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On Optimal Partial Broadcasting of Wireless Sensor Networks for Kalman Filtering

Qing-Shan Jia, *Member, IEEE* and Ling Shi, *Member, IEEE*

***Abstract*— State estimation using wireless sensor networks (WSNs) is an important technique in many commercial and military applications, in which a group of (nonidentical) sensors take noisy observations of system state and send back to a fusion center for state estimation through wireless broadcasting. In order to minimize the estimated state error covariance at a terminal stage at the fusion center, a partial broadcasting policy should tell which sensors to broadcast at each stage. The limited battery allows each sensor to broadcast only a few number of times. The limited wireless communication bandwidth allows only a few number of sensors to broadcast in the same time. Due to the aforementioned two couplings, the optimal partial broadcasting policy is not clear in general. Despite the abundant applications of partial broadcasting policies, theoretical analysis is rare. In this paper, we consider the scalar state estimation and provide a first study on the properties of optimal partial broadcasting policies. When there is no packet drop, a good-sensor-late-broadcast (GSLB) rule is shown to perform optimally. When there is a positive probability for packet drop, theoretical analysis suggests that the GSLB rule also has good performance.**

***Index Terms*— Wireless sensor network, partial broadcasting, Kalman filtering.**

I. INTRODUCTION

State estimation using wireless sensor networks (WSNs) has become an important technique in many commercial and military applications. Usually a group of (nonidentical) sensors take noisy observations of the system state and send back to a fusion center through wireless broadcasting. The fusion center combines all the information from the sensors and outputs a state estimation. Due to the limited battery at each sensor and the limited wireless communication bandwidth, usually only part of the sensors broadcast at a time. The policy that tells which sensor to broadcast at each time is called a partial broadcasting policy. The optimal partial broadcasting policy, which minimizes the estimated state error covariance at a terminal stage, is of interest.

There are at least three difficulties to find the optimal partial broadcasting policy. First, the limited battery capacity makes the decision making at different stages correlated. The more a sensor broadcasts during early stages of the lifetime, the less it can broadcast in the remaining stages. Thus a sensor needs to choose when to report its observation to the fusion center. Second, the limited communication bandwidth makes the decision making at different sensors correlated. Accurate sensors may have small batteries, while inaccurate sensors may have large batteries. The fusion center needs to command which sensors to broadcast at each stage. Third, the random packet drop not only degrades the amount of information that a sensor shares with the fusion center, but also makes the sequence of estimated state error covariance a stochastic sequence. This substantially complicates the theoretical analysis as will be discussed in section IV.

Due to the aforementioned difficulties, despite the abundant applications of partial broadcasting policies, theoretical analysis is rare. In this paper, we consider the finite-horizon discrete-time state estimation of a linear time-invariant scalar system and provide a first study on the properties of optimal partial broadcasting policies. When there is no packet drop, a good-sensor-late-broadcast (GSLB) rule is shown to perform optimally, which means sensors with large observation noise should not broadcast later than sensors with small observation noise. An algorithm is then presented to calculate the optimal policy. When there is a positive probability for packet drop, theoretical analysis also suggests that the GSLB rule has good performance.

The rest of the paper is organized as follows. A brief literature review is presented in section II. The problem is mathematically formulated in section III. The main results are shown in section IV, where subsection IV-A discusses the case of no packet drop and subsection IV-B discusses the case of packet drop. We briefly conclude in section V.

II. LITERATURE REVIEW

Broadcasting policy optimization is related to the sensor selection problem, where a central node selects a group of sensors to perform certain tasks. The sensor selection problem in general is equivalent to the Knapsack problem which is known to be NP-complete [1]. Many heuristics have been developed to solve this problem approximately such as selecting the most informative sensors [2], where the amount of information is quantified by entropy, distance measurement, or expected posterior distribution. Xiao et al. [3] developed an incremental selection heuristic to provide enough detection probability. Xu et al. [4] discussed different heuristics for prediction and wake-up mechanisms.

In order to consider the uncertainty in estimation and tracking, the sensor selection problem has been formulated as

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Qing-Shan Jia is with the Center for Intelligent and Networked Systems (CFINS), Department of Automation, TNLIST, Tsinghua University, Beijing 100084, China (Tel.: +86-10-62773006, fax: +86-10-62796115, email: jiaqs@tsinghua.edu.cn).

Ling Shi is with Department of Electronic and Computer Engineering, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong, China (Tel.: +852-23587055, fax: +852-23581485, email: eesling@ust.hk).

a partially observable Markov decision process (MDP) [5], [6] or a hierarchical MDP [7]. However, the large state space usually forbids an optimal solution. Approximate solutions were obtained instead.

