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A Tracking Controller for Linear Systems subject to Input Amplitude and Rate Constraints

Klaus Kefferpütz and Jürgen Adamy

Abstract— **We propose a new set point tracking controller for plants subject to simultaneous input amplitude and rate constraints. Short settling times are achieved by allowing the controller to saturate. The tracking controller can be deduced from a controller stabilizing the origin with an associated domain of attraction. Additionally, no assumptions concerning the rate of the reference signal are necessary.**

I. Introduction

The stabilization of linear systems subject to control and state constraints is a well studied problem. Different methods were derived to estimate the domain of attraction, e.g. [3], [10]. Recently, methods were developed for the stabilization of linear systems which are subject to simultaneous rate and amplitude constraints, e.g. [2], [7], [11].

Focus of these methods is driving an initial state to the origin contrary to the task of driving the system to another steady state. One approach to overcome the tracking problem is rewriting the state space model such that the tracking error becomes part of the state vector. Then the task is stabilizing the origin of the modified system. In this case, the maximal allowable set point is a function of the initial condition. Such methods were presented in [6], [9] for systems that are exclusively subject to input amplitude constraints. Additional rate constraints are considered in [15]. Therein an exosystem is used to provide the reference signal and therefore the set of possible reference signals is limited.

In [4] a non-saturating tracking controller is derived that can be deduced from the stabilizing controller. In [12] this method was extended to saturating inputs but rate constraints were not considered. In this paper we extend the concept of [4] to the case of systems with simultaneous amplitude and rate constraints employing saturating feedback controllers as derived in [2]. We also provide a convex design method.

In Section II the control problem under the assumption of a square system is formulated. Since the proposed controller can be inferred from a saturating state feedback controller which stabilizes the origin we refer to the stability of linear systems with state feedback controllers in the presence of simultaneous input amplitude and rate constraints in Section III. In Section IV we briefly state the results from [4] and in Section V we extend the concept to saturating controllers in the presence of rate constraints. After we refer to non-square systems in Section VI, a convex controller design method is provided in Section VII. Finally, we show the effectiveness of the proposed controller on a linear model of the McDonnell Douglas Tailless Advanced Fighter Aircraft (TAFA) [1].

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The authors are with the Institute of Automatic Control, Control Theory and Robotics Lab, Technische Universität Darmstadt, 64283 Darmstadt, Germany, klaus.kefferpuetz@rtr.tu-darmstadt.de and adamy@rtr.tu-darmstadt.de

II. Problem Statement

We consider systems with linear dynamics

$$\begin{aligned}\dot{\mathbf{x}}_o &= \mathbf{A}_o\mathbf{x}_o + \mathbf{B}_o\mathbf{u}_A\\ \mathbf{y}_o &= \mathbf{C}_o\mathbf{x}_o + \mathbf{D}_o\mathbf{u}_A\end{aligned} \tag{1}$$

with $\mathbf{A}_o \in \mathbb{R}^{n_o\times n_o}$, $\mathbf{B}_o \in \mathbb{R}^{n_o\times m}$, $\mathbf{C}_o \in \mathbb{R}^{q\times n_o}$ and $\mathbf{D}_o \in \mathbb{R}^{q\times m}$ subject to input amplitude and rate constraints $|u_{A,i}| \le 1$ and $|\dot{u}_{A,i}| \le 1$, $i = 1,\ldots,m$. The elements $u_{A,i}$ of the input vector $\mathbf{u}_A$ are provided by an amplitude and rate limited actuator, whose structure is depicted in Fig. 1. The differential equation

$$\dot{\mathbf{u}}_A = \boldsymbol{\sigma}\left(\mathbf{T}\left(\boldsymbol{\sigma}(\mathbf{u}) - \mathbf{u}_A\right)\right) \qquad \mathbf{u}_A(0) = \mathbf{u}_{A,0} \tag{2}$$

models the actuator dynamics. Therein we have $\mathbf{T} = \operatorname{diag}(\tau_1,\ldots,\tau_m)$, and $\boldsymbol{\sigma}(\cdot) = [\sigma(\cdot)\cdots\sigma(\cdot)]^T$ denotes the m-dimensional normalized saturation function with $\sigma(u) = \operatorname{sgn}(u)\min(1,|u|)$. Since the commanded input is constrained on $|u| \le 1$, $|u_{A,i}| \le 1$ holds. For $\tau_i \to \infty$ we obtain the ideal rate limiter [15]. Augmenting the state vector $\mathbf{x} = [\mathbf{x}_o^\top\ \mathbf{u}_A^\top]^\top$ we obtain the overall system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{\sigma}\left(\mathbf{K}_1\mathbf{x} + \mathbf{T}\boldsymbol{\sigma}(\mathbf{u})\right), \tag{3}$$

with

$$\mathbf{A} = \begin{bmatrix}\mathbf{A}_o & \mathbf{B}_o\\ \mathbf{0} & \mathbf{0}\end{bmatrix},\quad \mathbf{B} = \begin{bmatrix}\mathbf{0}\\ \mathbf{I}\end{bmatrix},\quad \mathbf{K}_1 = [\mathbf{0}\ \ -\mathbf{T}], \tag{4}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{K}_1 \in \mathbb{R}^{m\times n}$ and $n = n_o + m$. Note that nonunity saturation levels can be absorbed in the matrices $\mathbf{B}$, $\mathbf{T}$ and $\mathbf{K}_1$. Our objective is to design a saturating tracking controller that is able to drive the system to a new steady-state. To this end, we assume that the actuator output $\mathbf{u}_A$ is measurable. At the steady state,

$$\mathbf{A}_o\overline{\mathbf{x}}_o + \mathbf{B}_o\overline{\mathbf{u}}_A = \mathbf{0},$$

i.e, $\mathbf{A}\overline{\mathbf{x}} = \mathbf{0}$ holds, as well as $\dot{\overline{\mathbf{u}}}_A = \mathbf{0}$. The latter implies that $\overline{\mathbf{u}}_A = \overline{\mathbf{u}}$. If the system is square and the matrix

$$\mathbf{S} = \begin{bmatrix}\mathbf{A}_o & \mathbf{B}_o\\ \mathbf{C}_o & \mathbf{D}_o\end{bmatrix}$$

is invertible, the new steady-state can be calculated depending on an external (admissible) reference signal $\mathbf{r}$ provided that the reference signal is admissible. If the reference vector $\mathbf{r}$ is constant the error $\|\mathbf{y} - \mathbf{r}\|$ converges to zero for $t \to \infty$.

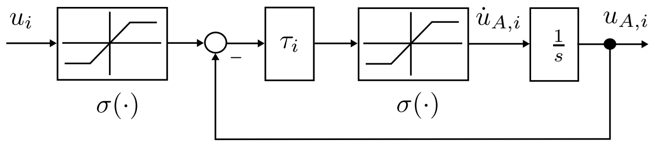

Fig. 1. Amplitude and rate constrained actuator.

