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Unified Approach to Trajectory Tracking and Set-Point Control for a Front-axle Driven Car-like Mobile Robot

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Abstract— **The paper presents a unified control framework for trajectory tracking and set-point motion tasks for a front-axle driven car-like robot. The concept relies on two crucial elements. The first one follows from reformulation of the vehicle kinematics into the form of the unicycle model for the vehicle body augmented with the front-wheel steering dynamics. The second one relies on application of the original Vector-Field-Orientation control strategy to the obtained unicycle subsystem. Physical inputs of the car-like vehicle are reconstructed from the inputs of the unicycle using the inverse procedure to the model reformulation conducted at the beginning. Performance and robustness of the proposed solution is illustrated by simulations.**

I. Introduction

Trajectory tracking and set-point regulation are two classical motion control problems widely treated for wheeled mobile robots in the automation and robotics literature. One can find several separate solutions for the one of mentioned tasks, mainly proposed for the most popular unicycle robot, [6]. The car-like kinematics – more difficult in control – has not attracted so much attention. Let us recall [3], [8], [7], [1], and [11] where the tracking task for car-like vehicle has been solved using different design techniques and mathematical tools. Set-point control for the kinematic car has been treated for example in [2] and [10]. None of the recalled propositions was able to simultaneously solve in a unified manner the two tasks under consideration. Unification seems to be desirable due to possibility of simpler control implementation and similar performance obtainable in a closed-loop system for both control tasks. To the authors' best knowledge, the only one proposition which solves the tracking and set-point stabilization in the unified manner is the transverse-function control approach presented for instance in [9]. The unified control design strategy introduced there can be used for systems which are described on Lie groups and leads to continuous controllers guaranteeing *practical* convergence of errors for a rich set of reference signals. It is worth to note that the system considered in the paper is not described on a Lie group. As a consequence left-invariance property is not satisfied and a change of coordinates involves change of control variables.

In this paper we propose an alternative, piecewise continuous in nature, Vector-Field-Orientation (VFO) control design strategy for a front-axle driven car-like mobile robot[1].

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[1]In contrast to almost all works cited above (except [3]), where the rear-axle driven kinematics were considered.

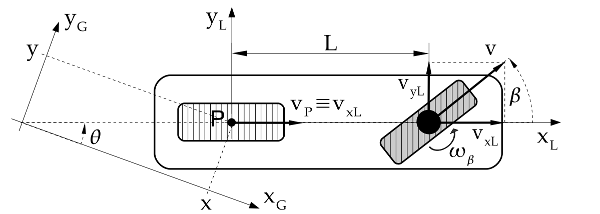

Fig. 1. Configuration coordinates of the car-like vehicle in a global frame

Utilization of the VFO design framework allows solving trajectory tracking and set-point tasks with asymptotic or *practical* convergence of errors applying a common controller structure in both cases. The VFO control strategy proposed in the paper feature an intuitive geometrical interpretation leading to simple parametric synthesis and *natural* (non-oscillatory) transients of the closed-loop system.

II. Problem statement

Consider the front-axle driven car-like kinematic structure depicted in Fig. 1 as a simple model of many practical vehicles like automobiles or working machines (e.g. forklifts). The vehicle kinematics is a driftless two-input system

$$\begin{bmatrix}\dot{\beta}\\ \dot{\theta}\\ \dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} u_1 + \begin{bmatrix}0\\ \frac{1}{L}\sin\beta\\ \cos\beta\cos\theta\\ \cos\beta\sin\theta\end{bmatrix} u_2, \tag{1}$$

where particular configuration coordinates are defined in Fig. 1 and have the following interpretation: $(x, y) \in \mathbb{R}^2$ – position coordinates of the *guidance point* P attached to the middle of the rear wheel/axle, $\theta \in \mathbb{R}$ – orientation of the vehicle body, $\beta \in [-\beta_m, \beta_m],\ \beta_m > 0$ – steering angle of the front wheel. Note that one can take into account three cases of β_m values: $\beta_m < \frac{\pi}{2}$ yielding kinematics of the curvature-limited motion capabilities, $\beta_m = \infty$ (implying $\beta \in \mathbb{R}$) which allows unconstrained pivoting of the front wheel, and $\beta_m = \frac{\pi}{2}$ as a *compromise* case yielding unlimited curvature of motion possible with simpler practical realization of the steering mechanism. Hereafter the latter case of $\beta_m = \frac{\pi}{2}$ is assumed.

The two control inputs of a vehicle are: $u_1 = \omega_\beta \in \mathbb{R}$ which is equivalent to the angular steering velocity of the front wheel, and $u_2 = v \in \mathbb{R}$ being a longitudinal velocity of the front driven wheel.

