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Thermo-Inspired Modeling and Analysis of Network Information Flows

Qing Hui

Abstract— **The aim of this research is to develop a fundamental thermo-inspired framework for modeling and analysis of network information flows. This framework is partially motivated by the treatment of energy flows in thermodynamic systems and Eulerian modeling of continuum network information flows as well as non-local biological aggregations. Capturing intrinsic energy or information flows inside these network systems is a crucial problem, partly because the energy or information flows within the network systems are the possible way to uncover the essence of stability and other dynamic properties of the network system and will definitely help us understand some fundamental phenomena exhibited by the natural and engineered systems. Furthermore, the existing theories for information flow modeling hardly address any systematic synthesis methods for building engineered complex systems, which may revolutionize the control theory and applications to network analysis and synthesis as well as diffusion processes.**

The proposed modeling and analysis framework in this paper is based on our recent research related to system thermodynamic theory in which energy flow is the central part of this theory and is consistent with some basic thermodynamic properties such as energy conservation and entropy nonconservation. More specifically, the proposed framework is inspired by the recently developed notion of *system thermodynamics* which results in model architectures involving the information flow propagated over a phase space according to certain thermodynamic laws.

I. Introduction

Advancement in communication and control has now allowed engineers to build large-scale networks of distributively controlled dynamic components (also called agents), in which the dynamic agents interact with each other via a communication network. Examples of such networked multiagent systems include groups of intelligent automobiles in smart highways [1], [2]; coordination of unmanned air vehicles (UAV's), unmanned ground vehicles (UGV's), and autonomous underwater vehicles (AUV's) for hazard mitigation and surveillance [3]–[6]; distributed mobile sensor networks for managing power levels of wireless and detection networks [7]–[9]; air and ground transportation systems for air traffic control and payload transport and traffic management [10]–[13]; swarms of air and space vehicle formations and maneuvers for command and control between heterogeneous air and space vehicles [14], [15]; and congestion control in communication and computer networks for information flow routing [16], [17].

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The author is with the Department of Mechanical Engineering, Texas Tech University, Lubbock, TX 79409-1021, USA (qing.hui@ttu.edu).

Understanding the effect and use of information in the natural and engineered multiagent systems requires two key elements: modeling (analysis of system behavior) and modification (synthesis of applied forces). Appropriate models must facilitate the study of the interaction between applied "forces" or "actions" and the resultant "effects" on system evolution. Today multiagent engineering systems, some of which are mentioned in the previous paragraph, can be seen as a first generation of a full-fledged large-scale distributedly controlled multiagent network. The current generation of distributed multiagent networks has provided an engineering solution to the problem of controlling large-scale networks of dynamic agents. Such distributed control solutions are inherently robust to individual agent failures and communication link failures and are scalable to a large number of agents due to the fact that no centralized control is involved and no full communication architecture is required.

However, emerging research on insect-like, bacteria-inspired, nano-scale robots and UAV's poses a great challenging to the first generation of multiagent control networks since such a network always consists of a gigantic number of agents, which is almost impossible to characterize each individual dynamics and to implement control laws on individual agents in this case. This scenario is similar to the study of fluid dynamics and statistical thermodynamics wherein the microscopic, collective behavior of the overall system is analyzed through Eulerian modeling and statistical methods. Instead of focusing on every particle in the physical system, the description of the flow of particles in a fixed spatial domain is adopted as an alternative to overcome this dimensioning disaster. This is more like an "integral" method for very large-scale swarms compared with the "differential" method for the first generation of multiagent networks. Hence, it is plausible to use some similar "integral" techniques for the analysis and synthesis of the next generation of very large-scale distributed multiagent networks [18].

This paper addresses this issue by developing a novel thermo-inspired analysis framework for Eulerian information flow models of very large-scale network information swarms. Dynamic models for very large-scale information swarms can be characterized by means of Lagrangian modeling and Eulerian modeling. In most literature on multiagent systems, Lagrangian modeling is a prevalent way to characterize each individual dynamics. In this case, each agent is simply abstracted as an ideal particle governed by ordinary differential or difference equations to do further analysis and synthesis.

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However, this setup is hard to capture the direction of information flows and to interpret the essence of collective behaviors such as information consensus. When the number of agents become enormous so that it is almost impossible to count the exact number, Lagrangian modeling fails to work, which, as we mentioned before, is a typical scenario in fluid mechanics.

To address the information flows in a very large array of network systems, here a novel Eulerian model is proposed to describe the local information for a distribution of swarms with an energy conservation equation by mimicking energy flow and flux here. It is important to note that the Lagrangian and Eulerian formulations are not isolated to each other. As a matter of fact, mathematically they can be connected by a so-called Fokker-Plank approximation [19]. However, for very large-scale network systems, it is more natural to use Eulerian models to describe the information propagation due to the dimensioning problem. The proposed continuum network information flow model not only can describe energy and information flows in thermodynamic systems and network systems, but also can be used to study of fluid mechanics and materials science.

