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Robust Nonlinear Least Squares via Consecutive LMI Optimizations

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Abstract— **An algorithm is developed for robust nonlinear least-squares optimization in which the function to be minimized has dependence on an uncertain parameter. The goal is to minimize the worst-case norm-square of the function, under the assumption that the uncertain parameter can take any value from a given compact region. The algorithm simply replaces the quadratic optimization in the Gauss-Newton update scheme with a robust linear matrix inequality (LMI) optimization step.**

I. Introduction

Robust optimization has evolved significantly in the last two decades and has strong relevance for engineering problems [1]. The key concern in robust optimization is to achieve robustness against uncertainty in the data of the considered problem. As a part of the related development, optimization based on linear matrix inequalities has spurred the field of robust control [6], since such problems can now be solved efficiently with the currently existing software [2], [7]. It is now well-established how the robust linear least-squares (LLS) problem can be reformulated in the form of an LMI optimization (see [6]). The reformulation is equivalent to the original problem when the dependency on the uncertain parameter is affine, whereas it usually involves some degree of conservatism in the general case of polynomial and rational dependency. This is because one then needs to employ some relaxations to arrive at a tractable LMI problem (see [5] and the references therein for various relaxation schemes).

It seems that the robust version of the nonlinear least-squares (NLS) problem has not yet received similar interest, understandably because of the complications with general nonlinear optimization (like the issues of local minima, convergence and computational load). The goal of this paper is to make an initial step in this direction and provide a robust NLS algorithm in the form of consecutive LMI optimizations. We first make a brief recap of the NLS problem and the Gauss-Newton algorithm in the next section. The robust version of the problem and the algorithm that we propose are provided in Section III. The paper is concluded after a simple illustrative example.

II. Nonlinear Least-Squares

We consider in this section the standard nonlinear least-squares (NLS) optimization problem

$$(\gamma_{\text{nls}}^{\circ})^2 \triangleq \min_{u\in\mathbb{R}^n} \|e(u)\|^2, \tag{1}$$

where $e:\mathbb{R}^n \to \mathbb{R}^m$ represents a vector-valued, differentiable nonlinear function. We also introduce

$$f(u) \triangleq \|e(u)\|^2 \tag{2}$$

to refer to the function to be minimized. As is usually the case for general nonlinear optimization problems, the goal is to develop an update scheme of the form

$$u^{k+1} = u^k + h^k, \tag{3}$$

where u^k represents the estimate of a minimizer, while h^k serves as the update, both at step k. The challenge is to build this scheme in such a way that u^k is assured to converge to a local minimizer as $k\to\infty$. In practice, we need to introduce a stopping criterion for such a consecutive scheme, which could for instance be $\|u^{k+1}-u^k\| < 10^{-\tau}$, where $\tau > 0$ determines the required precision.

The optimization methods basically differ according to how they obtain the update vector h^k. The Gauss-Newton method is based on approximating e with an affine function (and thus f with a quadratic function) in the neighborhood of u^k. Let us represent this linear estimate of e with

$$e(u^k+h) \approx e^k(h) \triangleq e(u^k) + J(u^k)h, \tag{4}$$

where J is the Jacobian of e defined as

$$J(u) \triangleq \frac{\partial e(u,\delta)}{\partial u} = \begin{bmatrix} \frac{\partial e_1(u,\delta)}{\partial u_1} & \cdots & \frac{\partial e_1(u,\delta)}{\partial u_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial e_m(u,\delta)}{\partial u_1} & \cdots & \frac{\partial e_m(u,\delta)}{\partial u_n} \end{bmatrix}. \tag{5}$$

With e^k denoting the affine estimate of e at step k, we have a quadratic estimate of f at the same step as

$$\begin{aligned} f(u^k+h) &\approx f^k(h) \triangleq \|e^k(h)\|^2 \\ &= e(u^k)^T e(u^k) + 2e(u^k)^T J(u^k)h + h^T J(u^k)^T J(u^k)h. \end{aligned} \tag{6}$$

In the Gauss-Newton algorithm, h^k is chosen as a minimizer of this function. The minimizers are to be sought among the solutions of

$$J(u^k)^T J(u^k) h^k = -J(u^k)^T e(u^k). \tag{7}$$

In order to enforce a unique, explicit solution, the Levenberg-Marquardt modification is employed by adding εh^k to the left-hand side, with a small $\varepsilon > 0$. In this case, the update direction is obtained explicitly as

$$h^k = -\left(\varepsilon I + J(u^k)^T J(u^k)\right)^{-1} J(u^k)^T f(u^k). \tag{8}$$

For further details, see [3], [4] and the references therein.

III. Robust Nonlinear Least-Squares

Let us now consider the general robust nonlinear least-squares (RNLS) optimization problem

$$\gamma^{\circ}_{\text{rnls}} \triangleq \min_{u\in\mathbb{R}^n} \left\{ \gamma : \|e(u,\delta)\|^2 \leq \gamma^2, \forall \delta \in \Delta \right\}. \qquad (9)$$

where $e : R^n \times \Delta \to R^m$ is a function that is differentiable with respect to its first argument and that has affine dependence on its second argument, which represents the uncertain parameter. The uncertainty is described by the compact set $\Delta \subset \mathbb{R}^l$, which is assumed to be the convex hull of finitely many extreme points collected in the set $\Delta_{\text{ex}} = \{\delta^1, \ldots, \delta^\eta\}$.