Rewriting (1) in a more general form as $\dot{\boldsymbol{q}} = \boldsymbol{g}_1 u_1 + \boldsymbol{g}_2(\boldsymbol{q}) u_2$ one can introduce the reference kinematics

$$\dot{\boldsymbol{q}}_t = \boldsymbol{g}_1 u_{1t} + \boldsymbol{g}_2(\boldsymbol{q}_t) u_{2t}, \tag{2}$$

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which allows defining an admissible reference signals $\boldsymbol{q}_t = [\beta_t\ \theta_t\ x_t\ y_t]^T$. For non-zero and bounded reference inputs u_{1t}, u_{2t} and assuming initial condition $\boldsymbol{q}_t(0)$ one obtains an admissible reference trajectory $\boldsymbol{q}_t(\tau)$ as a solution of equation (2). For our purposes we additionally assume that trajectory $\boldsymbol{q}_t(\tau)$ is sufficiently smooth such that

$$\dot{u}_{1t}(\tau), \dot{u}_{2t}(\tau), \ddot{u}_{2t}(\tau) \in \mathcal{L}_\infty, \tag{3}$$

and it is *persistently exciting* (PE) satisfying

$$\forall \tau \geqslant 0 \quad u_{2t}(\tau)\cos\beta_t(\tau) \neq 0. \tag{4}$$

Condition (4) means that longitudinal velocity of the point P of the reference vehicle never reverses its sign (velocity of point P is denoted in Fig. 1 as $v_P \equiv v_{xL} = v\cos\beta$). Solving (2) for $u_{1t} = u_{2t} \equiv 0$ yields a degenerated trajectory in a form of a constant set-point reference $\boldsymbol{q}_t = \boldsymbol{q}_t(0) = const$. Let us define the control design problem under consideration.

Problem 1: For $\boldsymbol{q}_t = [\beta_t\ \theta_t\ x_t\ y_t]^T \in [-\beta_m, \beta_m] \times \mathbb{R} \times \mathbb{R}^2$ being a solution of (2) determine a bounded feedback control $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{q}_t, \boldsymbol{q}, \cdot)$, $\boldsymbol{u} = [u_1\ u_2]^T$ for kinematics (1), which guarantees convergence of the configuration error

$$\boldsymbol{e}(\tau) = \begin{bmatrix} e_\beta(\tau) \\ e_\theta(\tau) \\ e_x(\tau) \\ e_y(\tau) \end{bmatrix} \triangleq \begin{bmatrix} \beta_t - \beta(\tau) \\ f_\theta(\theta_t - \theta(\tau)) \\ x_t - x(\tau) \\ y_t - y(\tau) \end{bmatrix} \tag{5}$$

in the sense that

$$\lim_{\tau\to\infty} \|\boldsymbol{e}(\tau)\| \leqslant \epsilon, \qquad \epsilon \geqslant 0, \tag{6}$$

where $f_\theta(\cdot) : \mathbb{R} \mapsto \mathbb{S}^1$, ϵ is some vicinity of the origin, and $\boldsymbol{q}_t := \boldsymbol{q}_t(\tau) = [\beta_t(\tau)\ \theta_t(\tau)\ x_t(\tau)\ y_t(\tau)]^T$ for trajectory tracking and $\boldsymbol{q}_t := [0\ \theta_t\ x_t\ y_t]^T$ for a set-point control task.

Condition defined in (6) includes two kind of convergence possibilities – asymptotic one if $\epsilon = 0$ and *practical* one if $\epsilon > 0$ (see [9]).

The rest of the paper describes a new solution to Problem 1 employing the VFO design framework presented recently in [4] for the unicycle model.

III. General control strategy

To utilize the VFO concept for the car-like kinematics, let us reformulate equations (1) introducing the fictitious inputs:

$$v_1 := (1/L)\sin\beta u_2, \qquad v_2 := \cos\beta u_2. \tag{7}$$

Comparing Fig. 1 it is evident that v_1 has a meaning of angular velocity of the vehicle platform, and v_2 is a longitudinal velocity of the guidance point P ($v_2 \equiv v_P$). Now, kinematics (1) can be rewritten and decomposed as:

$$\dot{\beta} = u_1, \tag{8}$$

$$\begin{bmatrix} \dot{\theta} \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} v_1 + \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} v_2. \tag{9}$$

Subsystem (9) describes the vehicle body kinematics, while (8) represents the steering angle dynamics. Since (9) has a form of the unicycle model, one can directly apply the VFO control approach to this subsystem – it will be conducted in the next subsection. Let us now assume that fictitious inputs v_1 and v_2 of (9) have been designed in the VFO framework.

In order to physically realize control action of the fictitious inputs by the original kinematics (1) one can recover, by combination of equations (7), the two simple relations:

$$u_2 = v_2\cos\beta + Lv_1\sin\beta, \tag{10}$$

$$\beta = \arctan\left(\frac{Lv_1}{v_2}\right) \quad \text{for} \quad \|\boldsymbol{v}\| \neq 0, \tag{11}$$

where $\boldsymbol{v} = [v_1\ v_2]^T$ and $\|\boldsymbol{v}\| = \sqrt{v_1^2 + v_2^2}$. The above relations describe how to define u_2 input and how to shape the time-behavior of β variable to obtain the control action of fictitious inputs v_1, v_2 designed before. In general, relation (11) cannot be met instantaneously due to dynamics (8). Hence, we propose to introduce an auxiliary steering variable $\beta_a \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ as follows:

$$\beta_a(\tau) \triangleq \arctan\left(\frac{Lv_1(\tau)}{v_2(\tau)}\right) \quad \text{for} \quad \|\boldsymbol{v}(\tau)\| \neq 0. \tag{12}$$

Defining now an auxiliary steering error

$$e_{\beta a}(\tau) \triangleq \beta_a(\tau) - \beta(\tau). \tag{13}$$

one can satisfy condition (11) by making the error $e_{\beta a}(\tau)$ converge to zero. It can be accomplished recalling again (8) and proposing the following simple steering law:

$$u_1 \triangleq k_\beta\, e_{\beta a} + \dot{\beta}_a, \tag{14}$$

where $k_\beta > 0$ is a design parameter and

$$\dot{\beta}_a = \frac{L(\dot{v}_1 v_2 - v_1 \dot{v}_2)}{L^2 v_1^2 + v_2^2} \quad \text{for} \quad \|\boldsymbol{v}(\tau)\| \neq 0 \tag{15}$$

is a time-derivative of the auxiliary steering variable given by (12). Particular forms of the right-hand-side terms in (15) result from the VFO design stage described further in the paper. For the tracking case (considered in Section IV-A) it can be shown that the time-derivatives of fictitious inputs in (15) are expressed as functions of subsequent time-derivatives of reference inputs justifying smoothness requirement given in (3).