We will develop a series of results to address the convergence and stability for an Eulerian swarm model. Similar to thermo-inspired analysis whose foundation is system thermodynamics [20], the proposed distributed analysis architectures are also predicated on system thermodynamics resulting in the model architectures involving the exchange of information between uniformly distributed swarms over an n-dimensional (not necessarily Euclidian) space that guarantee that the proposed distributed-parameter system is consistent with basic thermodynamic principles. Information consensus and semistability are shown using the well-known Sobolev embedding theorems and the notion of generalized (or weak) solutions. Finally, since the proposed system is guaranteed to satisfy basic thermodynamic principles, robustness to individual agent failures and unplanned individual agent behavior is automatically guaranteed.

II. Network Information Flow Models

In this paper, we consider an Eulerian information flow model involving a nonlocal spatio-temporal distribution of flow density. Specifically, consider the evolution equation for information flows defined over a compact connected set $\mathcal{V} \subset \mathbb{R}^n$ with a smooth boundary $\partial\mathcal{V}$ and volume $\mathrm{vol}\,\mathcal{V}$ characterized by the *conservation* equation [20], [21]

$$\frac{\partial u(x,t)}{\partial t} = -\nabla \cdot \phi(x, u(x,t), \nabla u(x,t)),$$

$$x \in \mathcal{V}, \quad t \geq t_0, \quad u(x,t_0) = u_{t_0}(x) \in \mathcal{X}, \quad x \in \mathcal{V}, \quad (1)$$

$$\phi(x, u(x,t), \nabla u(x,t)) \cdot \mathbf{n}(x) \geq 0, \quad x \in \partial\mathcal{V}, \quad t \geq t_0, \quad (2)$$

where $u : \mathcal{V} \times [0,\infty) \to \overline{\mathbb{R}}_+ \triangleq [0,\infty)$ denotes the density distribution at the point $x = [x_1, \ldots, x_n]^{\mathrm{T}} \in \mathcal{V}$ and time instant $t \geq t_0$, $\phi : \mathcal{V} \times [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ denotes a continuously differentiable *flux* function, ∇ denotes the nabla operator, "$\cdot$" denotes the dot product in $\mathbb{R}^n$, $\mathbf{n}^{\mathrm{T}}(x)$ denotes the outward normal vector to the boundary $\partial\mathcal{V}$ at $x \in \partial\mathcal{V}$, and $\mathcal{X}$ denotes a space of two-times continuously differentiable scalar functions defined on $\mathcal{V}$. Here, we assume that $\mathcal{V} = \{x \in \mathbb{R}^n : f(x) \leq 0\}$ and $\partial\mathcal{V} = \{x \in \mathbb{R}^n : f(x) = 0\}$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a given continuously differentiable function, and consequently, the outward normal vector to the boundary $\partial\mathcal{V}$ at $x \in \partial\mathcal{V}$ is given by $\mathbf{n}^{\mathrm{T}}(x) = \nabla f(x)$.

Equations (1) and (2) involve an information (or energy) flow equation for a uniformly distributed continuous system. Specifically, note that for any smooth, bounded region $\mathcal{V} \subset \mathbb{R}^n$, the integral $\int_{\mathcal{V}} u(x,t)\mathrm{d}\mathcal{V}$ denotes the total information (or energy) amount within $\mathcal{V}$ at time t. Hence, the rate of information change within $\mathcal{V}$ is governed by the flux function $\phi : \mathcal{V} \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, which controls the rate of information transmission through the boundary $\partial\mathcal{V}$. Hence, for each time t,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} u(x,t)\mathrm{d}\mathcal{V} = -\int_{\partial\mathcal{V}} \phi(x, u(x,t), \nabla u(x,t)) \cdot \mathbf{n}(x)\mathrm{d}\mathcal{S}_{\mathcal{V}},$$

where $\mathrm{d}\mathcal{S}_{\mathcal{V}}$ denotes an infinitesimal surface element of the boundary of the set $\mathcal{V}$. Using the divergence theorem, it follow that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}} u(x,t)\mathrm{d}\mathcal{V} &= -\int_{\partial\mathcal{V}} \phi(x, u(x,t), \nabla u(x,t)) \\ &\quad \cdot \mathbf{n}(x)\mathrm{d}\mathcal{S}_{\mathcal{V}} \\ &= -\int_{\mathcal{V}} \nabla \cdot \phi(x, u(x,t), \nabla u(x,t))\mathrm{d}\mathcal{V}.\end{aligned}$$

Since the region $\mathcal{V} \subset \mathbb{R}^n$ is arbitrary, it follows that the conservation equation over a unit volume within the continuum $\mathcal{V}$ involving the rate of information density change within the continuum is given by (1) and (2). The physical interpretation of (1) and (2) is straightforward. In particular, if $u(x,t)$ is an information (or energy) density at point $x \in \mathcal{V}$ and time $t \geq t_0$, then the conservation equation (1) describes the time evolution of the information (or energy) density $u(x,t)$ over the region $\mathcal{V}$, while the boundary condition in (2) involving the dot product implies that the information (or energy) of the system (1) and (2) can either be stored or transmitted but not supplied through the boundary of $\mathcal{V}$ from the environment.

