{G}}([t_{i_k}, t_{i_{k+1}}))$.

Lemma 3 (see in [7]): The matrix $H = \mathcal{L} + \Delta$ has all eigenvalues with positive real parts if and only if every node is reachable form the node 0 in $\bar{\mathcal{G}}$.

Remark 10: By Remark 7, $\sum_{q=i_k}^{i_{k+1}-1} \mathcal{L}_{\sigma(t_q)}$ can be view as a Laplacian of $\mathcal{G}([t_{i_k}, t_{i_{k+1}}))$. On the other hand, $\sum_{q=i_k}^{i_{k+1}-1} \Delta_{\sigma(t_q)}$ is a nonnegative diagonal matrix whose jth diagonal element is positive if and only if $(0, j) \in \bigcup_{q=i_k}^{i_{k+1}-1} \bar{E}_{\sigma(t_q)}$. Since $\sum_{q=i_k}^{i_{k+1}-1} H_{\sigma(t_q)} = \sum_{q=i_k}^{i_{k+1}-1} \mathcal{L}_{\sigma(t_q)} + \sum_{q=i_k}^{i_{k+1}-1} \Delta_{\sigma(t_q)}$, by Lemma 3, under Assumptions 2 and 4, the matrix $\sum_{q=i_k}^{i_{k+1}-1} H_{\sigma(t_q)}$ is positive definite. Therefore, $(\sum_{q=i_k}^{i_{k+1}-1} H_{\sigma(t_q)}) \otimes I_n$ is nonsingular.

Theorem 3: Under Assumptions 1, 2 and 4, the leader-following consensus problem is solvable by the distributed state feedback protocol (26) with the gain matrix $K = B^T P$, where P is the positive definite solution of (5).

Proof: By distributed state feedback control protocol (26), the closed-loop system of agent i is

$$\dot{x}_i = Ax_i + BB^T P \left(\sum_{j=1}^{N} a_{ij}(t)(x_j - x_i) + b_i(t)(x_0 - x_i) \right)$$

Let $\xi_i = x_i - x_0$, $\xi = [\xi_1^T, ..., \xi_N^T]^T$. Then the overall closed-loop system can be written in the form

$$\dot{\xi}(t) = \left(I_N \otimes A - H_{\sigma(t)} \otimes (BB^T P) \right) \xi(t) \qquad (32)$$

Under Assumptions 2 and 4, by Remark 10 and Theorem 1, we have $\lim_{t \to \infty} \xi(t) = 0$. Therefore, all the state $x_i(t)$ asymptotically converge to $x_0(t)$. The leader-following consensus is thus achieved. □

Remark 11: When $A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the plant (2) reduces to the controlled harmonic oscillator and the the leader is an undamped harmonic oscillator. Since A satisfies Assumption 1, by Theorem 3, the leader-following consensus problem is solvable by the state feedback protocol (26) with the control gain $K = \begin{pmatrix} 0 & 1 \end{pmatrix}$ provided that the dynamic graph satisfies Assumptions 2 and 4.

It is noted that, by Remark 5, the two inequalities (3) and (4) do not have a common positive definite solution P for any $\delta_{min} > 0$. Therefore, the approach in [12] cannot solve the same leader-following consensus problem.

References

[1] A. Ben-Israel, T. N. E. Greville, *Generalized Inverse: Theory and Applications*, second ed., Springer, NewYork, 2003.

[2] M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Trans. Automat. Control*, vol 43, no. 4, pp. 475-482, Apr. 1998.

[3] D. Cheng, J. Wang, and X. Hu, "An extension of LaSalle's invariance principle and its application to multi-agent consensus," *IEEE Trans. Automat. Control*, vol 53, no. 7, pp. 885-890, Jul. 2008.

[4] C. Godsil and G. Royle, *Algebraic Graph Theory,* New York: Springer-Verlag, 2001.

[5] J. P. Hespanha, "Uniform stability of switched linear systems: Extensions of LaSalle's invariance principle," *IEEE Trans. Automat. Control*, vol. 49, no. 4, pp. 470-482, Apr. 2004.

[6] Y. Hong, J. Hu, and L. Gao, "Tracking control for multiagent consensus with an active leader and variable topology," *Automatica*, vol 42, no. 7, pp. 1177-1182, 2006.

[7] J. Hu and Y. Hong, "Leader-following coordination of multi-agent systems with coupling time delays," *Physica A: Statistical Mechanics and its Applications*, vol 374, no. 2, pp. 853-863, 2007.

[8] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile agents using nearest neighbor rules," *IEEE Trans. Automat. Control*, vol 48, no. 6, pp. 988-1001, Jun. 2003.

[9] H. Jiang, "Hybrid adaptive and impulsive synchronisation of uncertain complex dynamical networks by the generalised Barbalats lemma," *IET Control Theory and Applications*, vol 3, no. 10, pp. 1330-1340, 2009.

[10] Z. Lin, *Coupled Dynamic Systems: From Structure Towards Stability and Stabilizability*, Ph.D. Dissertation, University of Toronto, Toronto, Canada, 2005.

[11] C. Ma and J. Zhang, "Necessary and sufficient conditions for consensusability of linear multi-agent systems," *IEEE Trans. Automat. Control*, vol. 55, no. 5, 1263-1268, May 2010.

[12] W. Ni and D. Cheng, "Leader-following consensus of multi-agent systems under fixed and switching topologies," *Systems and Control Letters*. vol. 59, no. 3, 209-217, 2010.

[13] T. Ooba and Y. Funahashi, "On a common quadratic Lyapunov function for widely distant system," *IEEE Trans. Automat. Control*, vol. 42, no. 12, 1697-1699, Dec. 1997.

[14] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Automat. Control*, vol. 49, no. 9, 1520-1533, Sept. 2004.

[15] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Automat. Control*. vol. 50, no. 5, 655-661, May 2005.

[16] W. Ren and R. W. Beard. *Distributed Consensus in Multi-vehicle Cooperative Control*, Communications and Control Engineering Series, London: Springer-Verlag, 2008.

[17] W. Ren, "Synchronization of coupled harmonic oscillators with local interaction," *Automatica*, vol. 44, no. 12, 3195-3200, 2008.

[18] J. J. Slotine and W. Li, *Applied Nonlinear Control*, Prentice Hall Englewood Cliffs, 1991.

[19] S. E. Tuna, "LQR-based coupling gain for synchronization of linear systems", Available at: http://arxiv.org/abs/0801.3390

[20] S. E. Tuna, "Conditions for synchronizability in arrays of coupled linear systems," *IEEE Trans. Automat. Control*, vol. 54, no. 10, 2416-2240, Oct. 2009.

[21] J. Wang, D. Cheng, and X. Hu, "Consensus of multi-agent linear dynamic systems," *Asian Journal of Control*, vol. 10, no. 2, 144-155, 2008.