The new steady state $\overline{\mathbf{x}}_o(\mathbf{r})$ and the corresponding input $\overline{\mathbf{u}}_A(\mathbf{r})$ for a reference signal $\mathbf{r}$ can be derived from

$$\begin{bmatrix} \overline{\mathbf{x}}_o(\mathbf{r}) \\ \overline{\mathbf{u}}_A(\mathbf{r}) \end{bmatrix} = \mathbf{S}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{r} \end{bmatrix}.$$

For compact notation we will use $\overline{\mathbf{x}}_o = \overline{\mathbf{x}}_o(\mathbf{r})$ and $\overline{\mathbf{u}}_A = \overline{\mathbf{u}}_A(\mathbf{r})$ for the reminder of the paper. If the system is square, the stationary state

$$\overline{\mathbf{x}} = \begin{bmatrix} \overline{\mathbf{x}}_o \\ \overline{\mathbf{u}}_A \end{bmatrix}$$

of the augmented systems as well as the stationary input $\overline{\mathbf{u}} = \overline{\mathbf{u}}_A$ are known. The case where $\mathbf{S}$ is not invertible is considered in Section VI.

III. Stabilizing feedback controller

The proposed tracking controller can be deduced from a controller stabilizing the origin of the system (3) and the related estimate of the domain of attraction. To state a lemma concerning the stability of linear systems subject to input constraints under nonlinear state feedback we need the set

$$\mathcal{G} = \{\mathbf{x} \in \mathbb{R}^n : v(\mathbf{x}) \leq 1\} \tag{5}$$

with the positive definite function $v(\mathbf{x})$. The latter implies $v(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$ and $v(\mathbf{0}) = 0$. Additionally, we need

Definition 1: The closed set $\mathcal{G}$ is said to be contractively invariant for the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, if $\dot{v}(\mathbf{x}) < 0$ holds for all $\mathbf{x} \in \mathcal{G} \backslash \{\mathbf{0}\}$.

A contractively invariant set is a domain of attraction. Defining the set $\mathcal{V} = \{\mathbf{v} \in \mathbb{N}^m : v_i \in \{1, 2, 3\}\}$, the matrix

$$\begin{aligned} &\mathbf{M}\left(\, \mathbf{v}, \mathbf{w}_1(\mathbf{x}), \mathbf{w}_2(\mathbf{x}), \mathbf{w}_3(\mathbf{x})\right) \\ &= \operatorname{diag}\left\{\delta(v_1-1), \delta(v_2-1), \dots, \delta(v_m-1)\right\} \mathbf{w}_1(\mathbf{x}) \\ &\quad + \operatorname{diag}\left\{\delta(v_1-2), \delta(v_2-2), \dots, \delta(v_m-2)\right\} \mathbf{w}_2(\mathbf{x}) \\ &\quad + \operatorname{diag}\left\{\delta(v_1-3), \delta(v_2-3), \dots, \delta(v_m-3)\right\} \mathbf{w}_3(\mathbf{x}) \end{aligned} \tag{6}$$

with

$$\delta(j) = \begin{cases} 1, & \text{if} \quad j = 0 \\ 0, & \text{if} \quad j \neq 0, \end{cases}$$

and the set

$$\mathcal{L}(\mathbf{h}(\mathbf{x})) = \{\mathbf{x} : |h_i(\mathbf{x})| \leq 1,\ i = 1, 2, \dots, m\}$$

in which the state feedback $\mathbf{h}(\mathbf{x})$ does not saturate, we can state a lemma from [11] concerning the stability of systems (4) under nonlinear state feedback. The lemma is an extension of the stability conditions derived in [2].

Lemma 1 ([8]): Given the dynamical system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{\sigma}\left(\mathbf{K}_1\mathbf{x} + \mathbf{T}\boldsymbol{\sigma}(\mathbf{u})\right)$ with the saturating state feedback $\mathbf{u} = \mathbf{k}_2(\mathbf{x})$ and the set $\mathcal{G}$. If there exist state feedbacks $\mathbf{u} = \mathbf{h}_1(\mathbf{x})$ and $\mathbf{u} = \mathbf{h}_2(\mathbf{x})$, such that $\mathcal{G} \subseteq \mathcal{L}(\mathbf{h}_1(\mathbf{x})) \cap \mathcal{L}(\mathbf{h}_2(\mathbf{x}))$ and

$$\begin{aligned} \frac{\partial v(\mathbf{x})}{\partial \mathbf{x}} &\left(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{M}(\mathbf{v}, \mathbf{h}_1(\mathbf{x}), \mathbf{K}_1\mathbf{x} + \mathbf{T}\mathbf{h}_2(\mathbf{x}),\right. \\ &\qquad \left.\mathbf{K}_1\mathbf{x} + \mathbf{T}\mathbf{k}_2(\mathbf{x}))\right) < 0 \end{aligned} \tag{7}$$

hold for all $\mathbf{x} \in \mathcal{G}$ and $\mathbf{v} \in \mathcal{V}$, then the set $\mathcal{G}$ is contractively invariant under the state feedback $\mathbf{u} = \mathbf{k}_2(\mathbf{x})$. Therefore, $\mathcal{G}$ is a domain of attraction.

In case a linear state feedback controller is employed, Lemma 1 simplifies to the lemma given in [2].

Lemma 2 ([2]): Given the dynamical system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{\sigma}\left(\mathbf{K}_1\mathbf{x} + \mathbf{T}\boldsymbol{\sigma}(\mathbf{u})\right)$ with the linear saturating state feedback $\mathbf{u} = \mathbf{K}_2\mathbf{x}$ and the domain of attraction $\mathcal{G}$, if there exist state feedbacks $\mathbf{H}_1\mathbf{x}$,$\mathbf{H}_2\mathbf{x}$ such that $\mathcal{G} \subseteq \mathcal{L}(\mathbf{H}_1\mathbf{x}) \cap \mathcal{L}(\mathbf{H}_2\mathbf{x})$ and

$$\begin{aligned} \frac{\partial v(\mathbf{x})}{\partial \mathbf{x}} &\left(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{M}(\mathbf{v}, \mathbf{H}_1\mathbf{x}, \mathbf{K}_1\mathbf{x} + \mathbf{T}\mathbf{H}_2\mathbf{x},\right. \\ &\qquad \left.\mathbf{K}_1\mathbf{x} + \mathbf{T}\mathbf{K}_2\mathbf{x})\right) < 0 \end{aligned}$$

hold for all $\mathbf{x} \in \mathcal{G}$ and $\mathbf{v} \in \mathcal{V}$, then the set $\mathcal{G}$ is contractively invariant under the state feedback $\mathbf{u} = \mathbf{K}_2\mathbf{x}$. Therefore, $\mathcal{G}$ is a domain of attraction.

Ellipsoids are commonly used estimates of the domain of attraction and so we use $v(\mathbf{x}) = \mathbf{x}^\top \mathbf{R}\mathbf{x}$ with the positive definite Matrix $\mathbf{R}$ for the reminder of the paper. Lemma 1 and 2 will be used later on to prove stability of the proposed saturating tracking controller. Note that state constraints can be easily employed by demanding $\mathcal{G} \subseteq \mathcal{X}$.

