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Feedforward Control of Inhomogeneous Linear First Order Distributed Parameter Systems

Florian Malchow and Oliver Sawodny

Abstract— **The paper presents a new approach for feedforward control of inhomogeneous distributed parameter systems. Inhomogeneous means that we consider a partial differential equation (PDE) with a spatially distributed control input. For the system we propose a transformation that homogenizes the PDE by transforming the input action on the boundary. Based on the transformation, a feedforward control approach is presented, allowing for an iterative computation of the control input from a desired output trajectory. To demonstrate the approach we apply it in detail to two examples: a continuous furnace and a counterflow heat exchanger.**

I. Introduction

Realizing desired output trajectories by open loop feedforward control is an important topic in control applications as well as in control theory. From an engineering point of view, the basic problem is the appropriate inversion of the in-/output dynamics. For linear and nonlinear systems with lumped parameters, described by ordinary differential equations (ODEs), many capable methods are known. See for example [1], [2], [3] and references therein.

Control design for distributed parameter systems (DPSs) on the other hand, can roughly be grouped in two different approaches. Methods applied subsequently to an approximation are called early-lumping. Usually the approximation includes the discretization of the partial differential equations (PDEs) to obtain ODEs or difference equations. Afterwards the well known methods for lumped parameter systems can be applied to the approximated system. The counterpart are the so-called late-lumping methods, with the control design being derived directly for the distributed parameter system. Thus the often physically motivated model is advantageously maintained throughout the entire control design process. In doing so, non physically motivated parameters, like discretization parameters are avoided. But the analysis is more complicated and usually late-lumping leads to control laws that cannot be implemented directly. So again approximation is necessary, but at a late stage instead of an early stage within the design process, which motivates the naming.

Most contributions in the field of control of distributed parameter systems focus on feedback control design. Though from lumped parameter systems it is known that a supplemental feedforward control can significantly increase the tracking performance; a fact that also holds for distributed parameter systems. See for example [4], where a two-degree-of-freedom tracking control approach of a boundary controlled diffusion convection reaction system is presented. The applied method bases on a formal power series parametrization of the system and is applicable to a broad class of one-dimensional, parabolic distributed parameter systems [5], [6]. A related method is shown in [7], which enables the treatment of parabolic PDEs on higher-dimensional spatial domains, with the feedforward control being computed iteratively, by solving an integral equation. In [8] a distributed parameter system, described by first order PDEs, modeling an heat exchanger is covered. The control law is derived explicitly, but it includes so-called distributed delays described by integrals over Bessel functions. These methods emerged from the field of flatness based control for distributed parameter systems; see [9], [10] for further details.

Indeed, feedforward control of distributed parameter systems is an active area of research, but most of the cited methods are specifically designed for boundary control, whereas spatially distributed inputs are little considered. That means the corresponding PDEs are homogeneous, the control input is in the boundary conditions. As a consequence the internal dynamics and zero dynamics usually are of order zero and have not to be taken into account. This is not the case with spatially distributed control inputs, where the internal dynamics influence is fundamental within the feedforward control design. The topic of internal dynamics and zero dynamics of distributed parameter systems has gained some attention recently, see for example [11], [12].

The rest of the paper is organized as follows: Section II describes the considered control problem and our method, to solve it. Section III clarifies the approach by demonstrating the application at two examples and Section IV gives the conclusions and a short outlook.

II. Feedforward Control Problem

In this Section, the feedforward control problem is defined and the method is described, which we propose for solving it. A linear, first order distributed parameter system with scalar control input $u(t)$ and scalar output $y(t)$ is considered. The control aim is to realize a desired output trajectory $y_d(t)$, such that $y(t) = y_d(t)$. Thus the control problem is the determination of the appropriate control input $u(t)$, i.e. the considered system is to be inverted.

Consider the linear, one dimensional, first order distributed

F. Malchow and O. Sawodny are with the Institute for System Dynamics at the University of Stuttgart, Pfaffenwaldring 9, D-70569 Stuttgart {malchow,sawodny}@isys.uni-stuttgart.de

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parameter system

$$\partial_t x + v\partial_z x + \alpha x = \beta(z)u(t), \quad z = (0, L),\ t > 0 \quad (1)$$

$$x(0,t) = 0, \quad x(z,0) = 0. \quad (2)$$

with positive flow rate v and homogeneous initial and boundary conditions. The right hand side of (1) is separated in a time dependent, scalar part $u(t)$, which is the control input and a spatially dependent part $\beta(z)$, we refer to as characteristic of the input.