III. Well-Posedness

We denote the information (or energy) distribution over the set $\mathcal{V}$ at time $t \geq t_0$ by $u_t \in \mathcal{X}$ so that for each $t \geq t_0$ the set of mappings generated by $u_t(x) \equiv u(x,t)$ for every $x \in \mathcal{V}$ gives the *flow* of (1) and (2). We assume that the function $\phi(\cdot,\cdot,\cdot)$ is continuously differentiable so that (1) and (2) admits a unique solution $u(x,t)$, $x \in \mathcal{V}$, $t \geq t_0$, and $u(\cdot,t) \in \mathcal{X}$, $t \geq t_0$, is continuously dependent on the initial information (or energy) distribution $u_{t_0}(x)$, $x \in \mathcal{V}$. It is well known, however, that nonlinear partial differential equations need not have smooth differentiable solutions (*classical solutions*), and one has to use the notion of Schwartz distributions that provides a framework in which the information (or energy) density function $u(x,t)$ may be differentiated in a generalized sense infinitely often [21]. In this case, one has a well-defined notion of solutions that have jump discontinuities, which propagate as shock waves. Thus, one has to deal with *generalized* or *weak* solutions wherein uniqueness is lost. In this case, the *Clausius-Duhem* inequality is invoked for identifying the physically relevant (i.e., thermodynamically admissible) solution [21], [22].

If u_{t_0} is a two-times continuously differentiable function with compact support and its derivative is sufficiently small on $[t_0, \infty)$, then the classical solution to (1) and (2) can break down at a finite time. As a consequence of this, one may only hope to find generalized (or weak) solutions to (1) and (2) over the semi-infinite interval $[t_0, \infty)$, that is, $\mathcal{L}_\infty$ functions $u(\cdot,\cdot)$ that satisfy (1) in the sense of distributions, which provides a framework in which $u(\cdot,\cdot)$ may be differentiated in a general sense infinitely often. It is important to note that we do *not* assume strict hyperbolicity of (1) and (2) since our interest in this paper is to address *semistability*, and hence, (1) and (2) cannot be hyperbolic. Thus, many results on well-posedness of solutions of (1) and (2) developed in the literature are not applicable in this case. Furthermore, the linearization method also fails to provide any stability information due to nonhyperbolicity. Global well-posedness of smooth solutions of nonhyperbolic partial differential equations of the form (1) and (2) remains an open problem in mathematics. Finally, the control aim here is to design a *distributed control* law so that the corresponding closed-loop system achieves semistability and *uniform information distribution* [20].

In this paper, $\mathcal{L}_2$ denotes the space of square-integrable Lebesgue measurable functions on $\mathcal{V}$ and the $\mathcal{L}_2$ operator norm $\|\cdot\|_{\mathcal{L}_2}$ on $\mathcal{X}$ is used for the definitions of Lyapunov, semi-, and asymptotic stability. Furthermore, we introduce the Sobolev spaces

$$\begin{aligned} \mathcal{W}_2^0(\mathcal{V}) &\triangleq \{u_t : \mathcal{V} \to \mathbb{R} : u_t \in \mathrm{C}^0(\mathcal{V}) \cap \mathcal{L}_2(\mathcal{V})\}_{\mathrm{co}} \\ &\quad \subset \mathcal{L}_2(\mathcal{V}), \\ \mathcal{W}_2^1(\mathcal{V}) &\triangleq \{u_t : \mathcal{V} \to \mathbb{R} : u_t \in \mathrm{C}^1(\mathcal{V}) \cap \mathcal{L}_2(\mathcal{V}), (\nabla u_t)^{\mathrm{T}} \\ &\quad \in \mathcal{L}_2(\mathcal{V})\}_{\mathrm{co}}, \end{aligned}$$

where $\mathrm{C}^r(\mathcal{V})$ denotes a function space defined on $\mathcal{V}$ with r-continuous derivatives and $\{\cdot\}_{\mathrm{co}}$ denotes completion of $\{\cdot\}$ in $\mathcal{L}_2$ in the sense of [23], with norms

$$\|u_t\|_{\mathcal{W}_2^0} \triangleq \|u_t\|_{\mathcal{L}_2} = \left[\int_{\mathcal{V}} u_t^2(x)\mathrm{d}\mathcal{V}\right]^{\frac{1}{2}}, \qquad (3)$$

$$\|u_t\|_{\mathcal{W}_2^1} \triangleq \left[\|u_t\|_{\mathcal{W}_2^0}^2 + D(u_t, u_t)\right]^{\frac{1}{2}}, \qquad (4)$$

defined on $\mathcal{W}_2^0(\mathcal{V})$ and $\mathcal{W}_2^1(\mathcal{V})$, respectively, where the gradient $\nabla u_t(x)$ in (4) is interpreted in the sense of a generalized gradient [23], and $D(u_t, u_t) \triangleq \int_{\mathcal{V}} \nabla u_t(x) \nabla^{\mathrm{T}} u_t(x) \mathrm{d}\mathcal{V}$ is the *Dirichlet integral* of u [24, p. 88]. Physically the Dirichlet integral term represents the potential energy in $\mathcal{V}$ of the *electrostatic field* $-\nabla u$. Note that since the solutions to (1) and (2) are assumed to be two-times continuously differentiable functions on a compact set $\mathcal{V}$ and ϕ is continuously differentiable, it follows that $u_t(x)$, $t \geq t_0$, belongs to $\mathcal{W}_2^0(\mathcal{V})$ and $\mathcal{W}_2^1(\mathcal{V})$.