IV. Non-saturating tracking control

In [4] a tracking controller for systems with linear dynamics under input and state constraints is derived. In this section we will state the major results of [4].

Before the tracking domain of attraction is defined, we refer to the admissible reference signals. In order to ensure that the available input amplitude allows for reaching the steady-state $\overline{\mathbf{x}}$ corresponding to the reference signal $\overline{\mathbf{r}}$ it must be ensured that $\overline{\mathbf{u}} = \boldsymbol{\sigma}(\overline{\mathbf{u}})$ holds. Additionally, the steady-state must be contained in the domain of attraction, i.e., $\overline{\mathbf{x}} \in \mathcal{G}$. This limits the set of possible reference signals $\overline{\mathbf{r}}$. Using the set of non-saturating input vectors

$$\mathcal{U} = \{\mathbf{u} \in \mathbb{R}^m : |u_i| \leq 1, i = 1, \dots, m\},$$

the set of admissible reference signals

$$\mathcal{R} = \{\mathbf{r} \in \mathbb{R}^q : \overline{\mathbf{u}}(\mathbf{r}) \in \mathcal{U}, \overline{\mathbf{x}}(\mathbf{r}) \in \mathcal{G}\}$$

is defined. Next, we specify an admissible reference signal.

Definition 2: Given $0 < \epsilon < 1$. A reference signal $\mathbf{r}(t)$ is said to be admissible, if it is continuous and

$$\lim_{t \to \infty} \mathbf{r}(t) = \mathbf{r}_\infty \in \mathcal{R}_\epsilon$$

with

$$\mathcal{R}_\epsilon = \{\mathbf{r} \in \mathbb{R}^q : (1+\epsilon)\,\overline{\mathbf{u}}(\mathbf{r}) \in \mathcal{U}, (1+\epsilon)\,\overline{\mathbf{x}}(\mathbf{r}) \in \mathcal{G}\}$$

is satisfied.

The scalar ϵ is introduced in order to avoid singularities in the control law. Because ϵ can be chosen arbitrary small, it does not influence the problem. Note that an admissible reference signal $\mathbf{r}(t)$ can take values outside the set $\mathcal{R}_\epsilon$, as long as $\mathbf{r}_\infty \in \mathcal{R}_\epsilon$ holds.

In order to restrict the reference signals to the admissible set $\mathcal{R}_\epsilon$ the function

$$P(\mathbf{r}) = \inf\left\{\gamma > 0 : \frac{1}{\gamma}\mathbf{r} \in \mathcal{R}_\epsilon\right\} \tag{8}$$

is used and so we obtain the restricted reference signal

$$\overline{\mathbf{r}} = \Gamma(\mathbf{r}) = \begin{cases} \mathbf{r}P(\mathbf{r})^{-1} & \text{if} \quad P(\mathbf{r}) > 1, \\ \mathbf{r} & \text{otherwise.} \end{cases} \tag{9}$$

Definition 3: The set $\mathcal{G} \subseteq \mathcal{X}$ is called a tracking domain of attraction, if there exists a (possibly nonlinear) state feedback controller

$$\mathbf{u}(t) = \mathbf{k}_\mathrm{T}(\mathbf{x}(t), \mathbf{r}(t))\,,$$

such that for any $\mathbf{x}(0) \in \mathcal{G}$ and any admissible reference signal $\mathbf{r}(t)$ the conditions $\mathbf{x}(t) \in \mathcal{G}$ and $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{r}_\infty$ hold.

Remark 1: We have slightly modified the definition given in [4] by dropping the claim $\mathbf{u}(t) \in \mathcal{U}$ to allow for a saturating controller, which is proposed in the next section.

Suppose that we have designed a possibly nonlinear non-saturating controller $\mathbf{k}(\mathbf{x}) \in \mathcal{U}$ satisfying $\mathbf{k}(c\mathbf{x}) = c\,\mathbf{k}(\mathbf{x})$[1] stabilizing the origin of system (1) with the associated domain of attraction $\mathcal{G}$. Because $\mathbf{r}(t) = \mathbf{0}$ is an admissible reference signal, any tracking domain of attraction is a domain of attraction $\mathcal{G}$. In [4] it is shown that every domain of attraction $\mathcal{G}$ is also a tracking domain of attraction. Therefore, the tracking controller can be deduced from the stabilizing and possibly nonlinear controller associated with the domain of attraction (5).

Since the tracking controller defined in [4] is only allowed to command input vectors $\mathbf{u} \in \mathcal{U}$, the available input energy is not fully utilized. In order to circumvent this disadvantage, we propose a saturating tracking controller.

V. Saturating tracking control for systems with amplitude and rate constraints

In this section we will show that the results from [4] remain valid also in case of a tracking controller $\mathbf{k}_\mathrm{T}(\mathbf{x}, \mathbf{r}) \notin \mathcal{U}$ that is allowed to saturate concerning the commanded input amplitude and the input rate. To derive the tracking controller and to prove stability we follow [4] and make use of the Minkowski-functional

$$V(\mathbf{x}) = \inf\left\{\gamma > 0 : \frac{1}{\gamma}\,\mathbf{x} \in \mathcal{G}\right\}. \tag{10}$$

Since $\gamma^{-1}\mathbf{x} \in \mathcal{G}$ holds if $v(\gamma^{-1}\mathbf{x}) = \gamma^{-2}\mathbf{x}^\top\mathbf{R}\mathbf{x} \le 1$ we get from (10)

$$V(\mathbf{x}) = \sqrt{\mathbf{x}^T\mathbf{R}\mathbf{x}}.$$

[1] Note that linearity additionally implies that the superposition property is fulfilled, i.e., $\mathbf{k}(\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{k}(\mathbf{x}_1) + \mathbf{k}(\mathbf{x}_2)$.

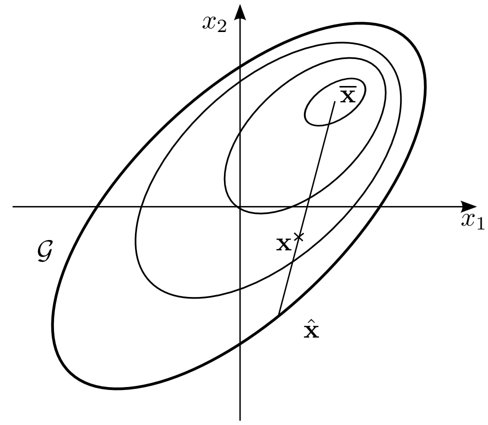

Fig. 2. Level sets of the function $V_*(\mathbf{x}, \overline{\mathbf{x}})$. The level set $V_*(\mathbf{x}, \overline{\mathbf{x}}) = 1$ is identical to the the boundary $\partial\mathcal{G} = \{\mathbf{x} : V(\mathbf{x}) = 1\} = \{\mathbf{x} : v(\mathbf{x}) = 1\}$