Remark 1: Definition (12) and time-derivative (15) are not determined for time instants $\underline{\tau}$ when $\|\boldsymbol{v}(\underline{\tau})\| = 0$. In this case one can introduce additional definitions, for example $\beta_a(\underline{\tau}) := \lim_{\tau\to\underline{\tau}} \beta_a(\tau)$ and $\dot{\beta}_a(\underline{\tau}) := 0$ activated for all $\underline{\tau}$ when $\|\boldsymbol{v}(\underline{\tau})\| = 0$. In practice, one may prefer replace the last condition by $\|\boldsymbol{v}(\underline{\tau})\| < \delta$ with $\delta > 0$ being a sufficiently small vicinity of zero.

Summarizing, the general control scheme for kinematics (1) consists of three steps: 1° determine the fictitious inputs v_1, v_2 for unicycle-like (vehicle body) subsystem (9) employing the VFO approach, 2° using the fictitious inputs compute the original vehicle input u_2 according to (10), the auxiliary signals (12)-(13), and time-derivative (15), 3° compute the original control input u_1 for kinematics (1) according to (14).

Let us now describe the step 1° of the above scheme. In order to explain application of the VFO control design to the unicycle-like kinematics (9), let us first introduce the so-called *convergence vector field*:

$$\boldsymbol{h} = [h_1 \ h_2 \ h_3]^T = \left[h_1 \ \boldsymbol{h}^{*T}\right]^T \in \mathbb{R}^3, \qquad (16)$$

defined in the tangent space of (9). Assume that (16) is properly constructed defining at every state point $\overline{\boldsymbol{q}} = [\theta \ x \ y]^T$ the *convergence velocity* vector $\boldsymbol{h}(\overline{\boldsymbol{q}}, \overline{\boldsymbol{q}}_t, \cdot)$ for system (9). The expression *convergence velocity* means here that $\boldsymbol{h}(\overline{\boldsymbol{q}}, \overline{\boldsymbol{q}}_t, \cdot)$ represents an instantaneous velocity vector which, if followed by system (9), guarantees its convergence to reference signals $\overline{\boldsymbol{q}}_t = [\theta_t \ x_t \ y_t]^T$. Therefore, one naturally is interested in satisfaction of the following relation: $\lim_{\tau\to\infty}(\dot{\overline{\boldsymbol{q}}}(\tau) - \boldsymbol{h}(\tau)) = \boldsymbol{0}$, which can be obtained by appropriately designed fictitious inputs v_1 and v_2. Using the right-hand-side terms of (9) this relation can be rewritten as follows:

$$\lim_{\tau\to\infty} \begin{bmatrix} v_1(\tau) - h_1(\tau) \\ v_2(\tau)\cos\theta(\tau) - h_2(\tau) \\ v_2(\tau)\sin\theta(\tau) - h_3(\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (17)$$

By simple combination of terms in (17) one obtains three design formulas – the first one which cannot be met instantaneously (due to the integral relation from the first row of (9)):

$$\lim_{\tau\to\infty} [\theta(\tau) - \text{Atan2c}\,(\text{sgn}(v_2)h_3(\tau), \text{sgn}(v_2)h_2(\tau))] = 0, \qquad (18)$$

where $\text{sgn}(\cdot) \in \{+1, -1\}$, $\text{Atan2c}\,(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ being a continuous version[2] of the $\text{Atan2}\,(\cdot,\cdot) : \mathbb{R}\times\mathbb{R} \mapsto (-\pi, \pi]$, and the next two formulas which can be satisfied instantaneously by direct definition of fictitious inputs:

$$v_1(\tau) \triangleq h_1(\tau), \qquad (19)$$

$$v_2(\tau) \triangleq h_2(\tau)\cos\theta(\tau) + h_3(\tau)\sin\theta(\tau). \qquad (20)$$

Equations (19) and (20) describe the general VFO control law for subsystem (9) which is valid for both considered motion tasks – tracking and set-point regulation. Relation (18) represents the so-called *orienting condition*. It explains how to reorient the second vector field of (9) – by change of θ variable – in order to match its orientation with that defined by $\boldsymbol{h}(\overline{\boldsymbol{q}}, \overline{\boldsymbol{q}}_t, \cdot)$. To meet (18) we introduce an *auxiliary orienting variable*

$$\theta_a(\tau) \triangleq \text{Atan2c}\,(\sigma h_3(\tau), \sigma h_2(\tau)) \in \mathbb{R}, \qquad (21)$$

with a *decision factor* $\sigma \in \{+1, -1\}$, which allows determining desired motion strategy for the vehicle (forward or backward motion) – it will be defined in the next section. Introducing now an auxiliary orienting error

$$e_{\theta a}(\tau) \triangleq \theta_a(\tau) - \theta(\tau), \qquad e_{\theta a}(\tau) \in \mathbb{R} \qquad (22)$$

we may consider satisfaction of (18) as equivalent to driving (22) to zero. From (9) it is evident that one can accomplish it using the first fictitious input v_1. Since (19) is valid, we have to appropriately design the first component of the convergence vector field. Let us propose to take:

$$h_1 \triangleq k_\theta\, e_{\theta a} + \dot{\theta}_a \qquad (23)$$

where $k_\theta > 0$ is a design parameter, and

$$\dot{\theta}_a = \frac{\dot{h}_3 h_2 - h_3 \dot{h}_2}{h_2^2 + h_3^2}, \quad h_2^2 + h_3^2 \neq 0 \qquad (24)$$

is a time-derivative of (21). By analogy to the form of (23) we propose to define the remaining part of the convergence vector field as follows:

$$\boldsymbol{h}^* \triangleq k_p\, \boldsymbol{e}^* + \boldsymbol{\nu}^* = \begin{bmatrix} k_p\, e_x + \nu_x \\ k_p\, e_y + \nu_y \end{bmatrix}, \qquad (25)$$

where $\boldsymbol{e}^* = [e_x \ e_y]^T$, $k_p > 0$ is a design parameter, and $\boldsymbol{\nu}^* = [\nu_x \ \nu_y]^T$ is a some feed-forward velocity term which will be determined in the next section.

Remark 2: In general, a structure of the convergence vector field affects quality of the transient stage for system (9). Although, one may propose several alternative definitions for $\boldsymbol{h}$*, we restrict further considerations to (23) and (25) as the simplest ones. Note that (23) is determined using the auxiliary error (22), rather than the orientation error* e_θ *introduced in (5). It is a key point of the VFO strategy.*

Equations (19)-(20) with propositions (23) and (25) possess an intuitive geometrical interpretation. The first input $v_1 = k_1\, e_{\theta a} + \dot{\theta}_a$ is responsible for reorientation of the second vector field $\boldsymbol{g}_2 = [0 \ \cos\theta \ \sin\theta]^T$ of kinematics (9) to match its orientation in $\mathbb{R}^2$ with a desired one determined by $\boldsymbol{h}^* = [h_2 \ h_3]^T$ (compare (21)). One may called it an *orienting control*, and θ an *orienting variable*. Simultaneously, the second input (*pushing control*) $v_2 = h_2\cos\theta + h_3\sin\theta \equiv \boldsymbol{h}^T\boldsymbol{g}_2$ *pushes* the sub-state $\boldsymbol{q}^* = [x \ y]^T$ proportionally to an instantaneous orthogonal projection of $\boldsymbol{h}$ onto $\boldsymbol{g}_2$ (pushing intensity is maximal only for $e_{\theta a} = 0$), see [4].

Having the general VFO control law given by (19)-(20) with (23) and (25), it remains only to define the two terms – decision factor σ from (21) and feed-forward velocity $\boldsymbol{\nu}^*$ from (25) – which have not been determined so far. These two terms will be proposed separately for the tracking and set-point tasks in the next section.

IV. VFO FOR TRACKING AND SET-POINT CONTROL

A. Control law for trajectory tracking

For the trajectory tracking case we propose to take the decision factor as follows

$$\sigma \triangleq \text{sgn}(v_{2t}) \overset{(7)}{=} \text{sgn}(u_{2t}\cos\beta_t), \qquad (26)$$

which guarantees that the controlled vehicle moves with a desired motion strategy (forward for $v_{2t} > 0$ or backward for $v_{2t} < 0$) already during a transient stage. Note that (26) does not change its value within the whole control time-horizon due to assumption (4).

[2] Function $\text{Atan2c}\,(\cdot,\cdot)$ has been introduced to ensure continuity of signals from (21) and (22) – more details can be found in [4].

We propose to choose the feed-forward term $\boldsymbol{\nu}^*$ as a reference velocity along the position trajectory:

$$\boldsymbol{\nu}^* \triangleq \dot{\boldsymbol{q}}_t^* = \begin{bmatrix}\dot{x}_t \\ \dot{y}_t\end{bmatrix} \overset{(9)}{=} v_{2t}\, \boldsymbol{g}_2^*(\theta_t) \overset{(7)}{=} u_{2t}\cos\beta_t \begin{bmatrix}\cos\theta_t \\ \sin\theta_t\end{bmatrix}, \tag{27}$$

which does not degenerate to zero at any time-instant since (4) holds. We are ready to formulate the first proposition.

Proposition 1: Let us define the set in the position error domain: $\mathcal{E}_T^* = \mathbb{R}^2 \setminus \{\boldsymbol{e}^* = -k_p^{-1}\dot{\boldsymbol{q}}_t^*\}$. Assuming that

$$\forall_{\tau \geqslant 0}\, \boldsymbol{e}^*(\tau) \in \mathcal{E}_T^* \quad \overset{(25,27)}{\Longrightarrow} \quad \forall_{\tau\geqslant 0}\, \|\boldsymbol{h}^*(\tau)\| \neq 0 \tag{28}$$

the feedback control law (10,14) with additional VFO definitions (19,20), (23,25) together with (26) and (27) applied to car-like kinematics (1) solves the Problem 1 with $\epsilon = 0$ for an admissible reference trajectory $\boldsymbol{q}_t(\tau)$ which satisfies (3) and (4).