Let the output be defined as

$$y(t) = \int_0^L h(\zeta)x(\zeta,t)d\zeta, \quad t \geq 0 \quad (3)$$

with the weighting function $h(z)$.

We assume, that the zero dynamics of system (1)-(3) are asymptotically stable. For linear, time invariant systems, this corresponds to the minimum phase property of the associated irrational transfer function. See [13] for an overview of this topic.

A. Transformation of the Distributed Parameter System

To allow for an inversion of the distributed parameter system, we introduce a transformation, that homogenizes the PDE. A prerequisite, with respect to the characteristic, is stated in the following Assumption:

Assumption 1: The Q-th derivative of the characteristic $\beta(z) \in C^Q$ is a linear combination of the derivatives up to Q, i.e. $\beta^{(Q)}(z) = \sum_{k=0}^{Q-1} b_{k+1}\beta^{(k)}(z),\ Q > 0, b_k \in \mathbb{R}$. Also $\beta^{(i)}, \beta^{(j)},\ i,j < Q$ are linear independent for $i \neq j$. We note that the trivial case $\beta^{(Q)} \equiv 0$ is included.

For system (1)-(3), we define the transformation as

$$x(z,t) = \tilde{x}(z,t) + \mathbf{\Phi}^T(z)\boldsymbol{\xi}(t). \quad (4)$$

With the spatial dependent vector $\mathbf{\Phi}^T(z) = \begin{bmatrix}\beta(z) & \beta'(z) & \dots & \beta^{Q-1}(z)\end{bmatrix}$ consisting of the characteristic and its derivatives and the time dependent vector $\boldsymbol{\xi}(t) = \begin{bmatrix}\xi_1(t) & \xi_2(t) & \dots & \xi_Q(t)\end{bmatrix}^T$. The minimum Q that satisfies Assumption 1 is picked. Inserting transformation (4) in PDE (1) and taking Assumption 1 into account yields

$$\begin{aligned}\partial_t\tilde{x} + v\partial_z\tilde{x} + \alpha\tilde{x} &= \beta(u - \alpha\xi_1 - \dot{\xi}_1 - vb_1\xi_Q) \\ &+ \beta'(-v\xi_1 - \alpha\xi_2 - \dot{\xi}_2 - vb_2\xi_Q) + \cdots \\ &+ \beta^{(Q-1)}(-v\xi_{Q-1} - \alpha\xi_Q - \dot{\xi}_Q - vb_Q\xi_Q).\end{aligned} \quad (5)$$

From the right hand side of (5), we obtain Q ODEs for the formal variables $\boldsymbol{\xi}(t)$, depending on the control input

$$\dot{\boldsymbol{\xi}}(t) = \begin{bmatrix} -\alpha & & & -vb_1 \\ -v & -\alpha & \cdots & -vb_2 \\ & & \ddots & \vdots \\ \mathbf{0} & & -v & -\alpha - vb_Q \end{bmatrix}\boldsymbol{\xi}(t) + \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}u(t) \quad (6)$$

with zero initial conditions $\boldsymbol{\xi}(0) = 0$. System (6) is controllable, the controllability matrix is upper triangular and has full rank.

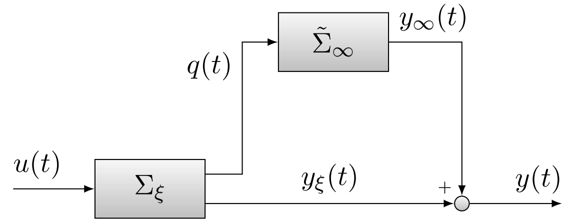


Fig. 1. System structure after transformation.

Inserting (6) in (5) yields the homogeneous PDE

$$\partial_t\tilde{x} + v\partial_z\tilde{x} + \alpha\tilde{x} = 0. \quad (7)$$

The corresponding initial and boundary conditions are obtained, by applying transformation (4) to equation (2):

$$\tilde{x}(z,0) = 0, \quad \tilde{x}(0,t) = -\mathbf{\Phi}^T(0)\boldsymbol{\xi}(t) = q(t). \quad (8)$$

Transforming the output, given by equation (3) yields

$$y(t) = \int_0^L h(\zeta)\tilde{x}(\zeta,t)d\zeta + \boldsymbol{c}^T\boldsymbol{\xi}(t) \quad (9)$$

with $\boldsymbol{c}^T = \int_0^L h(\zeta)\mathbf{\Phi}^T(\zeta)d\zeta$.