The gradients of the functions $v(\mathbf{x})$ and $V(\mathbf{x})$ have the same direction and it makes no difference which one is used as a Lyapunov function [3]. The following deformed version of (10) is used to overcome the tracking problem

$$V_*(\mathbf{x}, \overline{\mathbf{x}}) = \inf\left\{\gamma > 0 : \overline{\mathbf{x}} + \frac{1}{\gamma}(\mathbf{x} - \overline{\mathbf{x}}) \in \mathcal{G}\right\} \tag{11}$$

with $\overline{\mathbf{x}} \in \operatorname{int}\mathcal{G}$ where $\operatorname{int}\mathcal{G}$ denotes the interior of $\mathcal{G}$. Note that (11) is convex and for any $\mathbf{x} \in \mathcal{G}$ and any fixed $\overline{\mathbf{x}} \in \operatorname{int}\mathcal{G}$

$$V_*(\overline{\mathbf{x}}, \overline{\mathbf{x}}) = 0, \tag{12}$$
$$V_*(\mathbf{x}, \overline{\mathbf{x}}) < 1 \quad \text{if} \quad \mathbf{x} \in \mathcal{G}, \tag{13}$$
$$V_*(\mathbf{x}, \overline{\mathbf{x}}) = 1 \quad \text{if} \quad \mathbf{x} \in \partial\mathcal{G} \tag{14}$$

hold. Therefore, the function $V_*(\mathbf{x}, \overline{\mathbf{x}})$ seems to be a suitable Lyapunov function for the tracking problem. Next, we can state the tracking controller

$$\mathbf{u} = \mathbf{k}_\mathrm{T}(\mathbf{x}, \mathbf{r}) = \mathbf{k}_2(\hat{\mathbf{x}})\,V_*(\mathbf{x}, \overline{\mathbf{x}}) + (1 - V_*(\mathbf{x}, \overline{\mathbf{x}}))\,\overline{\mathbf{u}}. \tag{15}$$

Therein, the vector

$$\hat{\mathbf{x}} = \overline{\mathbf{x}} + \frac{1}{V_*(\mathbf{x}, \overline{\mathbf{x}})}(\mathbf{x} - \overline{\mathbf{x}}) \tag{16}$$

lies on the boundary of $\mathcal{G}$, i.e., $\hat{\mathbf{x}} \in \partial\mathcal{G}$ holds. Fig. 2 illustrates the level sets of the function $V_*(\mathbf{x}, \overline{\mathbf{x}})$ together with $\overline{\mathbf{x}}$ and $\hat{\mathbf{x}}$. Note that the level set $\{\mathbf{x} : V_*(\mathbf{x}, \overline{\mathbf{x}}) = 1\}$ is identical to the level set $\{\mathbf{x} : V(\mathbf{x}) = 1\} = \{\mathbf{x} : v(\mathbf{x}) = 1\}$ and therefore identical with $\partial\mathcal{G}$. The main result ensuring the stability of the tracking control is stated in

Theorem 1: Given a domain of attraction $\mathcal{G}$ of the system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{\sigma}\left(\mathbf{K}_1\mathbf{x} + \mathbf{T}\boldsymbol{\sigma}(\mathbf{u})\right)$ under the controller $\mathbf{u} = \mathbf{k}_2(\mathbf{x})$ that satisfies $\mathbf{k}_2(c\,\mathbf{x}) = c\,\mathbf{k}_2(\mathbf{x})$ as well as the conditions

$$\frac{\partial v(\mathbf{x})}{\partial \mathbf{x}}\left(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{M}(\mathbf{v}, \mathbf{h}_1(\mathbf{x}), \mathbf{K}_1\mathbf{x} + \mathbf{T}\mathbf{h}_2(\mathbf{x}),\right.$$
$$\left.\mathbf{K}_1\mathbf{x} + \mathbf{T}\mathbf{k}_2(\mathbf{x}))\right) < 0 \quad \forall \mathbf{x} \in \mathcal{G}, \mathbf{v} \in \mathcal{V}$$

and $\mathcal{G} \subseteq \mathcal{L}(\mathbf{h}_1(\mathbf{x})) \cap \mathcal{L}(\mathbf{h}_2(\mathbf{x}))$ for all $\mathbf{v} \in \mathcal{V}$. Then for any admissible reference signal $\mathbf{r}(t) = \overline{\mathbf{r}}$ together with the controller

$$\mathbf{u} = \mathbf{k}_\mathrm{T}(\mathbf{x}, \mathbf{r}) = \mathbf{k}_2(\hat{\mathbf{x}})V_*(\mathbf{x}, \overline{\mathbf{x}}) + (1 - V_*(\mathbf{x}, \overline{\mathbf{x}}))\overline{\mathbf{u}},$$

it follows that $\mathbf{x}(t) \in \mathcal{G}$ and $\lim_{t\to\infty} \mathbf{x}(t) = \overline{\mathbf{x}}(\overline{\mathbf{r}})$ hold for all $\mathbf{x}(0) \in \mathcal{G}$.

Proof: To prove Theorem 1 we have to show that for any $\mathbf{x} \in \mathcal{G} \setminus \{\overline{\mathbf{x}}\}$ and $\overline{\mathbf{x}} \in \operatorname{int} \mathcal{G}$

$$\dot{V}_*(\mathbf{x}, \overline{\mathbf{x}}) = \frac{\partial V_*(\mathbf{x}, \overline{\mathbf{x}})}{\partial \mathbf{x}} \dot{\mathbf{x}} < 0$$

holds. Suppose we have $\mathbf{x} \neq \overline{\mathbf{x}}$. Then we have to demand

$$\dot{V}_*(\mathbf{x}, \overline{\mathbf{x}}) = \frac{\partial V_*(\mathbf{x}, \overline{\mathbf{x}})}{\partial \mathbf{x}} \left(\mathbf{A}\mathbf{x} + \mathbf{B}\boldsymbol{\sigma}(\mathbf{K}_1\mathbf{x} + \mathbf{T}\boldsymbol{\sigma}(\mathbf{u}))\right) < 0. \tag{17}$$

In order to show (17) we derive an upper bound for $\dot{V}_*(\mathbf{x}, \overline{\mathbf{x}})$. To this end, we use a similar procedure as the one used in [2] to derive stability conditions of the equilibrium state $\mathbf{x} = \mathbf{0}$. First, we need the virtual controllers

$$\mathbf{h}_{\mathrm{T}1}(\mathbf{x}, \mathbf{r}) = \mathbf{h}_1(\hat{\mathbf{x}}) V_*(\mathbf{x}, \overline{\mathbf{x}}) \tag{18}$$

and

$$\mathbf{h}_{\mathrm{T}2}(\mathbf{x}, \mathbf{r}) = \mathbf{h}_2(\hat{\mathbf{x}}) V_*(\mathbf{x}, \overline{\mathbf{x}}) + (1 - V_*(\mathbf{x}, \overline{\mathbf{x}}))\overline{\mathbf{u}}. \tag{19}$$