IV. Thermo-Inspired Analysis

In this section, we develop a distributed controller that guarantees that the infinite-dimensional information flow model (1) and (2) has convergent flows to Lyapunov stable uniform equilibrium information density distributions determined by the system initial information density distribution. First, however, we establish several key definitions and stability results for nonlinear infinite-dimensional systems. Here, the state space is assumed to be a Banach space with fully nonlinear dynamics.

Let $\mathcal{B}$ be a Banach space with norm $\|\cdot\|_{\mathcal{B}}$. A *dynamical system* $\mathcal{G}$ on $\mathcal{B}$ is the triple $(\mathcal{B}, [t_0, \infty), s)$, where $s : [t_0, \infty) \times \mathcal{B} \to \mathcal{B}$ is such that the following axioms hold: i) (*Continuity*): $s(\cdot,\cdot)$ is jointly continuous, ii) (*Consistency*): $s(t_0, z_0) = z_0$ for all $t_0 \in \mathbb{R}$ and $z_0 \in \mathcal{B}$, and iii) (*Semigroup property*): $s(t+\tau, z_0) = s(\tau, s(t, z_0))$ for all $z_0 \in \mathcal{B}$ and $t, \tau \in [t_0, \infty)$. Given $t \in [0, \infty)$ we denote the *flow* $s(t, \cdot) : \mathcal{B} \to \mathcal{B}$ of $\mathcal{G}$ by $s_t(x_0)$ or s_t. Likewise, given $x \in \mathcal{B}$ we denote the *solution curve* or *trajectory* $s(\cdot, x) : [0, \infty) \to \mathcal{B}$ of $\mathcal{G}$ by $s^x(t)$ or s^x. The *positive limit set* of $x \in \mathcal{B}$ is the set $\omega(x)$ of points $z \in \mathcal{B}$ such that there exists an increasing sequence $\{t_i\}_{i=1}^{\infty}$ satisfying $s(t_i, x) \to z$ as $i \to \infty$. The image of $\mathcal{U} \subset \mathcal{B}$ under the flow s_t is defined by $s_t(\mathcal{U}) \triangleq \{y : y = s_t(x_0) \text{ for all } x_0 \in \mathcal{U}\}$. Finally, we define a *positive orbit* through the point $x \in \mathcal{B}$ as the motion along the curve $\mathcal{O}_x^+ \triangleq \{z \in \mathcal{B} : z = s(t, x), t \geq t_0\}$.

An *equilibrium point* of $\mathcal{G}$ is a point $z \in \mathcal{B}$ such that $s(t, z) = s(t_0, z)$ for all $t \geq t_0$. A set $\mathcal{M} \subseteq \mathcal{B}$ is *positively invariant* if $s_t(\mathcal{M}) \subseteq \mathcal{M}$ for all $t \geq 0$. The set $\mathcal{M}$ is *negatively invariant* if, for every $z \in \mathcal{M}$ and every $t \geq 0$, there exists $x \in \mathcal{M}$ such that $s(t, x) = z$ and $s(\tau, x) \in \mathcal{M}$ for all $\tau \in [0, t]$. The set $\mathcal{M}$ is *invariant* if $s_t(\mathcal{M}) = \mathcal{M}$, $t \geq 0$. Note that a set is invariant if and only if it is positively and negatively invariant.

Definition 4.1: Let $\mathcal{G}$ be a dynamical system on a Banach space $\mathcal{B}$ with norm $\|\cdot\|_{\mathcal{B}}$ and let $\mathcal{D}$ be a positively invariant set with respect to $\mathcal{G}$. An equilibrium point $x \in \mathcal{D}$ of $\mathcal{G}$ is *Lyapunov stable* if for every relatively open subset $\mathcal{N}_\varepsilon$ of $\mathcal{D}$ containing x, there exists a relatively open subset $\mathcal{N}_\delta$ of $\mathcal{D}$ containing x such that $s_t(\mathcal{N}_\delta) \subseteq \mathcal{N}_\varepsilon$ for all $t \geq t_0$. An equilibrium point $x \in \mathcal{D}$ of $\mathcal{G}$ is *semistable* if it is Lyapunov stable and there exists a relatively open subset $\mathcal{U}$ of $\mathcal{D}$ containing x such that for all initial conditions in $\mathcal{U}$, the trajectory $s(\cdot,\cdot)$ of $\mathcal{G}$ converges to a Lyapunov stable equilibrium point, that is, $\lim_{t\to\infty} s(t, z) = y$, where $y \in \mathcal{D}$ is a Lyapunov stable equilibrium point of $\mathcal{G}$ and $z \in \mathcal{U}$. Finally, an equilibrium point $x \in \mathcal{D}$ of $\mathcal{G}$ is *asymptotically stable* if it is Lyapunov stable and there exists a relatively open subset $\mathcal{U}$ of $\mathcal{D}$ containing x such that $\lim_{t\to\infty} s(t, z) = x$ for all $z \in \mathcal{U}$.