Figure 1 shows the structure of the transformed distributed parameter system. It is separated in two parts, one part is directly dependent of the input, while the other part is not. The direct part Σ_ξ is a system described by the ODEs (6), depending on $u(t)$. The indirect part $\tilde{\Sigma}_\infty$ is a boundary controlled distributed parameter system, described by the homogeneous PDE (7) and the conditions (8).

The output $y(t)$ can be described as sum of the direct output $y_\xi(t)$ and the indirect output $y_\infty(t)$. These are obtained from equation (9) as

$$y_\xi(t) = \boldsymbol{c}^T\boldsymbol{\xi}(t), \quad y_\infty(t) = \int_0^L h(\zeta)\tilde{x}(\zeta,t)d\zeta. \quad (10)$$

That reveals the benefit of the proposed transformation: instead of inverting the original distributed parameter system, we only have to invert the direct part, described by ODEs. This can be done using standard techniques like [2] or flatness based methods [3]. Still, the indirect part's influence has to be considered. This is done via an iterative algorithm, described in the following Section.

B. Iterative Computation of the Feedforward Control

Having a distributed parameter system, as specified in Figure 1, the control input $u(t)$ and the inner states $\boldsymbol{\xi}(t)$ of the direct part can be written as a functional depending on the direct parts output $y_\xi(t)$

$$u(t) = \Psi_u[y_\xi], \quad \boldsymbol{\xi}(t) = \mathbf{\Psi}_{\boldsymbol{\xi}}[y_\xi]. \quad (11)$$

Subsequently, the indirect part $\tilde{\Sigma}_\infty$, depending on $\boldsymbol{\xi}(t)$ can be solved to obtain $y_\infty(t)$

$$y_\infty(t) = \Psi_\infty[\boldsymbol{\xi}]. \quad (12)$$

Using equations (11) and (12), the actual computation of the feedforward control is done in the following iterative scheme. For the initial step the following holds

$$
\begin{aligned}
y_{\xi,0}(t) &= y_d(t) \\
u_0(t) = \Psi_u[y_{\xi,0}], \quad \boldsymbol{\xi}_0(t) &= \boldsymbol{\Psi}_{\boldsymbol{\xi}}[y_{\xi,0}] \\
y_{\infty,0}(t) &= \Psi_\infty[\boldsymbol{\xi}_0].
\end{aligned} \tag{13}
$$

So basically the influence of the indirect part to the real output is neglected in the initial step. However it is taken into account in the following steps. The k-th step is given as

$$
\begin{aligned}
y_{\xi,k}(t) &= -y_{\infty,k-1} \\
u_k(t) = \Psi_u[y_{\xi,k}], \quad \boldsymbol{\xi}_k(t) &= \boldsymbol{\Psi}_{\boldsymbol{\xi}}[y_{\xi,k}] \\
y_{\infty,k}(t) &= \Psi_\infty[\boldsymbol{\xi}_k].
\end{aligned} \tag{14}
$$

The complete feedforward control is given by the infinite series

$$
u(t) = \sum_{k=0}^{\infty} u_k(t) \tag{15}
$$

which yields the desired output

$$
\begin{aligned}
y(t) &= y_d(t) + y_{\infty,0}(t) + \sum_{k=1}^{\infty} y_{\infty,k}(t) - y_{\infty,k-1}(t) \\
&= y_d(t).
\end{aligned} \tag{16}
$$

C. Transformation of coupled PDEs

The proposed approach is not limited to distributed parameter systems of form (2). We show that it can also be applied to systems described by coupled PDEs.

Consider the linear, one dimensional first order distributed parameter system, described by two coupled PDEs

$$
\begin{aligned}
\partial_t x_1 + v_1 \partial_z x_1 + \alpha_1 x_1 &= -\lambda(x_1 - x_2) + \beta(z)u(t) \\
\partial_t x_2 + v_2 \partial_z x_2 + \alpha_2 x_2 &= \lambda(x_1 - x_2) \\
z \in (0, L), \quad & t > 0
\end{aligned} \tag{17}
$$

with nonzero flow rates v_1, v_2, positive coefficient λ and homogeneous initial and boundary conditions

$$
\begin{aligned}
x_1(z,0) = 0, \quad & x_2(z,0) = 0 \\
x_1(z_1,t) = 0, \quad & x_2(z_2,t) = 0
\end{aligned} \tag{18}
$$

and Assumption 1 holds for $\beta(z)$.