Both are non-saturating, since $\mathbf{h}_{\mathrm{T}2}(\mathbf{x}, \mathbf{r})$ is a convex combination of $\mathbf{h}_2(\hat{\mathbf{x}})$ and $\overline{\mathbf{u}} \in \mathcal{U}$. If $\mathcal{G} \subseteq \mathcal{L}(\mathbf{h}_2(\mathbf{x}))$ holds, the controller $\mathbf{h}_2(\hat{\mathbf{x}})$ is non-saturating on $\mathcal{G}$. By convexity we have $\mathbf{h}_{\mathrm{T}2}(\mathbf{x}, \mathbf{r}) \in \mathcal{U}$. Rewriting (17) and using the gradient $\nabla V_* = \partial V_*(\mathbf{x}, \overline{\mathbf{x}})/\partial \mathbf{x}$ we obtain

$$\nabla V_* \mathbf{A}\mathbf{x} + \sum_{i=1}^{m} \nabla V_* \mathbf{b}_i \sigma\big(\mathbf{k}_{1,i}^\top \mathbf{x} + \tau_i \left(\sigma(k_{\mathrm{T}i}(\mathbf{x}, \mathbf{r}))\right)\big) < 0, \tag{20}$$

where $\mathbf{b}_i$ denotes the i-th column of $\mathbf{B}$. To get a more compact notation we abbreviate

$$f_i(\mathbf{x}, \mathbf{r}) = k_{1,i}^\top \mathbf{x} + \tau_i \sigma\left(k_{\mathrm{T}i}(\mathbf{x}, \mathbf{r})\right).$$

First we eliminate the outermost saturation function. Consider the four cases in which $f_i(\mathbf{x}, \mathbf{r})$ is saturating, i.e., $|f_i(\mathbf{x}, \mathbf{r})| \geq 1$:

- $\nabla V_* \mathbf{b}_i \geq 0$ and
  $f_i(\mathbf{x}, \mathbf{r}) \leq -1$: $\nabla V_* \mathbf{b}_i \sigma(f_i(\mathbf{x}, \mathbf{r})) \leq \nabla V_* \mathbf{b}_i h_{\mathrm{T}1,i}(\mathbf{x}, \mathbf{r})$,
  $f_i(\mathbf{x}, \mathbf{r}) \geq \ \ 1$: $\nabla V_* \mathbf{b}_i \sigma(f_i(\mathbf{x}, \mathbf{r})) \leq \nabla V_* \mathbf{b}_i f_i(\mathbf{x}, \mathbf{r})$,
- $\nabla V_* \mathbf{b}_i \leq 0$ and:
  $f_i(\mathbf{x}, \mathbf{r}) \geq \ \ 1$: $\nabla V_* \mathbf{b}_i \sigma(f_i(\mathbf{x}, \mathbf{r})) \leq \nabla V_* \mathbf{b}_i h_{\mathrm{T}1,i}(\mathbf{x}, \mathbf{r})$,
  $f_i(\mathbf{x}, \mathbf{r}) \leq -1$: $\nabla V_* \mathbf{b}_i \sigma(f_i(\mathbf{x}, \mathbf{r})) \leq \nabla V_* \mathbf{b}_i f_i(\mathbf{x}, \mathbf{r})$.

In case $f_i(\mathbf{x}, \mathbf{r})$ is non-saturating, we have $\sigma(f_i(\mathbf{x}, \mathbf{r})) = f_i(\mathbf{x}, \mathbf{r})$. Combining the above results leads to

$$\begin{aligned} &\nabla V_* \mathbf{b}_i \sigma(f_i(\mathbf{x}, \mathbf{r})) \\ &\qquad \leq \max\{\nabla V_* \mathbf{b}_i h_{\mathrm{T}1,i}(\mathbf{x}, \mathbf{r}), \nabla V_* \mathbf{b}_i f_i(\mathbf{x}, \mathbf{r})\}. \end{aligned} \tag{21}$$

We proceed in the same way with the term $\sigma\left(k_{\mathrm{T}i}(\mathbf{x}, \mathbf{r})\right)$ in (20) and finally obtain

$$\begin{aligned} &\nabla V_* \mathbf{A}\mathbf{x} + \sum_{i=1}^{m} \nabla V_* \mathbf{b}_i \sigma\big(\mathbf{k}_{1,i}^\top \mathbf{x} + \tau_i \left(\sigma(k_i(\mathbf{x}, \mathbf{r}))\right)\big) \leq \\ &\nabla V_* \mathbf{A}\mathbf{x} + \sum_{i=1}^{m} \max\{\nabla V_* \mathbf{b}_i h_{\mathrm{T}1,i}(\mathbf{x}, \mathbf{r}),\ \ \nabla V_* \mathbf{b}_i \left(k_{1,i}^\top \mathbf{x}\right. \\ &\quad \left. + \tau_i h_{\mathrm{T}2,i}(\mathbf{x}, \mathbf{r})\right), \nabla V_* \mathbf{b}_i \left(k_{1,i}^\top \mathbf{x} + \tau_i k_{\mathrm{T}i}(\mathbf{x}, \mathbf{r})\right)\}. \end{aligned} \tag{22}$$

From (17), (22) and using $\mathbf{M}(\mathbf{v}, \mathbf{w}_1(\mathbf{x}), \mathbf{w}_2(\mathbf{x}), \mathbf{w}_3(\mathbf{x}))$ given by (6) we conclude that

$$\begin{aligned} &\dot{V}_* = \nabla V_* \cdot \dot{\mathbf{x}} \leq \nabla V_* \cdot \mathbf{M}\left(\mathbf{v}, \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{h}_{\mathrm{T}1}(\mathbf{x}, \mathbf{r}), \mathbf{A}\mathbf{x} + \right. \\ &\left. \mathbf{B}\mathbf{K}_1\mathbf{x} + \mathbf{B}\mathbf{T}\mathbf{h}_{\mathrm{T}2}(\mathbf{x}, \mathbf{r}),\ \ \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{K}_1\mathbf{x} + \mathbf{B}\mathbf{T}\mathbf{k}_{\mathrm{T}2}(\mathbf{x}, \mathbf{r})\right) < 0 \end{aligned}$$

for all $\mathbf{v} \in \mathcal{V}$.

Using (16) we substitute $\mathbf{x} = \hat{\mathbf{x}} V_*(\mathbf{x}, \overline{\mathbf{x}}) + (1 - V_*(\mathbf{x}, \overline{\mathbf{x}}))\overline{\mathbf{x}}$. Then we insert the tracking controller (15) and the non-saturating virtual controllers (18), (19) and get

$$\begin{aligned} \nabla V_* \mathbf{M}\,(\mathbf{v}, (\mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{h}_1(\hat{\mathbf{x}})) V_*(\mathbf{x}, \overline{\mathbf{x}}) + \ &\mathbf{A}\overline{\mathbf{x}}(1 - V_*(\mathbf{x}, \overline{\mathbf{x}})), \\ (\mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{K}_1\hat{\mathbf{x}} + \mathbf{B}\mathbf{T}\mathbf{h}_2(\hat{\mathbf{x}})) &V_*(\mathbf{x}, \overline{\mathbf{x}}) \\ + (\mathbf{A}\overline{\mathbf{x}} + \mathbf{B}\mathbf{K}_1\overline{\mathbf{x}} + \mathbf{B}\mathbf{T}\overline{\mathbf{u}})&(1 - V_*(\mathbf{x}, \overline{\mathbf{x}})) \\ (\mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{K}_1\hat{\mathbf{x}} + \mathbf{B}\mathbf{T}\mathbf{k}_2(\hat{\mathbf{x}})) &V_*(\mathbf{x}, \overline{\mathbf{x}}) \\ + (\mathbf{A}\overline{\mathbf{x}} + \mathbf{B}\mathbf{K}_1\overline{\mathbf{x}} + \mathbf{B}\mathbf{T}\overline{\mathbf{u}})&(1 - V_*(\mathbf{x}, \overline{\mathbf{x}}))) < 0 \end{aligned} \tag{23}$$