Some researchers focus on linear Gaussian state-space models. Alriksson et al. [8] used experiments to show that a distributed approach where communication only takes place between neighbors performed almost as well as the centralized Kalman filter. Shi et al. [9], [10] provided a systematic analysis of the tradeoff between the estimation quality and the communication and computation capacity of each node. Each sensor was configured as a local multi-hop tree at a centralized center. Joshi and Boyd [11] used convex optimization to approximately solve the sensor measurement selection problem. Ambrosino et al. [12] considered more details of the channel constraint. Sinopoli et al. [13] considered the effect of i.i.d. packet drop on state estimation, and studied the statistical convergence properties of the estimation error covariance. They showed that there exists a critical value for the arrival rate of the observations, beyond which a transition to an unbounded state error covariance occurs. Huang and Dey [14] and Xie and Xie [15] considered the effect of packet drop, the dynamics of which follows a Markov chain.

Savage and La Scala [16] considered the optimal scheduling of scalar Gauss-Markov systems with a terminal cost function. A single sensor is used to measure and track multiple targets. The overall measurement budget is limited. The question is when to measure which system at each stage so that the total estimated state error variance of all systems at a terminal stage is minimized. A simple index policy was shown to perform optimally in most cases. Although both their paper and this paper consider scalar Gauss-Markov system with terminal cost, the difference is clear. First, they used a single sensor to measure and track multiple targets. But in this paper we consider state estimation of a single system through multiple sensors. As a result, the objective functions are different. They minimized the total estimated state error variance of multiple systems at a terminal stage. While we minimize the expectation of the estimated state error variance of a single system at a terminal stage. Second, a limited total measure budget is considered in [16]. There is not any constraint on communication because a single sensor is used. However, in this paper the constraints are caused by the limited communication power of each sensor and the limited wireless communication bandwidth among the sensors. Third, data packet drops are considered in this paper as a natural consequence of wireless communications among sensors, but is not considered in [16].

Li et al. [17] considered the partial broadcasting of WSNs and developed a good-estimates-first-broadcast policy in order to minimize the one-step estimated state error covariance. This paper is different from their study because we consider the estimated state error variance at a finite terminal stage, but they consider a one-stage problem.

Open-loop schedules are considered in this paper, which are easy to implement and do not require much computing capabilities from each sensors. More generally, sensors could be scheduled in a closed-loop way, say based on the difference between the state estimate at the fusion center and the state estimate that could be obtained using full (or partial) sensor information. Feedback policies of this type have been examined in the literature on event-based sampling, say [18] and [19]. Imer and Basar [20] also considered a joint encoder (at the sensor) and decoder (at the fusion center) design problem for Gauss-Markov systems with average cost criteria. These feedback policies are useful when sensors have some computing capabilities.

III. Problem Formulation

Consider a discrete-time linear time-invariant scalar system

$$x_{k+1} = ax_k + w_k, \tag{1}$$

where $|a| \geq 1$; x_k is the system state at stage k with initial value $x_0 \sim N(0, \Pi_0)$ which has Gaussian distribution; $w_k \sim N(0, q)$ is the Gaussian disturbance; $E[w_k w_j^T] = q\delta_{kj}$, $q \geq 0$; δ is the Kronecker delta function, i.e., $\delta_{kj} = 1$, $k = j$ and $\delta_{kj} = 0$, $k \neq j$. A WSN of M synchronized sensors is used to monitor the state of the system. Each sensor i can take an observation of the system state at each stage,

$$y_k^{(i)} = cx_k + v_k^{(i)}, \tag{2}$$

where $c > 0$; $y_k^{(i)}$ is the observation; $v_k^{(i)} \sim N(0, r^{(i)})$ is the Gaussian observation noise; $E[v_k^{(i)} v_j^{(l)}] = r^{(i)}\delta_{kj}\delta_{il}$, $r^{(i)} \geq 0$. We assume x_0, w_k, and v_k are mutually uncorrelated. Without loss of generality, assume that $0 < r^{(1)} < r^{(2)} < \cdots < r^{(M)}$. At each time, some sensors are selected to broadcast their local observations $y_k^{(i)}$ back to the fusion center. Let $I_k = (I_k(1), \ldots, I_k(M))^T \in \mathcal{B}^M$ denote such a selection, where $I_k(i) = 1$ means sensor i is selected for broadcasting at time k; and $I_k(i) = 0$ means that sensor i is not selected for broadcasting at time k. Let $\mathbf{I}_k = (I_1, \ldots, I_k)$ denote the selection of sensors from time 1 to k. Then a partial broadcasting policy can be represented by $\mathbf{I}_N$, where N is the length of the finite-horizon of interest. The message broadcasted by a sensor will reach the fusion center with probability $0 < \lambda \leq 1$. Let $b_k \in \mathcal{B}^M$ represent whether the message broadcasted by the sensors at time k can reach the fusion center, and define $\mathbf{b}_N = (b_1, \ldots, b_N)$. Then the set of sensors whose observations reach the fusion center at time k is $s(I_k \odot b_k)$, where $I_k \odot b_k = (I_k(1)b_k(1), \ldots, I_k(M)b_k(M))^T$ and $s(I) = \{i | I(i) = 1\}$. Recall that the optimal estimate $\hat{x}_k$ for system in Eq. (1) given $y_k^{(i)}$, $i \in s(I_k \odot b_k)$ and the previous optimal estimate $\hat{x}_{k-1}$ in Eq. (2) is computed recursively from a Kalman filter through the following equations [21]