Similar to (3), the output is defined as

$$
y(t) = \sum_{k=1}^{2} \int_0^L h_k(\zeta) x_k(\zeta,t) d\zeta, \quad t \geq 0. \tag{19}
$$

For system (17)-(19), the transformation is defined as

$$
\begin{bmatrix} x_1(z,t) \\ x_2(z,t) \end{bmatrix} = \begin{bmatrix} \tilde{x}_1(z,t) \\ \tilde{x}_2(z,t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Phi}^T & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Phi}^T \end{bmatrix} \begin{bmatrix} \xi_1(t) \\ \vdots \\ \xi_{2Q}(t) \end{bmatrix}. \tag{20}
$$

Inserting transformation (20) in (17) and following the steps, described in Section II-A, yields the homogeneous PDE

$$
\begin{aligned}
\partial_t \tilde{x}_1 + v_1 \partial_z \tilde{x}_1 + \alpha_1 \tilde{x}_1 &= -\lambda(\tilde{x}_1 - \tilde{x}_2) \\
\partial_t \tilde{x}_2 - v_2 \partial_z \tilde{x}_2 + \alpha_2 \tilde{x}_2 &= \lambda(\tilde{x}_1 - \tilde{x}_2)
\end{aligned} \tag{21}
$$

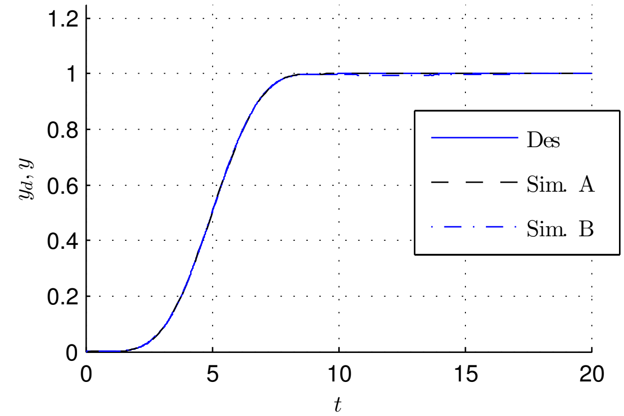


Fig. 2. Desired output trajectory $y_d(t)$ and simulated output $y(t)$ of example A and B.

with the boundary conditions

$$
\begin{aligned}
\tilde{x}_1(z_1,t) &= -\boldsymbol{\Phi}^T(z_1) \begin{bmatrix} \xi_1(t) & \dots & \xi_Q(t) \end{bmatrix}^T &= q_1(t) \\
\tilde{x}_2(z_2,t) &= -\boldsymbol{\Phi}^T(z_2) \begin{bmatrix} \xi_{Q+1}(t) & \dots & \xi_{2Q}(t) \end{bmatrix}^T &= q_2(t).
\end{aligned} \tag{22}
$$

Referring to Figure 1, the input to the boundary controlled distributed parameter system $\tilde{\Sigma}_\infty$ is $\mathbf{q}(t) = \begin{bmatrix} q_1(t) & q_2(t) \end{bmatrix}^T$.

The ODEs for the formal variables $\boldsymbol{\xi}(t)$ are obtained as

$$
\begin{aligned}
\dot{\xi}_1 &= -(\alpha_1 + \lambda)\xi_1 + \lambda \xi_{Q+1} - v_1 b_1 \xi_Q + u \\
\dot{\xi}_2 &= -v_1 \xi_1 - (\alpha_1 + \lambda)\xi_2 + \lambda \xi_{Q+2} - v_1 b_2 \xi_Q \\
&\dots \\
\dot{\xi}_Q &= -v_1 \xi_{Q-1} - (\alpha_1 + \lambda + v_1 b_Q)\xi_Q + \lambda \xi_{2Q} \\
\dot{\xi}_{Q+1} &= -(\alpha_2 + \lambda)\xi_{Q+1} + \lambda \xi_1 - v_2 b_1 \xi_{2Q} \\
\dot{\xi}_{Q+2} &= -v_2 \xi_{Q+1} - (\alpha_2 + \lambda)\xi_{Q+2} + \lambda \xi_2 - v_2 b_2 \xi_{2Q} \\
&\dots \\
\dot{\xi}_{2Q} &= -v_2 \xi_{2Q-1} - (\alpha_2 + \lambda + v_2 b_Q)\xi_{2Q} + \lambda \xi_Q
\end{aligned} \tag{23}
$$

with zero initial conditions $\boldsymbol{\xi}(0) = \mathbf{0}$. The controllability of System (23) can be shown for $\lambda \neq 0$ and $v_2 \neq 0$.