The next result gives a sufficient condition to guarantee semistability of the equilibria of $\mathcal{G}$. For the statement of this result, let $\mathcal{B}$ and $\mathcal{C}$ be Banach spaces and recall that $\mathcal{B}$ is *compactly embedded* in $\mathcal{C}$ if $\mathcal{B} \subset \mathcal{C}$ and a unit ball in $\mathcal{B}$ belongs to a compact subset in $\mathcal{C}$. Furthermore, define $\dot{V}(z) \triangleq \lim_{h\to 0^+} \frac{1}{h}[V s(t_0+h, z) - V(z)]$, $z \in \mathcal{B}$, for a given continuous function $V : \mathcal{B} \to \mathbb{R}$ and every $z \in \mathcal{B}$ such that the limit exists. The following result gives a Lyapunov-based test for semistability.

Theorem 4.1: Let $\mathcal{B}$ and $\mathcal{C}$ be Banach spaces such that $\mathcal{B}$ is compactly embedded in $\mathcal{C}$, and let $\mathcal{G}$ be a dynamical system defined in $\mathcal{B}$ and $\mathcal{C}$. Assume there exist continuous functions $V_{\mathcal{B}} : \mathcal{B} \to \mathbb{R}$ and $V_{\mathcal{C}} : \mathcal{C} \to \mathbb{R}$ such that $\dot{V}_{\mathcal{B}}$ and $\dot{V}_{\mathcal{C}}$ are defined on $\mathcal{B}_{\rm c}$ and $\mathcal{C}_{\rm c}$, respectively, where $\mathcal{B}_{\rm c} = \{z \in \mathcal{B} : V_{\mathcal{B}}(z) < \eta\}$ and $\mathcal{C}_{\rm c} = \{z \in \mathcal{C} : V_{\mathcal{C}}(z) < \eta\}$ for some $\eta > 0$ such that $\mathcal{B}_{\rm c} \subset \mathcal{C}_{\rm c}$. Furthermore, assume that $V_{\mathcal{B}}(s(t, z_0)) \leq V_{\mathcal{B}}(s(\tau, z_0))$ for all $t_0 \leq \tau \leq t$ and $z_0 \in \mathcal{B}_{\rm c}$, and $V_{\mathcal{B}}(s(t, z_0)) \leq V_{\mathcal{B}}(s(\tau, z_0))$ for all $t_0 \leq \tau \leq t$ and $z_0 \in \mathcal{C}_{\rm c}$. If $\mathcal{B}_{\rm c}$ is bounded and every point in the largest invariant subset $\mathcal{M}$ contained in $\mathcal{R}$ given by $\mathcal{R} \triangleq \{z \in \overline{\mathcal{C}_{\rm c}} : \dot{V}_{\mathcal{C}}(z) = 0\}$ is a Lyapunov stable equilibrium point of $\mathcal{G}$, then every equilibrium point in $\mathcal{M}$ is semistable.

The following result is a generalization of Theorem 4.1.

Theorem 4.2: Let $\mathcal{B}$ and $\mathcal{C}$ be Banach spaces such that $\mathcal{B}$ is compactly embedded in $\mathcal{C}$, and let $\mathcal{G}$ be a dynamical system defined in $\mathcal{B}$ and $\mathcal{C}$. Assume there exist lower semicontinuous functions $V_{\mathcal{B}} : \mathcal{B} \to \mathbb{R}$ and $V_{\mathcal{C}} : \mathcal{C} \to \mathbb{R}$ such that $\mathcal{B}_{\rm c} \subset \mathcal{C}_{\rm c}$, where $\mathcal{B}_{\rm c} = \{z \in \mathcal{B} : V_{\mathcal{B}}(z) < \eta\}$ and $\mathcal{C}_{\rm c} = \{z \in \mathcal{C} : V_{\mathcal{C}}(z) < \eta\}$ for some $\eta > 0$. Furthermore, assume that $V_{\mathcal{B}}(s(t, z_0)) \leq V_{\mathcal{B}}(s(\tau, z_0))$ for all $t_0 \leq \tau \leq t$ and $z_0 \in \mathcal{B}_{\rm c}$, and $V_{\mathcal{B}}(s(t, z_0)) \leq V_{\mathcal{B}}(s(\tau, z_0))$ for all $t_0 \leq \tau \leq t$ and $z_0 \in \mathcal{C}_{\rm c}$. Let $\mathcal{R}_\gamma \triangleq \bigcap_{c > \gamma} \overline{V^{-1}([\gamma, c])} \subseteq \overline{\mathcal{C}_{\rm c}}$ and let $\mathcal{M}_\gamma$ denote the largest invariant set contained in $\mathcal{R}_\gamma$. If $\mathcal{B}_{\rm c}$ is bounded and every point in the set $\mathcal{M} \triangleq \bigcup_{\gamma \in \mathbb{R}} \mathcal{M}_\gamma$ is a Lyapunov stable equilibrium point of $\mathcal{G}$, then every equilibrium point in $\mathcal{M}$ is semistable.