At the new steady-state, $\mathbf{A}\overline{\mathbf{x}} = \mathbf{0}$ and $\dot{\overline{\mathbf{u}}}_A = \mathbf{0}$ holds. Thus, (23) simplifies to

$$\begin{aligned} \nabla V_* \dot{\mathbf{x}} \leq \nabla V_* \mathbf{M}\,(\,\mathbf{v}, &(\mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{h}_1(\hat{\mathbf{x}})) V_*(\mathbf{x}, \overline{\mathbf{x}}), \\ &(\mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{K}_1\hat{\mathbf{x}} + \mathbf{B}\mathbf{T}\mathbf{h}_2(\hat{\mathbf{x}})) V_*(\mathbf{x}, \overline{\mathbf{x}}), \\ &\quad (\mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{K}_1\hat{\mathbf{x}} + \mathbf{B}\mathbf{T}\mathbf{k}_2(\hat{\mathbf{x}})) V_*(\mathbf{x}, \overline{\mathbf{x}})\ ) < 0. \end{aligned}$$

Note that $\hat{\mathbf{x}} \in \partial \mathcal{G}$ holds. On the boundary $\partial \mathcal{G}$ the gradients of $V(\mathbf{x})$ and $V_*(\mathbf{x}, \overline{\mathbf{x}})$ are aligned. Since the gradients of $v(\mathbf{x})$ and $V(\mathbf{x})$ are also aligned and we demanded that the conditions formulated in Lemma 1 are fulfilled, $\nabla V_* \cdot \dot{\mathbf{x}} < 0$ holds for all $\mathbf{x} \in \mathcal{G}$.

In case the system is square the condition $\mathbf{y}(t) \to \mathbf{r}_\infty$ as $t \to \infty$ is equivalent to $\mathbf{x}(t) \to \overline{\mathbf{x}}(\mathbf{r}_\infty)$, $\mathbf{u}(t) \to \overline{\mathbf{u}}(\mathbf{r}_\infty)$ [4]. Since $\dot{V}_*(\mathbf{x}, \overline{\mathbf{x}}) < 0$ we conclude that $V_*(\mathbf{x}, \overline{\mathbf{x}}) \to 0$ as $t \to \infty$ for all $\overline{\mathbf{x}} \in \operatorname{int} \mathcal{G}$ and $\mathbf{x} \in \mathcal{G}$. This implies that $\mathbf{x}(t) \to \overline{\mathbf{x}}(\mathbf{r}_\infty)$ and $\mathbf{k}_{\mathrm{T}}(\overline{\mathbf{x}}(\mathbf{r}_\infty), \mathbf{r}_\infty)) = \overline{\mathbf{u}}(\mathbf{r}_\infty)$. ∎

Remark 2: The controller (15) is not defined for $\mathbf{x} = \overline{\mathbf{x}}$ because of the term $\mathbf{k}_2(\hat{\mathbf{x}}) V_*(\mathbf{x}, \overline{\mathbf{x}})$. But as $\mathbf{k}_2(c\mathbf{x}) = c\,\mathbf{k}_2(\mathbf{x})$ holds, the function $\mathbf{x} \to \overline{\mathbf{x}}$ can be extended by continuity, i.e., $\mathbf{k}_2(\hat{\mathbf{x}}) V_*(\mathbf{x}, \overline{\mathbf{x}}) = \mathbf{k}_2(\hat{\mathbf{x}} V_*(\mathbf{x}, \overline{\mathbf{x}})) = \mathbf{0}$ for $\mathbf{x} \to \overline{\mathbf{x}}$.

VI. Non-square systems

In case the matrix $\mathbf{S}$ is not square or/and not invertible, one possibility could be to use the Moore-Penrose inverse $\mathbf{S}^+$. Unfortunately, the Moore-Penrose inverse does not guarantee, that the steady-state condition $\dot{\overline{\mathbf{x}}}_o = \mathbf{0}$ is met. To overcome this problem we define a least-squares problem with the equality constraint

$$\mathbf{L} \begin{bmatrix} \overline{\mathbf{x}}_o \\ \overline{\mathbf{u}}_A \end{bmatrix} = \begin{bmatrix} \mathbf{A}_o & \mathbf{B}_o \end{bmatrix} \begin{bmatrix} \overline{\mathbf{x}}_o \\ \overline{\mathbf{u}}_A \end{bmatrix} = \mathbf{0}.$$

The solution to this problem can be found in [14]. From there we obtain

$$\begin{aligned} \begin{bmatrix} \overline{\mathbf{x}}_o \\ \overline{\mathbf{u}}_A \end{bmatrix} = \tilde{\mathbf{S}} \begin{bmatrix} \mathbf{0} \\ \mathbf{r} \end{bmatrix} = \Big(\mathbf{S}^+ \\ + \left(\mathbf{S}^\top \mathbf{S}\right)^{-1} \mathbf{L}^\top \Big(\mathbf{L}\left(\mathbf{S}^\top \mathbf{S}\right)^{-1} \mathbf{L}^\top\Big)^{-1} \mathbf{L}\mathbf{S}^+\Big) \begin{bmatrix} \mathbf{0} \\ \mathbf{r} \end{bmatrix}. \end{aligned} \tag{24}$$

This solution minimizes the tracking error $||\overline{\mathbf{y}} - \overline{\mathbf{r}}||$, while ensuring $\dot{\overline{\mathbf{x}}}_o = \mathbf{0}$. In case $\mathbf{S}$ is invertible $\tilde{\mathbf{S}} = \mathbf{S}^{-1}$ holds.