$$\begin{aligned}
\hat{x}_{k|k-1} &= a\hat{x}_{k-1}, \\
P_{k|k-1} &= a^2 P_{k-1} + q, \\
P_k^{-1} &= P_{k|k-1}^{-1} + c^2 \sum\nolimits_{i \in s(I_k \odot b_k)} (r^{(i)})^{-1}, \\
K_k &= P_k c[\ldots, (r^{(i)})^{-1}, \ldots], i \in s(I_k \odot b_k), \\
\hat{x}_k &= \hat{x}_{k|k-1} + K_k([\ldots, y_k^{(i)}, \ldots]^T - c\hat{x}_{k|k-1}), \\
& \qquad\qquad i \in s(I_k \odot b_k),
\end{aligned}$$

where the recursion starts from $\hat{x}_0 = 0$ and $P_0 = \Pi_0$. If $s(I_k \odot b_k) = \varnothing$, then no observation reaches the fusion center at time k, and from [13]

$$\begin{aligned} \hat{x}_k &= a\hat{x}_{k-1}, \\ P_k &= a^2 P_{k-1} + q. \end{aligned}$$

In order to simplify the notation, define functions

$$\begin{aligned} h(x) &\equiv a^2 x + q, \\ \tilde{g}^I(x) &\equiv (1/x + R_I)^{-1}, \\ g^I(x) &\equiv \tilde{g}^I \circ h(x), \end{aligned}$$

where $R_I = c^2 \sum_{i \in s(I)} (r^{(i)})^{-1}$, and for functions $f_1, f_2 : \mathbb{S}_+ \to \mathbb{S}_+$, $f_1 \circ f_2(x) \equiv f_1(f_2(x))$. Then it is obvious that

$$P_k = g^{I_k \odot b_k}(P_{k-1}).$$

Further define functions

$$\begin{aligned} d(x) &\equiv x/(qx + a^2), \\ \tilde{f}^I(x) &\equiv x + R_I, \\ f^I(x) &\equiv \tilde{f}^I \circ d(x). \end{aligned}$$

Then it can be verified that $P_k^{-1} = f^{I_k \odot b_k}(P_{k-1}^{-1})$.

Given a broadcasting policy $\mathbf{I}_N$, the objective function is the expected estimated state error variance at time N, i.e., $E_{\mathbf{b}_N}[P_N(\mathbf{I}_N, \mathbf{b}_N)]$. There are two types of constraints. The limited battery at each sensor allows to broadcast only a limited number of times within the N stages, i.e., $\sum_{k=1}^N I_k(i) \leq C_i$, $i = 1, \ldots, M$. The limited communication bandwidth allows only a limited number of sensors to broadcast in the same time, i.e., $I_k^T I_k \leq B$, $k = 1, \ldots, N$. Now, the partial broadcasting policy optimization problem can be mathematically formulated as follows.

$$\min_{\mathbf{I}_N} E_{\mathbf{b}_N}\left[P_N(\mathbf{I}_N, \mathbf{b}_N)\right], \quad (3)$$

$$\text{s.t.} \quad \sum_{k=1}^N I_k(i) \leq C_i, i = 1, \ldots, M,$$

$$I_k^T I_k \leq B, k = 1, \ldots, N. \quad (4)$$

IV. Main Results

A. No packet drop

We start the discussion from this simple case and show that a simple rule performs optimally. First, we introduce this simple rule, which is called good-sensor-late-broadcast (GSLB) rule.

The Good-Sensors-Last-Broadcast (GSLB) Rule: Accurate sensors should broadcast later, i.e., if sensor i broadcasts at stage k, then a sensor j s.t. $r^{(i)} < r^{(j)}$ should not broadcast at stage $k+1, \ldots, N$.

In this subsection, we will show that when $\lambda = 1$, the GSLB rule performs optimally, which is summarized into the following theorem.

Theorem 1: When $\lambda = 1$, if a policy $\mathbf{I}_N$ violates the GSLB rule, there exists another policy $\mathbf{I}'_N$ that satisfies the GSLB rule and is no-worse than $\mathbf{I}_N$, i.e., $P_N(\mathbf{I}'_N) \leq P_N(\mathbf{I}_N)$.