Inserting (20) in the output equation (19) yields

$$
y_\infty(t) = \sum_{k=1}^{2} \int_0^L h_k(\zeta) \tilde{x}_k(\zeta,t) d\zeta, \quad t \geq 0. \tag{24}
$$

and

$$
y_\xi(t) = \mathbf{c}^T \boldsymbol{\xi}(t), \quad \mathbf{c}^T = \begin{bmatrix} \int_0^L h_1(\zeta) \boldsymbol{\Phi}(\zeta) d\zeta \\ \int_0^L h_2(\zeta) \boldsymbol{\Phi}(\zeta) d\zeta \end{bmatrix}^T. \tag{25}
$$

III. Examples

We demonstrate the proposed approach with two examples. Both are described by first order PDEs. Example A is a simple transport process, described by a single PDE. Example B consists of two transport systems with opposing flow directions, described by two coupled PDEs. Figure 3 depicts the structure of both considered distributed parameter systems.

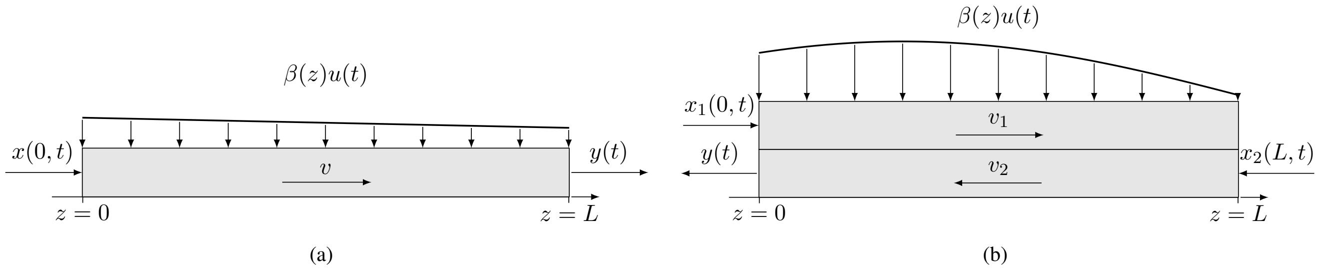


Fig. 3. Schematic diagram of example A continuous furnace (a) and example B counter-flow heat exchanger (b)

The desired output trajectory $y_d(t)$ is piecewise defined as

$$y_d(t) = \begin{cases} 0 & t \le t_0, \\ \gamma(t) & t_0 < t < T, \\ 1 & t \ge T. \end{cases} \tag{26}$$

The function $\gamma(t)$ is a polynomial of order eleven, with the first five derivatives being zero at $t = t_0$ and $t = T$. Figure 2 illustrates $y_d(t)$ with $t_0 = 0s$ and $T = 10s$.

A. Continuous Furnace

In this scenario we derive a feedforward control for the one dimensional, first order distributed parameter system (1)-(2). A typical application for this kind of distributed parameter system is a continuous furnace [14], [15], [16], [17]. Then $x(z,t)$ is a temperature at point z and time t, v a flow or feed-through velocity and α a heat loss coefficient. The heat input is the product of the control input $u(t)$ and the spatial characteristic $\beta(z)$: $u(t)$ represents the intensity of gas burners or heating lamps, whereas $\beta(z)$ includes the spatial dependence, due to geometric factors. Using (3) with the Dirac delta function $h(z) = \delta(z - L)$, the output is defined as

$$y(t) = x(L,t), \tag{27}$$

which corresponds to the outlet temperature in the case of a continuous furnace.

The spatial characteristic is given as

$$\beta(z) = 1/2 - z/6 \tag{28}$$

and with Assumption 1 we determine $Q = 2$. This can be easily verified since the second derivative is $\beta''(z) = b_1\beta(z) + b_2\beta'(z)$ with $b_1 = b_2 = 0$.