Convergence analysis (Sketch). Analysis is done in five steps: S1 to S5. S1: It can be seen that employing (14) to model (1) gives an exponential convergence of $e_{\beta a}(\tau)$ to zero as $\tau \to \infty$. Using now (12) one can write the following useful relation (valid for $\|\boldsymbol{v}\| \neq 0$):

$$\lim_{\beta\to\beta_a}\left[\tan\beta - \frac{Lv_1}{v_2}\right] = 0, \tag{29}$$

which will be utilized in the sequel. S2: Let us consider convergence of $e_{\theta a}(\tau)$. In order to do this let us combine the second equation of (1) with (10) to write, that for $\beta \to \beta_a$ (according to S1) one gets:

$$\begin{aligned}\dot{\theta} &= (1/L)(\tan\beta v_2)\cos^2\beta + v_1\sin^2\beta \overset{(29)}{=} \\ &= (1/L)Lv_1\cos^2\beta + v_1\sin^2\beta = v_1.\end{aligned}$$

Substituting now (19) into above relation with definition (23) gives $\dot{\theta} = k_\theta e_{\theta a} + \dot{\theta}_a$. As a consequence $e_{\theta a}(\tau) \to 0$ for $\tau \to \infty$. S3: In this step we are interested in behavior of the position error $\boldsymbol{e}^*(\tau)$. First note that substitution of (10) into (1) and utilization of (29) implies: $\dot{x} = \cos\theta v_2$ and $\dot{y} = \sin\theta v_2$. The last two relations together with equation $\dot{\theta} = v_1$ obtained in S2 define unicycle-like model (9). Therefore we can now proceed our analysis using fictitious inputs (19) and (20) applied to subsystem (9). Rewriting position error as $\boldsymbol{e}^* = \boldsymbol{q}_t^* - \boldsymbol{q}^*$ where $\boldsymbol{q}_t^* = [x_t\ y_t]^T$, one obtains $\dot{\boldsymbol{e}}^* = \dot{\boldsymbol{q}}_t^* - \dot{\boldsymbol{q}}^*$ and from (25) and (27) also $\dot{\boldsymbol{q}}_t^* = \boldsymbol{h}^* - k_p\boldsymbol{e}^*$. Combining the last two equations one can write an auxiliary equation of perturbed position-error dynamics as follows:

$$\dot{\boldsymbol{e}}^* = -k_p\boldsymbol{e}^* + \boldsymbol{r}(\boldsymbol{e}^*, e_{\theta a}, \cdot). \tag{30}$$

It can be shown (see [4] for details) that a norm of the perturbation term $\boldsymbol{r}(\boldsymbol{e}^*, e_{\theta a}, \cdot)$ can be expressed as follows: $\|\boldsymbol{r}(e_{\theta a},\cdot)\|^2 = \|\boldsymbol{h}^*\|^2(1 - \cos^2 e_{\theta a})$. It is evident that $\lim_{e_{\theta a}\to 0}\|\boldsymbol{r}(e_{\theta a},\cdot)\| = 0$ and due to corollary from step S2 we have: $\lim_{\tau\to\infty}\|\boldsymbol{r}(\tau)\| = 0$. Using now the lemmas from [5] related to the stability of perturbed systems (pages 350-355) one can conclude about boundedness and asymptotic convergence of $\boldsymbol{e}^*(\tau)$ to zero for $\tau \to \infty$. S4: Recalling (21), (25), (26) and (27) with the convergence result form step S3 one can write: $\theta_a|_{\boldsymbol{e}^*=\boldsymbol{0}} = \text{Atan2c}\left(\text{sgn}(v_{2t})\dot{y}_t, \text{sgn}(v_{2t})\dot{x}_t\right)$. Since $\theta_t = \text{Atan2}\left(\text{sgn}(v_{2t})\dot{y}_t, \text{sgn}(v_{2t})\dot{x}_t\right)$ (according to (9)), one obtains

$$\theta_a|_{\boldsymbol{e}^*=\boldsymbol{0}} \mod 2\pi = \theta_t \tag{31}$$

and together with corollary from step S2 one concludes: $\lim_{\tau\to\infty} e_\theta(\tau) = 0$. S5: The last step relates to behavior of the steering angle error $e_\beta(\tau)$. Utilizing the fact that $v_1 = v_{1t} = \dot{\theta}_t$ and $v_2 = v_{2t} = \dot{x}_t\cos\theta_t + \dot{y}_t\sin\theta_t$ for $e_{\theta a} = e_\theta = 0$ and $\boldsymbol{e}^* = \boldsymbol{0}$, one can rewrite in this case the auxiliary steering error (13) as follows:

$$\begin{aligned} e_{\beta a} &= \beta_a - \beta = \arctan\left(\frac{Lv_1}{v_2}\right) - \beta = \\ &= \arctan\left(\frac{L\dot{\theta}_t}{\dot{x}_t\cos\theta_t + \dot{y}_t\sin\theta_t}\right) - \beta \overset{(1)}{=} \\ &\overset{(1)}{=} \arctan\left(\frac{u_{2t}\sin\beta_t}{u_{2t}\cos\beta_t}\right) - \beta \overset{(5)}{=} \beta_t - \beta = e_\beta.\end{aligned}$$

The above expression together with the convergence result taken from step S1 allow concluding $\lim_{\tau\to\infty} e_\beta(\tau) = 0$.