We will prove Theorem 1 through three steps. First, we will show that more broadcastings are always better (Lemma 2). Then the implication is that sensor i should broadcast exactly C_i times by stage N, assuming N is sufficiently large such that this is possible. Second, sensors should broadcast as late as possible (Lemma 3). An implication is that exactly B sensors should broadcast in the last several stages, assuming $B \leq M$. Third, exchanging an early broadcasting of a good sensor with a late broadcasting of a bad sensor is always beneficial (Lemma 4). This then implies that bad sensors should not broadcast later than good sensors. We now follow the three steps to prove Theorem 1.

Lemma 1: $f^{I+e_i}(x) > f^I(x)$, $x > 0$, $I(i) = 0$, e_i is the vector with only the i-th component being 1 and the rest being 0.

Proof: We have

$$\begin{aligned} f^{I+e_i}(x) &= d(x) + R_{I+e_i} = d(x) + R_I + R_{e_i} \\ &= f^I(x) + R_{e_i} > f^I(x), \end{aligned}$$

where the last inequality is due to the fact that $R_{e_i} > 0$. ■

Lemma 1 means that $P_k^{-1}(I_k + e_i) > P_k^{-1}(I_k)$, if $I_k(i) = 0$. In order to show that adding one more broadcasting of sensor i at time k is also beneficial to P_N, we need the following properties.

Property 1: If $x_1 > x_2 > 0$, $d(x_1) > d(x_2)$.

Proof: Note that

$$\frac{d}{dx}\left(d(x)\right) = \frac{a^2}{(qx + a^2)^2} > 0.$$

Thus $d(x)$ is monotonically strictly increasing w.r.t. x. ■

Property 2: If $x_1 > x_2 > 0$, $f^I(x_1) > f^I(x_2)$, $\forall I \in \mathcal{B}^M$.

Proof: By definition, $f^I(x) = d(x) + R_I$. From Property 1, we know that $d(x)$ is monotonically increasing w.r.t. x. Thus $f^I(x)$ is monotonically increasing w.r.t x. This completes the proof. ■

Combining Lemma 1 and Property 2, we can see that $P_N^{-1}(I_k + e_i) > P_N^{-1}(I_k)$, which implies $P_N(I_k + e_i) < P_N(I_k)$. Then we have

Lemma 2: $P_N(I_k + e_i) < P_N(I_k)$, $x > 0$, $I(i) = 0$.

Lemma 2 implies that sensor i should broadcast exactly C_i times by time N. The first step towards proving Theorem 1 is completed.

We have

Property 3: $d(x) + \alpha \geq d(x + \alpha)$, $x > 0$, $\alpha > 0$.

Proof: By definition, we have

$$d(x) + \alpha - d(x+\alpha) = \frac{x}{qx + a^2} + \alpha - \frac{x + \alpha}{q(x+\alpha) + a^2}. \quad (5)$$

After some deduction, the right-hand-side (RHS) of Eq. (5) equals to

$$\frac{\alpha q^2 x^2 + \alpha(q(q\alpha + a^2) + qa^2)x + \alpha a^2(q\alpha + a^2 - 1)}{(qx + a^2)(q(x+\alpha) + a^2)}. \quad (6)$$

Note that $a \geq 1$. Thus $\alpha a^2(q\alpha + a^2 - 1) \geq 0$. So, Eq. (6)$\geq 0$. This completes the proof. ■

Lemma 3: $f^{I_2+e_i} \circ f^{I_1}(x) \geq f^{I_2} \circ f^{I_1+e_i}(x)$, $x > 0$, $I_1(i) = I_2(i) = 0$.

Proof: By definition, we have

$$\begin{aligned} f^{I_2+e_i} \circ f^{I_1}(x) &= d(f^{I_1}(x)) + R_{I_2} + R_{e_i}, \\ f^{I_2} \circ f^{I_1+e_i}(x) &= d(f^{I_1}(x) + R_{e_i}) + R_{I_2}. \end{aligned}$$

Then we have

$$\begin{aligned} & f^{I_2+e_i} \circ f^{I_1}(x) - f^{I_2} \circ f^{I_1+e_i}(x) \\ = \;& d(f^{I_1}(x)) + R_{e_i} - d(f^{I_1}(x) + R_{e_i}) \geq 0, \quad (7) \end{aligned}$$

where the last inequality follows from Property 3. ■

Postponing the broadcasting is always beneficial to P_N. The second step towards proving Theorem 1 is completed.

For the third step of the proof of Theorem 1, we have

Property 4: $d(x+\alpha)+\beta \geq d(x+\beta)+\alpha$, $\alpha < \beta$, $x > 0$.