Next, we give an extension of the Krasovskii-LaSalle invariant set theorem to infinite-dimensional dynamical systems. This result can be found at [25].

Lemma 4.1 ([25]): Consider a dynamical system $\mathcal{G}$ defined on a Banach space $\mathcal{B}$. Let $\mathcal{B}_{\rm c} \subset \mathcal{B}$ be a closed set, and assume there exists a continuous function $V : \mathcal{B}_{\rm c} \to \mathbb{R}$ such that $\dot{V}(z) \leq 0$, $z \in \mathcal{B}_{\rm c}$. Furthermore, let $\mathcal{R} \triangleq \{z \in \mathcal{B}_{\rm c} : \dot{V}(z) = 0\}$, and let $\mathcal{M}$ denote the largest invariant set (with respect to the dynamical system $\mathcal{G}$) contained in $\mathcal{R}$. Then for every $z_0 \in \mathcal{B}_{\rm c}$ such that $\mathcal{O}^+_{z_0} \subset \mathcal{B}_{\rm c}$ and $\mathcal{O}^+_{z_0}$ is contained in a compact subset of $\mathcal{B}$, $s(t, z_0) \to \mathcal{M}$ as $t \to \infty$.

Using the above result, one can obtain a Lyapunov-based test for semistability without using compact embedding.

Theorem 4.3: Consider a dynamical system $\mathcal{G}$ defined on a Banach space $\mathcal{B}$. Let $\mathcal{B}_{\rm c} \subset \mathcal{B}$ be a closed set, and assume there exists a continuous function $V : \mathcal{B}_{\rm c} \to \mathbb{R}$ such that $\dot{V}(z) \leq 0$, $z \in \mathcal{B}_{\rm c}$. Furthermore, let $\mathcal{R} \triangleq \{z \in \mathcal{B}_{\rm c} : \dot{V}(z) = 0\}$, and let $\mathcal{M}$ denote the largest invariant set (with respect to the dynamical system $\mathcal{G}$) contained in $\mathcal{R}$. If every point in $\mathcal{M}$ is a Lyapunov stable equilibrium point of $\mathcal{G}$, then for every $z_0 \in \mathcal{B}_{\rm c}$ such that $\mathcal{O}^+_{z_0} \subset \mathcal{B}_{\rm c}$ and $\mathcal{O}^+_{z_0}$ is contained in a compact subset of $\mathcal{B}$, every equilibrium point in $\mathcal{M}$ is semistable.

The following assumptions are needed for the main results of the paper. For the statement of these assumptions, $\phi : \mathcal{V} \times \overline{\mathbb{R}}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the system information (or energy) flow within the continuum $\mathcal{V}$, that is, $\phi(x, u(x,t), \nabla u(x,t)) = [\phi_1(x, u(x,t), \nabla u(x,t)), \ldots, \phi_n(x, u(x,t), \nabla u(x,t))]^{\rm T}$, where $\phi_i(\cdot,\cdot,\cdot)$ denotes the information (or energy) flow through a unit area per unit time in the x_i direction for all $i = 1, \ldots, n$, and $\nabla u(x,t) \triangleq [D_1 u(x,t), \ldots, D_n(x,t)]$, $x \in \mathcal{D}$, $t \geq t_0$, denotes the gradient of $u(\cdot, t)$ with respect to the spatial variable x.

Assumption 1: For every $x \in \mathcal{V}$ and unit vector $\mathbf{u} \in \mathbb{R}^n$, $\phi(x, u_t(x), \nabla u_t(x)) \cdot \mathbf{u} = 0$ if and only if $F(\nabla u_t(x)\mathbf{u}, u_t(x)) = 0$, where $F : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is a continuous function satisfying $F(0, \cdot) = 0$.

Assumption 2: For every $x \in \mathcal{V}$ and unit vector $\mathbf{u} \in \mathbb{R}^n$, $\phi(x, u_t(x), \nabla u_t(x)) \cdot \mathbf{u} > 0$ if and only if $F(\nabla u_t(x)\mathbf{u}, u_t(x)) < 0$, and $\phi(x, u_t(x), \nabla u_t(x)) \cdot \mathbf{u} < 0$ if and only if $F(\nabla u_t(x)\mathbf{u}, u_t(x)) > 0$.

Note that Assumption 1 implies that $\phi_i(x, u_t(x), \nabla u_t(x)) = 0$ if and only if $F(D_i u_t(x), u_t(x)) = 0$, $x \in \mathcal{V}$, $i = 1, \ldots, n$, while Assumption 2 implies that $\phi_i(x, u_t(x), \nabla u_t(x)) F(D_i u_t(x), u_t(x)) \leq 0$, $x \in \mathcal{V}$, $i = 1, \ldots, n$. The physical interpretation of Assumption 1 is that if the flux function ϕ in a certain direction is zero, then information or energy density change in this direction is not possible. This statement is reminiscent of the *zeroth law of thermodynamics*, which postulates that temperature equality is a necessary and sufficient condition for thermal equilibrium. Assumption 2 implies that information or energy flows from information rich or more energetic regions to information poor or less energetic regions and is reminiscent of the *second law of thermodynamics*, which states that heat (energy) must flow in the direction of lower temperatures. For further details of these assumptions, see [20].