VII. Design of a Linear saturating tracking controller

In this section we state the design of a tracking controller that is employed in the following section. Using Lemma 2 and the Lyapunov function $v(\mathbf{x}) = \mathbf{x}^\top \mathbf{R}\mathbf{x}$ we obtain

$$\mathcal{G} = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top \mathbf{R}\mathbf{x} \leq 1\right\}. \tag{25}$$

We use the LMI[2] design proposed in [2] but with another objective function. First, we briefly state the LMI conditions. Therein, we use the common substitutions $\mathbf{Q} = \mathbf{R}^{-1}$, $\mathbf{Y}_1 = \mathbf{H}_1\mathbf{Q}$, $\mathbf{Y}_2 = \mathbf{H}_2\mathbf{Q}$ and $\mathbf{Z} = \mathbf{K}_2\mathbf{Q}$. The LMIs ensuring that $\dot{v}(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathcal{G}$ are

$$\begin{aligned}\mathbf{Q}\mathbf{A}^\top + \mathbf{A}\mathbf{Q} + \mathbf{B}\mathbf{M}(\mathbf{v}, \mathbf{Y}_1, \mathbf{K}_1\mathbf{Q} + \mathbf{T}\mathbf{Y}_2, \mathbf{K}_1\mathbf{Q} + \mathbf{T}\mathbf{Z})\\ + \mathbf{M}(\mathbf{v}, \mathbf{Y}_1, \mathbf{K}_1\mathbf{Q} + \mathbf{T}\mathbf{Y}_2, \mathbf{K}_1\mathbf{Q} + \mathbf{T}\mathbf{Z})^\top \mathbf{B}^\top \prec 0\end{aligned} \tag{26}$$

for all $\mathbf{v} \in \mathcal{V}$. To ensure $\mathcal{G} \subseteq \mathcal{L}(\mathbf{H}_1\mathbf{x}) \cap \mathcal{L}(\mathbf{H}_2\mathbf{x})$ we use the LMIs

$$\begin{bmatrix} \mathbf{Q} & \mathbf{y}_{j,i} \\ \mathbf{y}_{j,i}^\top & 1 \end{bmatrix} \succeq 0, \tag{27}$$

for $i = 1, \ldots, m$, $j = 1, 2$ where $\mathbf{y}_{j,i}^\top$ denotes the i-th row of the matrix $\mathbf{Y}_j$. In order to ensure that all states of interest are included in the domain of attraction, we use the convex polyhedron $\mathcal{X}_0$ with N vertices $\mathbf{x}_{0,j}$ and demand $\mathcal{X}_0 \subseteq \mathcal{G}$, leading to the LMI

$$\begin{bmatrix} 1 & \mathbf{x}_{0,j}^\top \\ \mathbf{x}_{0,j} & \mathbf{Q} \end{bmatrix} \succeq 0, \qquad j = 1, \ldots, N. \tag{28}$$

We suggest to choose the estimate of the rate of convergence [5] of the virtual system $\dot{\mathbf{x}} = \hat{\mathbf{A}}\mathbf{x} = (\mathbf{A} + \mathbf{B}\mathbf{K}_1 + \mathbf{B}\mathbf{T}\mathbf{K}_2)\mathbf{x}$ as an objective function. The rate of convergence is defined as the largest α such that for all trajectories

$$\lim_{t\to\infty} e^{\alpha t}||\mathbf{x}(t)|| = 0$$

holds. If for the positive definite matrix $\mathbf{Q}$ and a positive $\underline{\alpha}$ the inequality $\mathbf{Q}\hat{\mathbf{A}}^\top + \hat{\mathbf{A}}\mathbf{Q} \prec -2\underline{\alpha}\mathbf{Q}$ is met, then $\underline{\alpha} < \alpha$ is a lower bound of the rate of convergence of the artificial system. We replace the condition in (26) referring to the case $\mathbf{v} = [3\ 3\ \ldots\ 3]^\top$

$$\begin{aligned}\mathbf{Q}\mathbf{A}^\top + \mathbf{A}\mathbf{Q} + (\mathbf{K}_1\mathbf{Q} + \mathbf{T}\mathbf{Z})^\top \mathbf{B}^\top\\ + \mathbf{B}(\mathbf{K}_1\mathbf{Q} + \mathbf{T}\mathbf{Z}) \prec -2\underline{\alpha}\mathbf{Q}\end{aligned} \tag{29}$$

and maximize $\underline{\alpha}$ to achieve a fast controller. In order to avoid numerical difficulties resulting from unnecessary large elements in $\mathbf{Z}$, we restrict the saturating controller to $|u_i| = |k_{2i}^\top \mathbf{x}| \leq \beta$. This leads to the LMIs

$$\begin{bmatrix} \mathbf{Q} & \mathbf{z}_i \\ \mathbf{z}_i^\top & \beta^2 \end{bmatrix} \succeq 0, \qquad i = 1, \ldots, m. \tag{30}$$

Next, we solve the optimization problem

$$\max_{\underline{\alpha}, \mathbf{Q}, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Z}} \underline{\alpha}, \quad \text{s. t.} \tag{31}$$

$$(26),\ (27),\ (28),\ (29),\ (30).$$

[2]For an introduction to LMIs see [5].

This optimization problem can be easily solved using current LMI-Solvers. In this paper we used the interface YALMIP [13] together with the solver SDPT3 [16].

Solving the optimization problem (31) we obtain the controller $\mathbf{K}$. From this solution we infer the tracking controller $\mathbf{k}_{\mathrm{T}}(\mathbf{x}, \mathbf{r})$. Therefore, we need the Minkowski-function

$$\begin{aligned}V_*(\mathbf{x}, \overline{\mathbf{x}}) &= \inf\left\{\gamma > 0 : \overline{\mathbf{x}} + \gamma^{-1}\mathbf{e}_x \in \mathcal{G}\right\}\\ &= \frac{\overline{\mathbf{x}}^\top \mathbf{R}\mathbf{e}_x + \sqrt{(\overline{\mathbf{x}}^\top \mathbf{R}\mathbf{e}_x)^2 + \mathbf{e}_x^\top \mathbf{R}\mathbf{e}_x(1 - \overline{\mathbf{x}}^\top \mathbf{R}\overline{\mathbf{x}})}}{1 - \overline{\mathbf{x}}^\top \mathbf{R}\overline{\mathbf{x}}}.\end{aligned}$$

with $\mathbf{e}_x = \mathbf{x} - \overline{\mathbf{x}}$. Inserting $V_*(\mathbf{x}, \overline{\mathbf{x}})$ in (15) and exploiting the property $c\mathbf{k}_2(\mathbf{x}) = \mathbf{k}_2(c\mathbf{x})$ we obtain the tracking controller

$$\mathbf{k}_{\mathrm{T}}(\mathbf{x}, \mathbf{r}) = \mathbf{K}_2\mathbf{x} + (1 - V_*(\mathbf{x}, \overline{\mathbf{x}}))(\overline{\mathbf{u}} - \mathbf{K}_2\overline{\mathbf{x}}).$$

Finally, we restrict the reference signals to the set of admissible reference vectors. Using the maximum norm and $\overline{\mathbf{u}}_r = \mathbf{S}_u\mathbf{r}$ we get

$$P(\mathbf{r}) = (1 + \epsilon)\max\left\{(\mathbf{r}^\top \mathbf{S}^\top \mathbf{R}\mathbf{S}\mathbf{r})^{1/2}, \|\frac{\overline{u}_{r,i}}{u_{max,i}}\|_\infty\right\}$$

where $\mathbf{S}_x$ and $\mathbf{S}_u$ consist of the associated rows and columns of $\tilde{\mathbf{S}}$ from (24).