Proof: By definition, we have

$$\begin{aligned} & (d(x+\alpha)+\beta) - (d(x+\beta)+\alpha) \\ = \;& \left(\frac{x+\alpha}{q(x+\alpha)+a^2} + \beta\right) - \left(\frac{x+\beta}{q(x+\beta)+a^2} + \alpha\right) \quad (8) \end{aligned}$$

After some deduction, we have the RHS of Eq. (8) equals to

$$\frac{(\beta-\alpha)(q^2(x+\alpha)(x+\beta) + qa^2(2x+\alpha+\beta) + a^4 - a^2)}{(q(x+\alpha)+a^2)(q(x+\beta)+a^2)}. \quad (9)$$

Because $a \geq 1$, $a^4 - a^2 \geq 0$. Thus Eq. (9)≥ 0. ■

Lemma 4: $f^{I_2+e_i} \circ f^{I_1+e_j}(x) \geq f^{I_2+e_j} \circ f^{I_1+e_i}(x)$, $x > 0$, $i < j$, $I_1(i) = I_1(j) = I_2(i) = I_2(j) = 0$.

Proof: By definition, we have

$$\begin{aligned} f^{I_2+e_i} \circ f^{I_1+e_j}(x) &= d(d(x) + R_{I_1} + R_{e_j}) + R_{I_2} + R_{e_i}, \\ f^{I_2+e_j} \circ f^{I_1+e_i}(x) &= d(d(x) + R_{I_1} + R_{e_i}) + R_{I_2} + R_{e_j}. \end{aligned}$$

Note that $R_{e_i} > R_{e_j}$. Then following Property 4, we have

$$f^{I_2+e_i} \circ f^{I_1+e_j}(x) - f^{I_2+e_j} \circ f^{I_1+e_i}(x) \geq 0.$$

■

Lemma 4 implies that exchanging an early broadcasting of a good sensor with a late broadcasting of a bad sensor is always beneficial. We can now prove Theorem 1.

Proof: (of Theorem 1) If a policy $\mathbf{I}_N$ violates the GSLB rule, we can first add broadcasting of sensor i if it has not broadcasted C_i times by time N. Second, we postpone the broadcasting of all the sensors as late as possible, while keeping the relative order among the broadcastings not changed. Third, starting from the earliest broadcasting of sensor 1, if any other sensor broadcasts later, exchange the two broadcastings. Repeat the exchange operation for sensors $2, \ldots, M$. When completed, we obtain a policy $\mathbf{I}'_N$, which satisfies the GSLB rule. Lemmas 2-4 ensure that the above three steps of modification of $\mathbf{I}_N$ does not increase P_N, i.e., $P_N(\mathbf{I}'_N) \leq P_N(\mathbf{I}_N)$. This completes the proof. ■

It is not difficult to show that following the three-step of modifications in the proof of Theorem 1, the resultant policy is the optimal policy $\mathbf{I}^*_N$. This leads to Algorithm 1 that constructs the optimal policy. Note that Algorithm 1 executes exactly $\min\{NB, \sum_{i=1}^M C_i\}$ times, which is very fast and can be easily implemented in practice.

Algorithm 1 Construct the Optimal Policy When $\lambda = 1$.

```
Initialization: n = 0, k = N, I_N = 0.
Iterative allocation:
for i = 1 to M do
  for j = 1 to C_i do
    I_k(i) = 0, n = (n mod B) + 1.
    if n = B then
      k = k - 1.
      if k < 1 then
        Stop and output I_N.
      end if
    end if
  end for
end for
Output I_N.
```

B. Packet drop

When packet dropping happens with positive probability, i.e., $\lambda < 1$, the message sent by a sensor may not reach the fusion center. Then the estimated state error variance is a random variable, which complicates the analysis. In order to show similar results as in subsection IV-A, we need to take a sample path view, i.e., to compare the performance of different policies on each (pair) of sample paths. This technique will be further explained in the following analysis.

We still follow three steps to show the benefit of the GSLB rule when $\lambda < 1$. First, more broadcastings are beneficial.

Lemma 5: $E_b[f^{(I+e_i)\odot b}(x)] > E_b[f^{I\odot b}(x)]$, $x > 0$, $I(i) = 0$.

Proof: Note that

$$\begin{aligned} E_b[f^{(I+e_i)\odot b}(x)] &= \frac{1}{|\mathcal{B}^M|} \sum_{b \in \mathcal{B}^M} f^{(I+e_i)\odot b}(x), \\ E_b[f^{I\odot b}(x)] &= \frac{1}{|\mathcal{B}^M|} \sum_{b \in \mathcal{B}^M} f^{I\odot b}(x). \end{aligned}$$

Note that $(I+e_i)\odot b = I\odot b + e_i \odot b$. Then from Lemma 1, we have

$$\begin{aligned} f^{(I+e_i)\odot b}(x) &> f^{I\odot b}(x), \text{ if } b(i) = 1, \\ f^{(I+e_i)\odot b}(x) &= f^{I\odot b}(x), \text{ if } b(i) = 0. \end{aligned}$$

Since $\Pr\{b(i) = 1\} = \lambda > 0$, we then have

$$\sum_{b \in \mathcal{B}^M} f^{(I+e_i)\odot b}(x) > \sum_{b \in \mathcal{B}^M} f^{I\odot b}(x).$$

Then we have $E_b[f^{(I+e_i)\odot b}(x)] > E_b[f^{I\odot b}(x)]$. ■

However, since $E[P_k^{-1}] \neq 1/E[P_k]$, Lemma 5 only shows $E[P_k^{-1}(I_k+e_i)] > E[P_k^{-1}(I_k)]$, but not

$$E[P_k(I_k+e_i)] < E[P_k(I_k)]. \quad (10)$$

Fortunately, Eq. (10) also holds.