In this paper, we assume that the solution $u(x,t), x \in \mathcal{V}$, $t \geq t_0$, to (1) and (2) is nonnegative for all nonnegative initial information density distributions $u_{t_0}(x) \geq 0$, $x \in \mathcal{V}$.

Next, we state that if no information flow is allowed into or out of $\mathcal{V}$ (i.e., the boundary $\partial\mathcal{V}$ is insulated), then (1) and (2) is Lyapunov stable.

Lemma 4.2: Consider the dynamical system given by (1) and (2). Assume that Assumptions 1 and 2 hold. If

$$\phi(x, u(x,t), \nabla u(x,t)) \cdot \mathbf{n}(x) = 0, \quad x \in \partial\mathcal{V}, \quad t \geq t_0, \quad (5)$$

and

$$F(\nabla u_t(x)\mathbf{u}, u_t(x)) \cdot (\nabla u_t(x)\mathbf{u}) \geq 0, \quad x \in \mathcal{V}, \quad (6)$$

for any unit vector $\mathbf{u} \in \mathbb{R}^n$, then $u(x,t) \equiv \alpha$, $\alpha \geq 0$, is Lyapunov stable.

Next, we claim that the total $\mathcal{L}_2$ norm of the energy of (1) and (2) is nonincreasing.

Lemma 4.3: Consider the dynamical system given by (1) and (2). Assume that Assumptions 1, 2, and (6) hold. If either $u(x,t) = 0$ for all $x \in \partial\mathcal{V}$ and $t \geq t_0$ or (5) holds, then $\|u_t\|_{\mathcal{W}_2^0} \leq \|u_\tau\|_{\mathcal{W}_2^0}$ for all $t_0 \leq \tau \leq t$.

Next, we present necessary and sufficient conditions for semistability of the information flow model (1) and (2).

Theorem 4.4: Consider the dynamical system given by (1) and (2). Assume that Assumptions 1, 2, and (6) hold.

Furthermore, assume that

$$\Delta u_t(x)\nabla\cdot\phi(x,u_t(x),\nabla u_t(x)) +\nabla u_t(x)\phi(x,u_t(x),\nabla u_t(x))\leq 0, \quad x\in\mathcal{V},\quad u_t\in\mathcal{W}_2^1(\mathcal{V}), \tag{7}$$

where $\Delta\triangleq\nabla\cdot\nabla$ denotes the Laplace operator. Then for every $\alpha\geq 0$, $u(x,t)\equiv\alpha$ is a semistable equilibrium state of (1) and (2) if and only if (5) holds. In this case, $u(x,t)\to\frac{1}{\mathrm{vol}\mathcal{V}}\int_{\mathcal{V}}u_{t_0}(x)\mathrm{d}\mathcal{V}$ as $t\to\infty$ for every initial condition $u_{t_0}\in\mathcal{W}_2^1(\mathcal{V})$ and every $x\in\mathcal{V}$; moreover, $\frac{1}{\mathrm{vol}\mathcal{V}}\int_{\mathcal{V}}u_{t_0}(x)\mathrm{d}\mathcal{V}$ is a semistable equilibrium state of (1) and (2).

Theorem 4.4 shows that the information flow model (1) and (2) with Assumptions 1, 2, and (6) has convergent flows to Lyapunov stable uniform equilibrium information density distributions determined by the system initial information density distribution. This phenomenon is known as *equipartition of energy* [20] in system thermodynamics and *information consensus* or *protocol agreement* [26], [27] in cooperative network systems. The following result is a direct consequence of Theorem 4.4.

Corollary 4.1: Consider the dynamical system $\mathcal{G}$ given by (1) and (2). Assume that Assumption 1, 2, and (6) hold, and

$$\Delta u_t(x)\nabla\cdot\phi(x,u_t(x),\nabla u_t(x))\leq 0, \quad x\in\mathcal{V},\quad u_t\in\mathcal{W}_2^1(\mathcal{V}). \tag{8}$$

Then for every $\alpha\geq 0$, $u(x,t)\equiv\alpha$ is a semistable equilibrium state of (1) and (2) if and only if (5) holds. In this case, $u(x,t)\to\frac{1}{\mathrm{vol}\mathcal{V}}\int_{\mathcal{V}}u_{t_0}(x)\mathrm{d}\mathcal{V}$ as $t\to\infty$ for every initial condition $u_{t_0}\in\mathcal{W}_2^1(\mathcal{V})$ and every $x\in\mathcal{V}$; moreover, $\frac{1}{\mathrm{vol}\mathcal{V}}\int_{\mathcal{V}}u_{t_0}(x)\mathrm{d}\mathcal{V}$ is a semistable equilibrium state of (1) and (2).