The control objective is to realize the desired output trajectory $y(t) = y_d(t)$ shown in Figure 2. To that we perform transformation (4), what yields the homogenized PDE

$$\partial_t\tilde{x} + v\partial_z\tilde{x} + \alpha\tilde{x} = 0. \tag{29}$$

with the boundary condition

$$\tilde{x}(0,t) = -[\beta(0)\ \beta'(0)]\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix}. \tag{30}$$

The ODEs of the formal variables are

$$\dot{\boldsymbol{\xi}}(t) = \begin{bmatrix}-\alpha & 0\\ -v & -\alpha\end{bmatrix}\boldsymbol{\xi}(t) + \begin{bmatrix}1\\ 0\end{bmatrix}u(t), \quad \boldsymbol{\xi}(0) = 0 \tag{31}$$

We define

$$y_\xi(t) = [\beta(L)\ \beta'(L)]\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\end{bmatrix}, \quad y_\infty(t) = \tilde{x}(L,t). \tag{32}$$

Now the distributed parameter system is transformed into the structure illustrated in Figure 1.

The feedforward control approach includes the inversion of system (31) with respect to $y_\xi(t)$. This can easily be done, for example with a flatness based approach. Introducing the flat output $y^* = \xi_2$ for system Σ_ξ yields directly a parametrization of $\boldsymbol{\xi}(t)$ and $u(t)$

$$\begin{aligned} \xi_1(t) &= -\alpha/v y^*(t) - 1/v\dot{y}^*(t),\\ \xi_2(t) &= y^*(t),\\ u(t) &= (\alpha^2 y^*(t) + 2\alpha\dot{y}^*(t) - \ddot{y}^*(t))/v. \end{aligned} \tag{33}$$

To a given reference trajectory for $y_\xi(t)$, we can calculate the corresponding flat output $y^*(t)$ by solving the zero dynamics

$$\dot{y}^*(t)\beta(L)/v + y^*(t)(\alpha/v\beta(L) - \beta'(L)) = -y_\xi(t). \tag{34}$$

With equations (33) and (34), system Σ_ξ can be inverted. So the control input $u(t)$ and the formal variables $\boldsymbol{\xi}(t)$ are dependent of $y_\xi(t)$. This corresponds to equation (11). In the example (29) and (30) can even be solved, which yields

$$y_\infty(t) = -e^{-\alpha L/v}[\beta(0)\ \beta'(0)]\begin{bmatrix}\xi_1(t - L/v)\\ \xi_2(t - L/v)\end{bmatrix}. \tag{35}$$

So we can compute $y_\infty(t)$ from the formal variables $\boldsymbol{\xi}(t)$, which is postulated in equation (12). Finally we can apply the algorithm (13),(14) to calculate the feedforward control, as specified in equation (15).

A simulation with the parameter values given in Table I is performed. The infinite series (15) is evaluated up to the fourth element. The desired trajectory of the output (26) is matched (see Figure 2). Figure 4 shows the feedforward control $u(t)$ and the spatio-temporal profile $x(z,t)$ is illustrated in Figure 5. It can be seen that there is still some control action necessary, although the output remains constant on its new level. This clearly indicates the presence of internal dynamics.

In this paper we skip the question of convergence for this example and refer to [15], where a sufficient condition for the convergence of (15) is derived. It is to mention that this example can be extended by time varying parameters and still be solved in the same way.

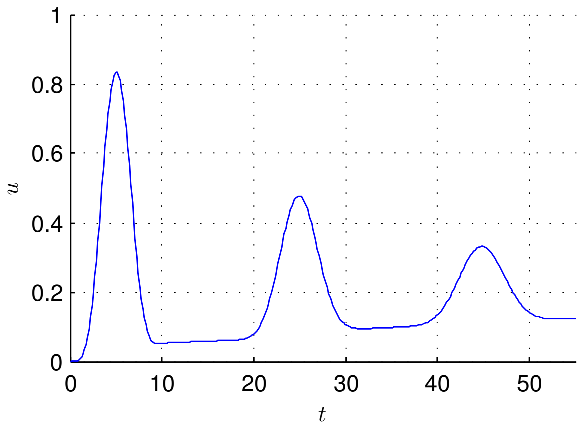


Fig. 4. Feedforward control continous furnace.