Lemma 6: $E_b[g^{(I+e_i)\odot b}(x)] < E_b[g^{I\odot b}(x)]$, $x > 0$, $I(i) = 0$.

Proof: Note that

$$g^{(I+e_i)\odot b}(x) = (f^{(I+e_i)\odot b}(1/x))^{-1},$$
$$g^{I\odot b}(x) = (f^{I\odot b}(1/x))^{-1}.$$

Lemma 5 shows that $f^{(I+e_i)\odot b}(1/x) \geq f^{I\odot b}(1/x)$, where the inequality is strict if $b(i) = 1$. Then we have

$$g^{(I+e_i)\odot b}(x) \leq g^{I\odot b}(x). \quad (11)$$

Since $\Pr\{b(i) = 1\} = \lambda > 0$, we have $E_b[g^{(I+e_i)\odot b}(x)] < E_b[g^{I\odot b}(x)]$. ■

Lemma 6 implies that the additional broadcasting of a sensor at time k reduces $E[P_k]$. It turns out that $E[P_N]$ is also reduced, but we need some additional analysis to show this.

Property 5: If $x_1 > x_2 > 0$, $g^I(x_1) > g^I(x_2)$, $\forall I$.

Proof: Note that

$$g^I(x_1) = (f^I(1/x_1))^{-1},$$
$$g^I(x_2) = (f^I(1/x_2))^{-1}.$$

Property 2 shows that $f^I(1/x_1) < f^I(1/x_2)$. Thus we have $g^I(x_1) > g^I(x_2)$. ■

Theorem 2: $E_{\mathbf{b}_N}[g^{I_N\odot b_N} \circ \cdots \circ g^{I_{k+1}\odot b_{k+1}} \circ g^{(I_k+e_i)\odot b_k}(x)] < E_{\mathbf{b}_N}[g^{I_N\odot b_N} \circ \cdots \circ g^{I_{k+1}\odot b_{k+1}} \circ g^{I_k\odot b_k}(x)]$, $x > 0$, $I_k(i) = 0$.

Proof: Eq. (11) shows that $g^{(I_k+e_i)\odot b_k}(x) \leq g^{I_k\odot b_k}(x)$, where the inequality is strict if $b_k(i) = 1$. Then following Property 5, we have

$$g^{I_N\odot b_N} \circ \cdots \circ g^{I_{k+1}\odot b_{k+1}} \circ g^{(I_k+e_i)\odot b_i}(x)$$
$$\leq g^{I_N\odot b_N} \circ \cdots \circ g^{I_{k+1}\odot b_{k+1}} \circ g^{I_k\odot b_k}(x).$$

Because $\Pr\{b_k(i) = 1\} = \lambda > 0$, we then have $E_{\mathbf{b}_N}[P_N(I_k + e_i, \mathbf{b}_N)] < E_{\mathbf{b}_N}[P_N(I_k, \mathbf{b}_N)]$. ■

Comparing the above analysis with the first step towards proving Theorem 1, we can see that the analysis is complicated by the fact that P_N is a random variable when $\lambda < 1$.

Now, one may wish to follow similar analysis to show that postponing a broadcasting is always beneficial, i.e.,

$$g^{(I_{k+1}+e_i)\odot b_{k+1}} \circ g^{I_k\odot b_k}(x)$$
$$\leq g^{I_{k+1}\odot b_{k+1}} \circ g^{(I_k+e_i)\odot b_k}(x), \forall b_k, b_{k+1}. \quad (12)$$

Unfortunately, Eq. (12) does not hold if $b_k(i) = 1$ and $b_{k+1}(i) = 0$. The reason is in this sample path, if sensor i broadcasts at time k, the message will reach the fusion center. But if sensor i broadcasts at time $k+1$, the packet will drop. Fortunately, we are still able to show that

Theorem 3: $E_{\mathbf{b}_N}[g^{I_N\odot b_N} \circ \cdots \circ g^{(I_{k+1}+e_i)\odot b_{k+1}} \circ g^{I_k\odot b_k}(x)] \leq E_{\mathbf{b}_N}[g^{I_N\odot b_N} \circ \cdots \circ g^{I_{k+1}\odot b_{k+1}} \circ g^{(I_k+e_i)\odot b_k}(x)]$, $x > 0$, $I_k(i) = I_{k+1}(i) = 0$.