Remark 3: Violation of assumption (28) at some time instant $\overline{\tau}$ implies that (21) and (24) are not determined at $\overline{\tau}$. In geometrical interpretation violating (28) relates to a situation when the controlled vehicle moves exactly along the same direction as a reference vehicle but with opposite body-orientation. It is a rare (non-attracting) and non-persistent case in practice but theoretically may happen during a transient stage (and only then). Naturally, convergence of position error $\boldsymbol{e}^*$ is preserved also in this case. To cope with the indeterminacy of terms (21) and (24), we propose to introduce additional definitions for θ_a and $\dot{\theta}_a$ in a small ε-vicinity of point $\boldsymbol{h}^* = \boldsymbol{0}$ as follows:

$$\theta_a \triangleq \theta_a(\tau^-) \quad \textit{and} \quad \dot{\theta}_a \triangleq 0 \qquad \textit{for} \quad \|\boldsymbol{h}^*\| < \varepsilon, \tag{32}$$

with $0 < \varepsilon < \inf_\tau |u_{2t}(\tau)\cos\beta_t(\tau)|$, and τ^- being determined as a time instant when $\|\boldsymbol{h}^*(\tau^-)\| = \varepsilon$. Note that other definitions together with the control input ones remain unchanged. Since the indeterminacy point is non-attracting and non-persistent, all the convergence results obtained above stay valid.

B. Control law for set-point regulation

In the case of set-point regulation we propose to choose the decision factor as a sign of an initial position error of x axis of the local frame attached to the reference vehicle:

$$\sigma \triangleq \text{sgn}(e_{x0}^L), \tag{33}$$

where $e_{x0}^L \equiv e_x^L(\tau = 0)$ and $e_x^L = e_x\cos\theta_t + e_y\sin\theta_t$. However, the choice (33) is not very critical – for many conditions σ may be chosen by the user arbitrarily from the set $\{+1, -1\}$ (according to the required motion strategy of reaching the set-point by the vehicle). Since for a constant set-point the reference velocity $\dot{\boldsymbol{q}}_t^* \equiv \boldsymbol{0}$, we propose to define the feed-forward term $\boldsymbol{\nu}^*$ equal to the so-called *virtual reference velocity* $\dot{\boldsymbol{q}}_{tv}^*$, namely:

$$\boldsymbol{\nu}^* \triangleq \dot{\boldsymbol{q}}_{tv}^* \triangleq -\eta\,\sigma\,\|\boldsymbol{e}^*\|\,\boldsymbol{g}_2^*(\theta_t), \qquad \eta \in (0, k_p), \tag{34}$$

where η is an additional design parameter, σ is given by (33), and $\| \boldsymbol{e}^* \| = \sqrt{e_x^2 + e_y^2}$, $\boldsymbol{g}_2^*(\theta_t) = [\cos\theta_t \ \sin\theta_t]^T$. Note that $\dot{\boldsymbol{q}}_{tv}^* \to \boldsymbol{0}$ as $\| \boldsymbol{e}^* \| \to 0$, but $\dot{\boldsymbol{q}}_{tv}^*$ is non-zero during a transient stage. The last feature reveals to be useful in shaping the vehicle motion in a neighborhood of a reference point, where some kind of *directing effect* can be enforced by the user with intensity depending on value of η (directing effect will be explained by simulation results in Section V). We can now formulate the second proposition.

Proposition 2: Let us define the set in the position error domain: $\mathcal{E}_P^* = \mathbb{R}^2 \setminus \{\boldsymbol{e}^* = \boldsymbol{0}\}$. *Assuming that*

$$\forall_{\tau \geqslant 0} \, \boldsymbol{e}^*(\tau) \in \mathcal{E}_P^* \quad \overset{(25,34)}{\Longrightarrow} \quad \forall_{\tau \geqslant 0} \, \| \boldsymbol{h}^*(\tau) \| \neq 0 \qquad (35)$$

the feedback control law (10,14) with additional VFO definitions (19,20), (23,25) together with (33) and (34) applied to car-like kinematics (1) solves the Problem 1 with $\epsilon = 0$ *for a reference set-point* $\boldsymbol{q}_t$.

Convergence analysis (Sketch). Analysis will be performed by analogy to the previous one conducted after Proposition 1. The first two steps S1 and S2 are the same implying also in this case asymptotic convergence of $e_{\beta a}$ and $e_{\theta a}$ errors. S3: Following similar considerations as in the former proof one can derive an auxiliary equation of the positional error dynamics:

$$\dot{\boldsymbol{e}}^* = -k_p \boldsymbol{e}^* + \boldsymbol{r}(\boldsymbol{e}^*, e_{\theta a}, \cdot) - \dot{\boldsymbol{q}}_{vt}^* \qquad (36)$$

with perturbing term $\boldsymbol{r}(\boldsymbol{e}^*, e_{\theta a}, \cdot)$ such that $\| \boldsymbol{r}(e_{\theta a}, \cdot) \|^2 = \| \boldsymbol{h}^* \|^2 (1 - \cos^2 e_{\theta a})$. Defining the positive definite function $V := (1/2)\boldsymbol{e}^{*T}\boldsymbol{e}^*$ one can show that its time-derivative along the solution of (36) satisfies:

$$\dot{V} \leqslant -[k_p(1-\gamma) - \eta(1+\gamma)] \, \| \boldsymbol{e}^* \|^2 = -\zeta(\gamma) \, \| \boldsymbol{e}^* \|^2 \,, \quad (37)$$