Remark 3: The non-saturating controller for comparison purposes can be designed solving the optimization problem

$$\max_{\underline{\alpha}, \mathbf{Q}, \mathbf{Y}} \underline{\alpha}, \quad \text{s. t.} \tag{32}$$

$$\begin{bmatrix} \mathbf{Q} & \mathbf{y}_i \\ \mathbf{y}_i^\top & 1 \end{bmatrix} \succeq 0, \quad \begin{bmatrix} \mathbf{Q} & \mathbf{g}_i \\ \mathbf{g}_i^\top & 1 \end{bmatrix} \succeq 0, \quad i = 1, \ldots, m,$$
$$\mathbf{Q}\mathbf{A}^\top + \mathbf{A}\mathbf{Q} + \mathbf{G}^\top \mathbf{B}^\top + \mathbf{B}\mathbf{G} \prec -2\underline{\alpha}\mathbf{Q},$$
$$\text{and } (28).$$

where we used the substitutions $\mathbf{Q} = \tilde{\mathbf{R}}^{-1}$, $\mathbf{G} = \mathbf{K}_1\mathbf{Q} + \mathbf{T}\mathbf{Y}$ and $\mathbf{Y} = \tilde{\mathbf{K}}_2\mathbf{Q}$. $\mathbf{y}_i$ and $\mathbf{g}_i$ denote the i-th row of $\mathbf{Y}$ and $\mathbf{G}$ respectively. The first LMI ensures the the commanded input amplitude is non-saturating and the second LMI ensures that the input rate is non-saturating.

VIII. Example

In order to show the effectiveness of the proposed method, we use the linearized longitudinal dynamics of the McDonnell Douglas Tailless Advanced Fighter Aircraft (TAFA) [1],

$$\dot{\mathbf{x}}_o = \begin{bmatrix} -1 & 1 \\ 6 & -2 \end{bmatrix}\mathbf{x}_o + \begin{bmatrix} 0 \\ 8 \end{bmatrix} u_A,$$

where x_1 is the deviation of the angle of attack rad and $y = x_2$ is the body axis pitch rate rad/s. The latter is our controlled variable. An amplitude and rate constrained actuator provides the elevator deflection angle from the trim flight condition as input u_A, which is limited to $|u_a| \leq 20/180\pi rad$ and $|\dot{u}_a| \leq 40/180\pi rad/s$. The linearized system has a stable pole at $\lambda_1 = -4$ and an unstable pole at $\lambda_2 = 1$. We want to guarantee an operating region that includes possible initial conditions contained in the set

$$\mathcal{X}_0 = \left\{\mathbf{x} :\ |x_{o1}| \leq \frac{20}{180}\pi\, rad, |x_{o2}| \leq \frac{25}{180}\pi\, rad/s\right\}.$$

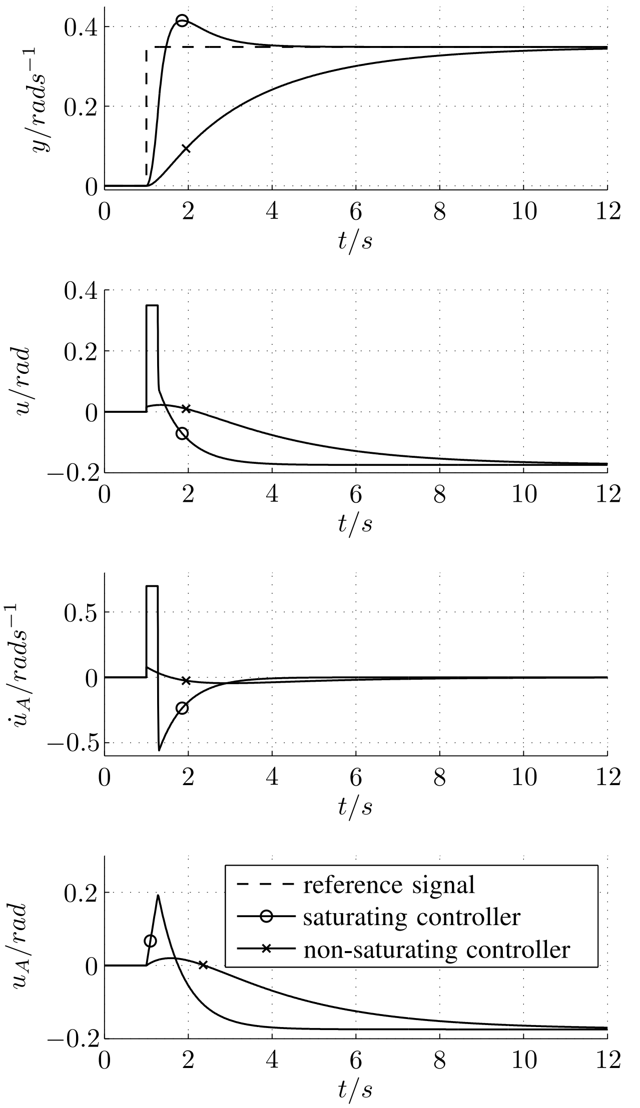

Fig. 3. Evolution of reference signal r and system output y for a saturating and a non-saturating controller (first). The commanded elevator deflection angle u (second), the input rate $\dot{u}_A$ (third) and the actual elevator deflection angle u_A (fourth). The initial state was chosen as $\mathbf{x}_0 = [0\ 0]^\top$.

We choose $\tau = 5$ assume that the elevator is initially not elongated, i.e., $u_{A,0} = 0$. We designed the saturating controller as explained in Section VII with $\beta = 50$ resulting in

$$\mathbf{K}_2 = \begin{bmatrix} -31.5926 & -10.5239 & -26.0708 \end{bmatrix},$$
$$\mathbf{R} = \begin{bmatrix} 1.7484 & 0.5826 & 0.9388 \\ 0.5826 & 0.1943 & 0.3127 \\ 0.9388 & 0.3127 & 0.8029 \end{bmatrix}.$$

The non-saturating controller is designed solving the optimization problem (32) and we obtain

$$\tilde{\mathbf{K}} = \begin{bmatrix} -0.2423 & -0.0808 & 0.5049 \end{bmatrix},$$
$$\tilde{\mathbf{R}} = \begin{bmatrix} 4.1704 & 1.2982 & 5.3482 \\ 1.2982 & 0.5063 & 1.7600 \\ 5.3482 & 1.7600 & 13.2112 \end{bmatrix}.$$

The evolutions of the system output, the commanded input u, the actual input rate $\dot{u}_A$ and u_A for the initial state $\mathbf{x}_0 = [0\ \ 0]^\top$ are depicted in Fig. 3. As reference signal we choose a step function of height $\bar{\mathbf{r}} = 20/180\pi$ at $t = 1s$. Obviously the saturating tracking controller has a shorter settling time compared to the non-saturating controller because the available input amplitude and rate is exploited. The commanded input amplitude as well as the input rate are saturating.

IX. Conclusions

We proposed a set point tracking controller for systems subject to simultaneous amplitude and rate constraints. The commanded input as well as the input rate are allowed to saturate. To this end, we extended the non-saturating tracking controller proposed in [4] for systems that are exclusively subject to amplitude constraints to the case of additional rate constraints. We provided a saturating controller based on linear state feedback [2] that can be designed using LMIs. Finally, we showed the effectiveness of the saturating controller in comparison with a non-saturating controller.

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