Condition (8) implies that for an information (or energy) density distribution $u_t(x)$, $x\in\mathcal{V}$, the information (or energy) flow $\phi(x,u_t(x),\nabla u_t(x))$ at $x\in\mathcal{V}$ is proportional to the information (or energy) density at this point. Note that for a linear information (or energy) flow model where $\phi(x,u_t(x),\nabla u_t(x))=-k[\nabla u_t(x)]^{\mathrm{T}}$ and $k>0$ is a conductivity constant, condition (8) is automatically satisfied since $\Delta u_t(x)\nabla\cdot\phi(x,u_t(x),\nabla u_t(x))=-k[\Delta u_t(x)]^2\leq 0$, $x\in\mathcal{V}$.

V. Advection-Diffusion Models

The nonlinear partial differential equation (1) describes a general conservation equation which includes many important swarming models discussed in the literature. See, for example, [28]. In this section, we turn our attention to a specific form of (1) involving the *advection-diffusion* model [28] defined over a compact connected set $\mathcal{V}\subset\mathbb{R}^n$ with a smooth boundary $\partial\mathcal{V}$ and volume $\mathrm{vol}\,\mathcal{V}$ given by

$$\frac{\partial\rho(x,t)}{\partial t} = -\nabla\cdot(\rho(x,t)v(x,t)) +\nabla\cdot(\rho(x,t)B(x,t)), \tag{9}$$

$$\rho(x,t_0) = \rho_{t_0}(x),\quad x\in\mathcal{V},\quad t\geq t_0, \tag{10}$$

where $\rho:\mathcal{V}\times[0,\infty)\to\overline{\mathbb{R}}_+$ denotes the density distribution of mobile agents at the point $x=[x_1,\ldots,x_n]^{\mathrm{T}}\in\mathcal{V}$ and time instant $t\geq t_0$, $v:\mathcal{V}\times[0,\infty)\to\mathbb{R}^n$ is a density-dependent advection velocity, and $B:\mathcal{V}\times[0,\infty)\to\mathbb{R}^{n\times n}$ is a diffusion operator. Here, we consider the case where $v(x,t)$ is given by

$$v(x,t)=\nabla(K(x)*\rho(x,t)),\quad x\in\mathcal{V},\quad t\geq t_0, \tag{11}$$

where $*$ denotes the convolution operator and a smooth, nonnegative $K(\cdot)$ satisfies $0<\int_{\mathbb{R}^n}K(x)\mathrm{d}x<\infty$ and $K(x)=K(-x)$ for all $x\in\mathbb{R}^n$, and

$$B(x,t)=\nabla\rho(x,t) \tag{12}$$

for all $x\in\mathcal{V}$ and $t\geq t_0$.

Next, we use an energy-based method to study the asymptotic behavior of (9) and (10). To define the energy, we first rewrite the equation (9) in a slightly different form

$$\frac{\partial\rho(x,t)}{\partial t} = \nabla\cdot(\rho(x,t)\nabla(\rho(x,t) -K(x)*\rho(x,t))). \tag{13}$$

We now define the energy

$$E(\rho)=\frac{1}{2}\int_{\mathcal{V}}\left(\rho^2-\rho K*\rho\right)\mathrm{d}\mathcal{V}, \tag{14}$$

where the first term arises from avoidance and the second from aggregation. This energy is dissipated under (9) and (10), which is given by the following result.

Lemma 5.1: Assume that (5) holds. Then the energy defined by (14) satisfies $\dot{E}\leq 0$.

Note that (14) is a non-convex functional composed of a positive avoidance term $\frac{1}{2}\int_{\mathcal{V}}\rho^2\mathrm{d}\mathcal{V}$ and a negative aggregation term $-\frac{1}{2}\int_{\mathcal{V}}\rho K*\rho\mathrm{d}\mathcal{V}$ which have different nonlinear dependence on ρ and different length scales. Next, we show that it is possible for the advection-diffusion equation (9) and (10) to achieve semistability under certain circumstances.

Theorem 5.1: Consider the dynamical system given by (9) and (10). Assume that $v(x,t)$ satisfies (11) and $B(x,t)$ satisfies (12) and all the solutions of (9) and (10) are bounded. Furthermore, assume that (5) holds and

$$\nabla\rho\cdot\nabla(K*\rho)\leq|\nabla\rho|^2,\quad \rho\in\mathcal{W}_2^0(\mathcal{V}). \tag{15}$$

Then $\rho(x,t)\equiv\beta(x)$ is a semistable equilibrium state of (9) and (10), where $\beta(\cdot)$ is nonnegative and satisfies $\nabla(K*\beta)=\nabla\beta$.

VI. Conclusions

We construct a continuum model for network information aggregations in which the information flows under the thermodynamic principles. Existence and uniqueness of solutions, stability and asymptotic behavior of solutions, and energy-based thermodynamic analysis have been investigated by merging different techniques and knowledge from PDE, ODE, control theory, and dynamical systems theory. There are still a lot of open problems regarding this model such as relaxing the conditions guaranteeing existence, uniqueness, and stability of solutions. Finally, numerical implementation of the proposed model is an ongoing research topic.

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