where $\gamma = \gamma(e_{\theta a}) := \sqrt{1 - \cos^2 e_{\theta a}}$. Note that $\zeta(\gamma)$ is a positive definite function for $\gamma(e_{\theta a}) < (k_p - \eta)/(k_p + \eta)$. Since $\eta \in (0, k_p)$ (see (34)), $\gamma(e_{\theta a}) \in [0, 1]$ and $e_{\beta a}(\tau) \to 0$ (due to S2) the last inequality will be met for all $\tau \geqslant \tau_\gamma$, where τ_γ is a finite time instant, which always exists. Hence, one can conclude about boundedness and asymptotic convergence of $\boldsymbol{e}^*(\tau)$ to zero for $\tau \to \infty$. S4: Since $\boldsymbol{r}(e_{\theta a}, \cdot) \triangleq \boldsymbol{h}^* - \dot{\boldsymbol{q}}^*(\theta)$, it is clear that for $e_{\theta a} \to 0$ (due to S2) we have $\dot{\boldsymbol{q}}^* \to \boldsymbol{h}^*$. Recalling now model (1) one can write $\tan\theta = \dot{y}/\dot{x}$, thus for $e_{\theta a} = 0$ one gets:

$$\tan\theta = \left(\frac{-k_p e_y}{\eta\sigma \, \| \boldsymbol{e}^* \|} + \sin\theta_t \right) / \left(\frac{-k_p e_x}{\eta\sigma \, \| \boldsymbol{e}^* \|} + \cos\theta_t \right). \qquad (38)$$

The above formula implies: $\tan\theta \to \tan\theta_t$ as $\boldsymbol{e}^* \to \boldsymbol{0}$. It can be also shown that at the limit for $\boldsymbol{e}^* \to \boldsymbol{0}$ holds: $\boldsymbol{g}_2^{*T}(\theta)\boldsymbol{g}_2^*(\theta_t) > 0$. The last relation together with (38) and with the convergence result of S3 allows concluding that $e_\theta(\tau) \to 0$ as $\tau \to \infty$. S5: As a result of the *directing effect* introduced by a particular definition of the virtual reference velocity in (34), the first fictitious input v_1 terminally (that is as $\boldsymbol{e}^* \to \boldsymbol{0}$) tends to zero faster than the pushing input v_2. This rate difference is greater for the less value of $k_p - \eta$ (due to higher intensity of the directing effect). Therefore, an argument of $\arctan(\cdot)$ function in definition (12) tends to zero as $\boldsymbol{e}^* \to \boldsymbol{0}$ implying (together with the convergence result from S3) that $\beta_a(\tau) \to 0$ as $\tau \to \infty$. Since $\beta_t = 0$ from definition, and recalling the corollary from S1, one may finally conclude $\lim_{\tau\to\infty} e_\beta(\tau) = 0$.

Remark 4: Assumption (35) excludes the stabilized point from the domain of proper determination of the VFO controller. It comes again from definitions (21) and (24) which are not defined for $\boldsymbol{e}^* = \boldsymbol{0}$. *Since the point* $\boldsymbol{e}^* = \boldsymbol{0}$ *is reachable only at infinity (according to the above analysis), the VFO set-point controller belongs to the so-called* almost stabilizers. *Although in practice, one may need a well defined controller also at this point. Introducing definitions*

$$\theta_a \triangleq \theta_a(\tau_\kappa), \quad \dot{\theta}_a \triangleq 0, \quad \beta_a = \dot{\beta}_a \triangleq 0, \quad u_2 \triangleq 0 \qquad (39)$$

being active for $\tau \geqslant \tau_\kappa$ *where* $\| \boldsymbol{e}^*(\tau \geqslant \tau_\kappa) \| < \kappa$ *with some assumed* $\kappa > 0$, *leads to the ultimate boundedness of error (5) solving Problem 1 with* $\epsilon = \epsilon(\kappa, e_\theta(\tau_\kappa)) > 0$ *and with* $\lim_{\tau\to\infty} e_\beta(\tau) = 0$.

V. Numerical verification

Two numerical tests have been conducted using model (1) with $L = 0.2$ m. Values for the VFO parameters have been chosen as follows: $k_\beta = 10$, $k_\theta = 5$, $k_p = 2$, and $\eta = 1.5$ (all in units of 1/s). The first test (SimA) verifies control quality for a trajectory tracking task with a reference trajectory generated by numerical integration of (2) for an initial condition $\boldsymbol{q}_t(0) = \boldsymbol{0}$ and reference inputs: $u_{1t}(\tau) = 0.6\sin(2\tau)$ rad/s, $u_{2t} = 0.4$ m/s (forward motion strategy). An initial condition of the controlled robot was $\boldsymbol{q}_0 = [\frac{-\pi}{3} \ \frac{-\pi}{3} \ 0.2 \ 0.5]^T$. The second test (SimB) verifies control quality for a set-point regulation task, where a reference point has been chosen as follows: $\boldsymbol{q}_t = [0 \ 0 \ -0.5 \ 0]^T$. In this case an initial condition for the controlled vehicle was $\boldsymbol{q}_0 = [\frac{-\pi}{3} \ \frac{-\pi}{3} \ 0.4 \ 1.0]^T$. According to proposition (33) the vehicle approach to the reference posture has been forced in this case in a backward motion strategy.