Proof: We will need a different technique to show this. The idea is to construct a different sample path $\mathbf{b}'_N$. The only difference between $\mathbf{b}_N$ and $\mathbf{b}'_N$ are that $b'_k(i) = b_{k+1}(i)$, $b'_{k+1}(i) = b_k(i)$. Note that

$$g^{(I_{k+1}+e_i)\odot b'_{k+1}} \circ g^{I_k\odot b'_k}(x)$$
$$= (f^{(I_k+e_i)\odot b'_{k+1}} \circ f^{I_k\odot b'_k}(1/x))^{-1},$$
$$g^{I_{k+1}\odot b_{k+1}} \circ g^{(I_k+e_i)\odot b_k}(x)$$
$$= (f^{I_{k+1}\odot b_{k+1}} \circ f^{(I_k+e_i)\odot b_k}(1/x))^{-1}.$$

Case 1: $b_k(i) = 1$. We have

$$(I_{k+1} + e_i) \odot b'_{k+1} = I_{k+1} \odot b_{k+1} + e_i,$$
$$(I_k + e_i) \odot b_k = I_k \odot b_k + e_i.$$

Then from Lemma 3, we have

$$f^{(I_{k+1}+e_i)\odot b'_{k+1}} \circ f^{I_k\odot b'_k}(1/x)$$
$$\geq f^{I_{k+1}\odot b_{k+1}} \circ f^{(I_k+e_i)\odot b_k}(1/x).$$

Case 2: $b_k(i) = 0$. We have

$$(I_{k+1} + e_i) \odot b'_{k+1} = I_{k+1} \odot b_{k+1},$$
$$(I_k + e_i) \odot b_k = I_k \odot b_k.$$

Then we have

$$f^{(I_{k+1}+e_i)\odot b'_{k+1}} \circ f^{I_k\odot b'_k}(1/x)$$
$$= f^{I_{k+1}\odot b_{k+1}} \circ f^{(I_k+e_i)\odot b_k}(1/x).$$

Combing the above two cases, we have

$$f^{(I_{k+1}+e_i)\odot b'_{k+1}} \circ f^{I_k\odot b'_k}(1/x)$$
$$\geq f^{I_{k+1}\odot b_{k+1}} \circ f^{(I_k+e_i)\odot b_k}(1/x).$$

Combining the above equations together, we then have

$$g^{(I_{k+1}+e_i)\odot b'_{k+1}} \circ g^{I_k\odot b'_k}(x) \leq g^{I_{k+1}\odot b_{k+1}} \circ g^{(I_k+e_i)\odot b_k}(x).$$

Note that for every $\mathbf{b}_N$, such a $\mathbf{b}'_N$ can be constructed as above. And all such $\mathbf{b}'_N$'s are not repeated. Since $\Pr\{b_k(i) = 1\} = \lambda > 0$, combining with Property 5, we have $E_{\mathbf{b}'_N}[g^{I_N\odot b'_N} \circ \cdots \circ g^{(I_{k+1}+e_i)\odot b'_{k+1}} \circ g^{I_k\odot b_k}(x)] \leq E_{\mathbf{b}_N}[g^{I_N\odot b_N} \circ \cdots \circ g^{I_{k+1}\odot b_{k+1}} \circ g^{(I_k+e_i)\odot b_k}(x)]$. ■

For the third step towards showing the good performance of the GSLB rule, we need to show that exchanging an early broadcasting of a good sensor with a late broadcasting of a bad sensor is beneficial. One may wish to follow the above technique to show this. Unfortunately, this does not work. To be specific, let $\mathbf{I}_N$ denote the policy, in which sensor i broadcasts at time k and sensor j broadcasts at $k+1$, $i < j$. Let $\mathbf{I}'_N$ be the policy constructed in the above way, in which sensor i broadcasts at time $k+1$ and sensor j broadcasts at k. Let $\mathbf{b}_N$ be a randomness matrix. We can construct another randomness matrix $\mathbf{b}'_N$, in which the only difference between $\mathbf{b}_N$ and $\mathbf{b}'_N$ are $b'_k(i) = b_{k+1}(i), b'_k(j) = b_{k+1}(j), b'_{k+1}(i) = b_k(i), b'_{k+1}(j) = b_k(j)$. When $b_k(i) = 0$ and $b_{k+1}(j) = 1$, in policy $\mathbf{I}_N$, the early broadcasting of sensor i does not reach the fusion center, while the late broadcasting of sensor j reaches. Then in policy $\mathbf{I}'_N$, $b'_{k+1}(i) = 0$ and $b'_k(j) = 1$. This means only the early broadcasting of sensor j reaches the fusion center. Since a later broadcasting is better, $P_N(\mathbf{I}'_N, \mathbf{b}'_N) > P_N(\mathbf{I}_N, \mathbf{b}_N)$, which means exchanging the broadcasting of two sensors degrades the performance on this pair of sample paths. Fortunately, when considering the expected value, we have

Theorem 4: $E[f^{(I_{k+1}+e_i)\odot b_{k+1}} \circ f^{(I_k+e_j)\odot b_k}(x)] \geq E[f^{(I_{k+1}+e_j)\odot b_{k+1}} \circ f^{(I_k+e_i)\odot b_k}(x)]$, $x > 0$, $i < j$, $I_k(i) = I_k(j) = I_{k+1}(i) = I_{k+1}(j) = 0$.