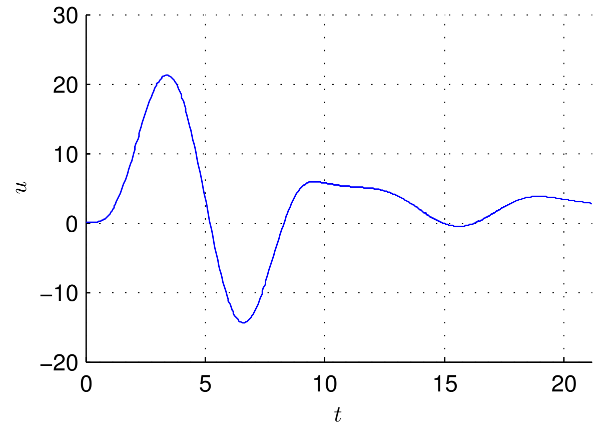


Fig. 6. Feedforward control counterflow heat exchanger.

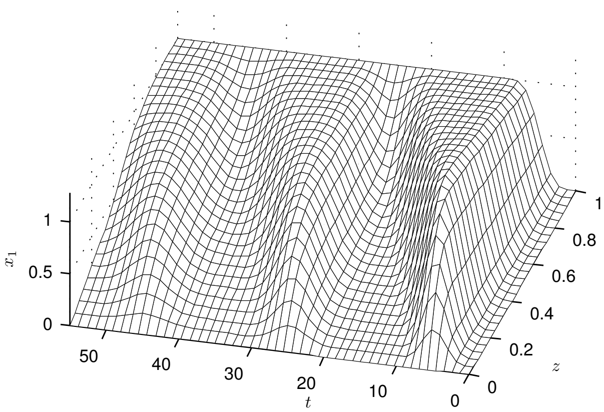


Fig. 5. Profile $x(z,t)$ with $y(t) = x(1,t)$.

B. A counterflow heat exchanger

In the second example, we consider system (17)-(18), given by two coupled, linear first order PDEs. The flow velocity v_1 is positive, while v_2 is negative, which means they are directed opposing. A typical application is a counter flow heat exchanger, see [8] for a similar problem with boundary control. We consider homogeneous boundary conditions, defined at the inlet

$$x_1(0,t) = 0, \;\; x_2(L,t) = 0. \tag{36}$$

As illsutrated in Figure 3, the output is the value at the outlet of the second layer. From (19) with $h_1(z) = 0, \;\; h_2(z) = \delta(z)$, we get

$$y(t) = x_2(0,t). \tag{37}$$

The spatial characteristic is

$$\beta(z) = \sin\left(2\pi z/(3L) + 0.3\pi\right). \tag{38}$$

Using Assumption 1, we determine $Q = 2$ and $\beta''(z) = b_1\beta(z) + b_2\beta'(z)$ with $b_1 = -(2\pi/3L)^2, \;\; b_2 = 0$.

Applying transformation (20) yields the homogenized PDEs

$$\begin{aligned} \partial_t\tilde{x}_1 + v_1\partial_z\tilde{x}_1 + \alpha_1\tilde{x}_1 &= -\lambda(\tilde{x}_1 - \tilde{x}_2) \\ \partial_t\tilde{x}_2 + v_2\partial_z\tilde{x}_2 + \alpha_2\tilde{x}_2 &= \lambda(\tilde{x}_1 - \tilde{x}_2) \end{aligned} \tag{39}$$

with the boundary conditions

$$\begin{aligned} \tilde{x}_1(0,t) &= -\mathbf{\Phi}(0)\begin{bmatrix}\xi_1(t) & \xi_2(t)\end{bmatrix}^T \\ \tilde{x}_2(L,t) &= -\mathbf{\Phi}(L)\begin{bmatrix}\xi_3(t) & \xi_4(t)\end{bmatrix}^T. \end{aligned} \tag{40}$$

Accordingly to (23), the ODEs for the formal variables $\boldsymbol{\xi}(t)$ are determined as

$$\begin{aligned} \dot{\xi}_1 &= (\lambda - \alpha_1)\xi_1 - v_1\xi_2 - \lambda\xi_3 + u(t) \\ \dot{\xi}_2 &= (\lambda - \alpha_1)\xi_2 + v_1\left(2\pi/(3L)\right)^2\xi_1 - \lambda\xi_4 \\ \dot{\xi}_3 &= (\lambda - \alpha_2)\xi_3 - v_2\xi_2 - \lambda\xi_1 \\ \dot{\xi}_4 &= (\lambda - \alpha_2)\xi_4 + v_2\left(2\pi/(3L)\right)^2\xi_1 - \lambda\xi_2 \end{aligned} \tag{41}$$

with zero initial conditions $\boldsymbol{\xi}(0) = \mathbf{0}$.