The results of simulations SimA and SimB are presented in Figs. 2 and 3. Let us note fast error convergence, *natural* vehicle movement, and boundedness of control signals and steering variables β_a and β obtained in both tests, also in the case of a vehicle switchback illustrated in the bottom plot of Fig. 2. It is worth noting that due to the features of VFO control strategy, [4], in the case of set-point regulation the fictitious inputs v_1, v_2 decay without oscillations, with v_1 tending to zero faster than v_2. It implies that the auxiliary variable (12) tends to zero when $\boldsymbol{e}^* \to \boldsymbol{0}$ (see Fig. 3). Robustness of the closed-loop system to uncertainty of the vehicle kinematic parameter L and to feedback measurement noises has been examined. Three simulation runs have been conducted using parameter L_s in (10), (12), and (15) instead of L taking: $L_s := L$ (nominal case), $L_s := 1.5L$ (overestimated case), and $L_s := 0.5L$ (underestimated case). Additionally, the white noises with standard deviation $s_d = 0.001$ have been added to all the feedback measurements. The results obtained are presented in Fig. 4 as time plots of

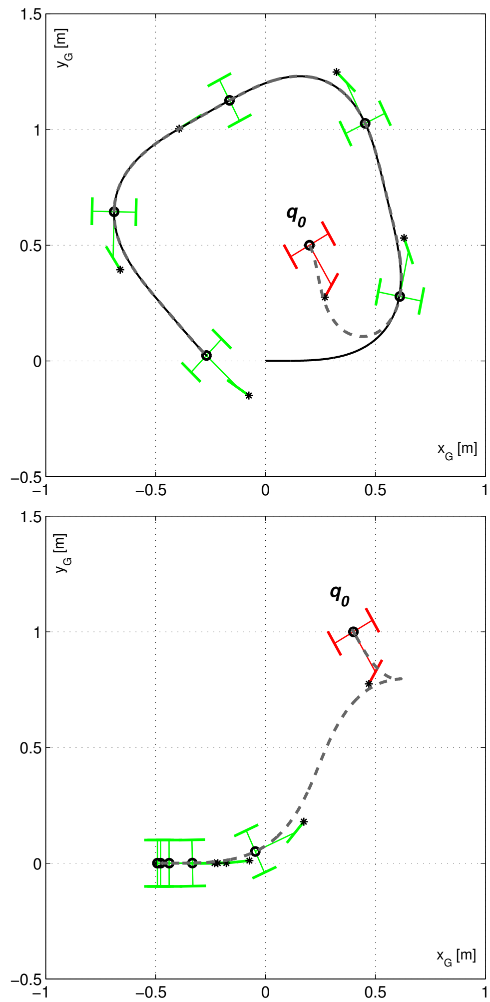

Fig. 2. VFO control results: SimA-tracking (top), SimB-regulation (bottom)

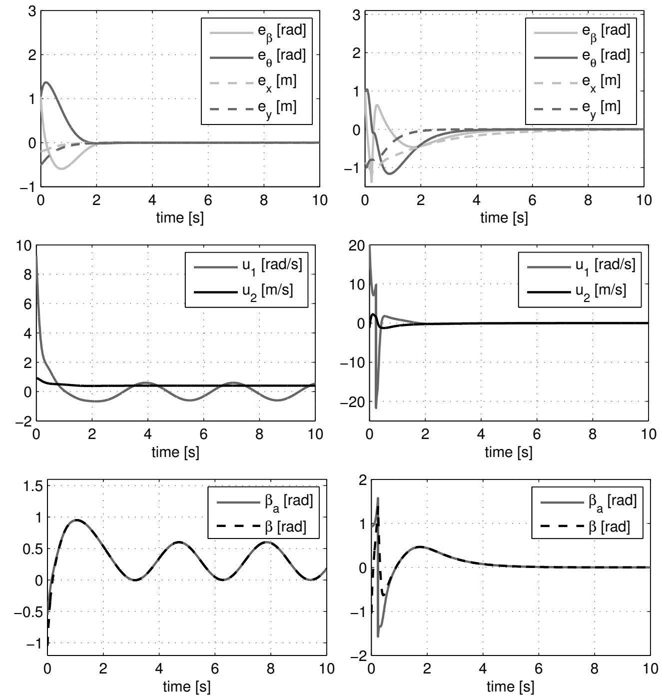

Fig. 3. VFO control results: SimA-tracking (left), SimB-regulation (right)

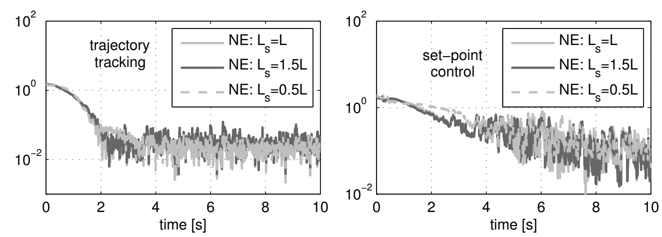

Fig. 4. Robustness results for SimA (left) and for SimB (right)

$\|\boldsymbol{e}(\tau)\|$. For all selected conditions stability of the closed-loop system has been preserved with practical convergence of errors. For SimB the vicinity $\kappa := 0.02\,\mathrm{m}$ has been used in the procedure described in Remark 4. During set-point control the most sensitive to noises near the point $\boldsymbol{e}^* = \mathbf{0}$ are the auxiliary signals (12) and (21), which can terminally lead to some kind of *wriggling* behavior of the vehicle. This effect can be attenuated by reducing the gains k_β and k_θ and enlarging vicinity κ with a cost of worse control precision in a final stage (as a matter of practical compromise).

VI. Final Remarks and Conclusions

Control design concept presented in this paper can be treated as an extension of the VFO methodology application to the car-like kinematics. Utilization of the VFO concept allows treating the trajectory tracking and set-point control tasks in a unified manner. The VFO control approach, presented recently in [4] for the unicycle, has been applied here after reformulation of the original car-like robot kinematics into the body-posture subsystem model, of the unicycle type, and the front-wheel steering dynamics. Future research plans encompass experimental validation of the presented algorithm, as well as solution of the problem for a rear-axle driven car-like kinematics.

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