Proof: For any given $(b_k, b_{k+1}) \in \mathcal{B}^2$, construct (b'_k, b'_{k+1}) s.t. $b'_k(i) = b_{k+1}(i), b'_k(j) = b_{k+1}(j), b'_{k+1}(i) =$

$b_k(i), b'_{k+1}(j) = b_k(j), b'_t(l) = b_t(l), l \neq i, j, t = k, k+1$. It is easy to verify that when $(b_k(i), b_{k+1}(j)) = (0, 0)$, we have

$$\begin{aligned} & f^{(I_{k+1}+e_i)\odot b'_{k+1}} \circ f^{(I_k+e_j)\odot b'_k}(x) \\ = \; & f^{(I_{k+1}+e_j)\odot b_{k+1}} \circ f^{(I_k+e_i)\odot b_k}(x). \end{aligned}$$

When $(b_k(i), b_{k+1}(j)) = (1, 1)$, we have

$$\begin{aligned} & f^{(I_{k+1}+e_i)\odot b'_{k+1}} \circ f^{(I_k+e_j)\odot b'_k}(x) \\ \geq \; & f^{(I_{k+1}+e_j)\odot b_{k+1}} \circ f^{(I_k+e_i)\odot b_k}(x). \end{aligned}$$

When $(b_k(i), b_{k+1}(j)) \in \{(0, 1), (1, 0)\}$, we have

$$\begin{aligned} & \sum_{(b_k(i), b_{k+1}(j)) \in \{(0,1),(1,0)\}} f^{(I_{k+1}+e_i)\odot b'_{k+1}} \circ f^{(I_k+e_j)\odot b'_k}(x) \\ & - \sum_{(b_k(i), b_{k+1}(j)) \in \{(0,1),(1,0)\}} f^{(I_{k+1}+e_j)\odot b_{k+1}} \circ f^{(I_k+e_i)\odot b_k}(x) \geq 0 \end{aligned} \quad (13)$$

Note that $\Pr\{b_k(i) = 0, b_{k+1}(j) = 1\} = \Pr\{b_k(i) = 1, b_{k+1}(j) = 0\} = \lambda(1-\lambda)$. Combining the above three cases, we have

$$\begin{aligned} & E[f^{(I_{k+1}+e_i)\odot b_{k+1}} \circ f^{(I_k+e_j)\odot b_k}(x)] \\ \geq \; & E[f^{(I_{k+1}+e_j)\odot b_{k+1}} \circ f^{(I_k+e_i)\odot b_k}(x)]. \end{aligned}$$

■

Note that Theorem 4 means that $E[P_{k+1}^{-1}(\mathbf{I}_N)] \leq E[P_{k+1}^{-1}(\mathbf{I}'_N)]$. However, we have not been able to show $E[P_N^{-1}(\mathbf{I}_N)] \leq E[P_N^{-1}(\mathbf{I}'_N)]$, neither $E[P_N(\mathbf{I}_N)] \geq E[P_N(\mathbf{I}'_N)]$. Thus we have not been able to show that GSLB rule leads to the optimal policy. But the above analysis implies that GSLB rule might give good performance, if not optimal. How to theoretically quantify the performance loss, if any, will be a future research topic.

V. Conclusion

In this paper, we consider the discrete-time Kalman filtering of a linear time-invariant scalar system using WSNs, where each sensor has limited communication budget and the WSN has a limited wireless communication bandwidth. First, when there is no packet drop, the good-sensor-late-broadcast (GSLB) rule is shown to provide the optimal performance. An algorithm is developed to obtain the optimal policy within $\sum_{i=1}^{M} C_i$ steps of calculations, which is easy to use in practice. Second, when there is a positive probability for packet drop, we show that the GSLB rule improves $E[P_k^{-1}]$. Though we have not been able to show that the GSLB rule reduces $E[P_k]$, the theoretical analysis suggests that the GSLB rule might have good performance, if not optimal. How to theoretically quantify the performance loss, if any, is a future work. It is also an important future research topic to consider the vector system state, and random delay in the wireless communication instead of packet dropping. Note that the case of deterministic packet delay has been addressed in [22].

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