With the introduction of δ dependence in e, we loose the opportunity of obtaining the update direction by an explicit formula or by a standard minimization over h. Since (9) is a min-max problem, it would be reasonable to obtain the update direction by solving a min-max problem of the form

$$h^k = \arg\min_{h\in\mathbb{R}^n} \{\gamma : f^k(h,\delta) \triangleq \|e^k(h,\delta)\|^2 \leq \gamma^2, \forall \delta \in \Delta\}, \quad (10)$$

where e^k is now a function of δ given by

$$e^k(h,\delta) \triangleq e(u^k,\delta) + J(u^k,\delta)h. \qquad (11)$$

By applying the Schur-complement lemma, we can reformulate (10) as a robust LMI optimization of the form

$$h^k = \arg\min_{h\in\mathbb{R}^n} \left\{ \gamma : \begin{bmatrix} \gamma I & e^k(h,\delta) \\ e^k(h,\delta)^T & \gamma \end{bmatrix} \succcurlyeq 0, \forall \delta \in \Delta \right\}. \qquad (12)$$

Thanks to the affine dependence on δ, this problem can be rendered tractable as

$$h^k = \arg\min_{h} \left\{ \gamma : \begin{bmatrix} \gamma I & e^k(h,\delta^j) \\ e^k(h,\delta^j)^T & \gamma \end{bmatrix} \succcurlyeq 0, j = 1, \ldots, \eta \right\}. \qquad (13)$$

We have thus developed an RNLS algorithm that is formed by consecutive LMI optimizations. In each step, one has to solve an LMI problem over $n+1$ variables subject to η LMI constraints of size $(m+1)\times(m+1)$. It is also possible to include further constraints expressed in the form of LMIs in u. For this, one only needs to add those constraints to the problem in (13) with $u = u^k + h$ (e.g. the range constraint $u \leq u_{\max}$ should be added as $h \leq u_{\max} - u^k$).

IV. Example

We consider the RNLS problem for the simple function

$$e(u,\delta) = q(1+k_q\delta)u^2 + s(1+k_s\delta)u - t, \qquad (14)$$

with the data $q = 1, s = 1, k_q = 0, k_s = 2, t = 6$ and the uncertainty range identified as $\delta \in \Delta = [-1,1]$. Figure 1 provides the plots of this function within the range $u \in [-4,3]$ for several δ values (see the top plot). We can detect three local minima within the considered range obtained at $u = 2$, $u = 0$ and $u = -3$, among which $u = 2$ is the global minimizer. The bottom plot shows the worst-case values of $|e(u,\delta)|^2$ (i.e. maximum over $\delta \in [-1,1]$) versus u, as obtained from the top plot. The consecutive estimates of local minimizers (u^k with $k \geq 1$) obtained with different initial estimates are

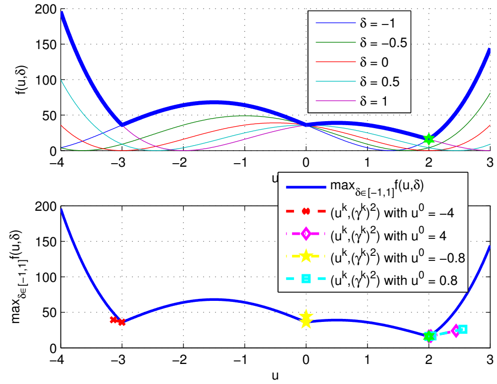

Fig. 1. Robust NLS optimization example.

also displayed in this plot. We observe convergence to the closest local optima in all the cases.

V. Concluding Remarks

We have provided a scheme for robust nonlinear least-squares optimization in the form of consecutive LMI optimizations. Although successful convergence is observed to the local minima in our simple examples, it is not straightforward to adapt the local convergence proof of the Gauss-Newton scheme, which is in fact established under some further assumptions on the function (see [3]). The scheme can be applied in the case of polynomial or rational parameter dependency as well. Nevertheless, this would require the use of somewhat complicated relaxation schemes [5] and make the convergence analysis even more challenging. In some more complicated examples, we observed better convergence behavior when the update is implemented as $u^{k+1} = u^k + \alpha_k h^k$, with a step size $\alpha_k \in (0,1]$ that is smaller than one. Though it is possible to add an extra minimization over the step size, this would also be somewhat more complicated than the ones in the common optimization schemes.

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References

[1] A. Ben-Tal and A. S. Nemirovski L. El Ghaoui. *Robust Optimization*. Princeton University Press, 2009.

[2] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali. *LMI Control Toolbox: For Use with MATLAB*. The Mathworks, Inc., Massachusets, 1995.

[3] Jr. J. E. Dennis and R. B. Schnabel. *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*. Classics in Applied Mathematics. SIAM, Philadelphia, PA, 2nd edition, 1996.

[4] O. Tingleff K. Madsen, H.B. Nielsen. Methods for non-linear least squares problems. Informatics and Mathematical Modelling, Technical University of Denmark, Denmark, April 2004.

[5] C. W. Scherer. LMI relaxations in robust control. *European Journal of Control*, 12(1):3–29, 2006.

[6] C. W. Scherer and S. Weiland. Linear matrix inequalities in control, February 2005.

[7] J. F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11–12:625–653, 1998. Available from `http://sedumi.mcmaster.ca/`.