Further, the following holds

$$y_\infty(t) = \tilde{x}_2(0,t) \tag{42}$$

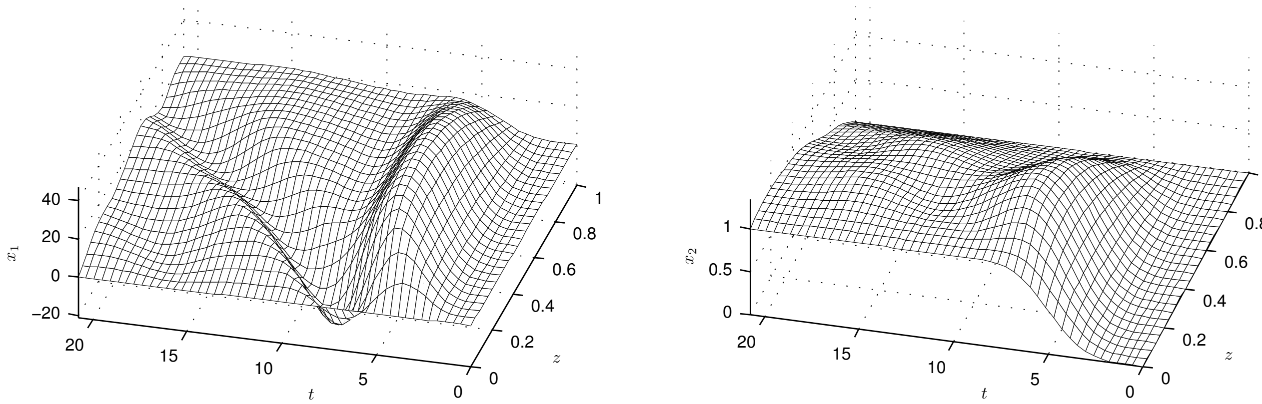
$$y_\xi(t) = \begin{bmatrix}0 & 0 & \beta(L) & \beta'(L)\end{bmatrix}\boldsymbol{\xi}(t). \tag{43}$$

Referring to Figure 1, the finite dimensional part Σ_ξ is defined through (41) and (43), whereas the infinite dimensional part $\tilde{\Sigma}_\infty$ is given by (39), (40) and (42). The feedforward control can be calculated as in the previous example.

A simulation with the parameter values given in Table I was performed. The infinite series (15) was evaluated up to the fourth element. The desired trajectory of the output (26) is matched (see Figure 2). The feedforward control $u(t)$ is plotted in Figure 6 and the spatio-temporal profiles $x_{1/2}(z,t)$ are plotted in Figure 7.

IV. Conclusions and Outlook

The feedforward control of linear first order distributed parameter systems with inhomogeneous PDE is considered. A late-lumping approach is proposed that allows for the inversion of the considered system. The control input can



(a) (b)

Fig. 7. Temporal-spatial profile of $x_1(z,t)$ (a) and $x_2(z,t)$ (b) with $y(t) = x_2(0,t)$.

TABLE I

PARAMETER VALUES USED IN EXAMPLE A AND EXAMPLE B.

Example A		Example B	
α	0.04	$\alpha_{1,2}$	0.05
v	0.05	$v_1,\ v_2$	$0.05, -0.1$
		λ	0.01
L	1	L	1

iteratively be computed in terms of an infinite series to a desired trajectory of the output.

The approach is based on a transformation of the distributed parameter system, it is separated in a direct part and an indirect part. The direct part is described by a finite dimensional linear system. Whereas the indirect part is described by a boundary controlled distributed parameter system with a homogeneous PDE. The feedforward control is obtained from inverting the direct part, which can be done using standard methods. Yet the influence of the indirect part cannot be neglected and is incorporated in the presented approach. This results in an iterative algorithm, with the complete feedforward control law given in terms of an infinite series.

We note that the applicability of the proposed method is not limited to first order nor one-dimensional distributed parameter systems. The transfer to linear second order parabolic and hyperbolic systems can be done straight forward. As it is the case for linear time variant systems or systems in two or three spatial dimensions. Yet the main objective of our current research is aimed towards general necessary conditions for the convergence of the feedforward control law, that is described by an infinite series.

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