2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

"Night Comes to the Cretaceous" and Other Tales of the Decibel

William Messner

Abstract— **This education paper presents three *entertainment problems* that may be of use to educators. These problems illustrate how to use the decibel scale for back of the envelope calculation without a calculator. "Night Comes to the Cretaceous" estimates the size the meteor believed to have caused the extinction of the dinosaurs from the diameter of the Chicxulub crater in the Yucatan, the crater excavated by the detonation of the first hydrogen bomb, and the radius of the Earth's orbit around the Sun. "Euler Would Be Proud" shows how to easily determine powers of e and how to determine natural logarithms without a calculator. "The Amazing Number 1.6" demonstrates that the doubling of the number of transistors on a chip every 18 months of Moore's Law corresponds to an annual compound growth rate of 60% and then shows the implications of this growth rate on computational power over periods of decades. This paper uses the by-hand rational approximations to the multiplicative factors corresponding to integer decibel values between 1 and 20 and the rational decibel approximations corresponding to integer factors between 1 and 10 that appeared in [2].**

TABLE I
DECIBEL FACTOR APPROXIMATIONS

dB value	factor	error (%)	*dB* value	factor	error (%)
1	1.125	0.2	11	3.6	1.5
2	1.25	-0.7	12	4	0.5
3	1.4	-0.9	13	4.5	0.7
4	1.6	1.0	14	5	-0.3
5	1.8	1.2	15	5.6	-0.4
6	2	0.2	16	6.25	-0.9
7	2.25	0.5	17	7	-1.2
8	2.5	-0.5	18	8	0.7
9	2.8	-0.7	19	9	1.0
10	3.2	1.2	20	10	0

TABLE II
DECIBEL APPROXIMATIONS CORRESPONDING TO INTEGER FACTORS

factor	*dB* value	error (%)	factor	*dB* value	error (%)
1	0	0	6	15.5	0.4
2	6	-0.3	7	17	0.6
3	9.5	-0.4	8	18	0.3
4	12	-0.3	9	19	-0.4
5	14	0.1	10	20	0

I. INTRODUCTION

The decibel scale is the logarithmic scale ubiquitously employed in plotting the magnitude of frequency responses (e.g. [1]). By-hand derivations of easy-to-remember approximations to the multiplicative factors corresponding to integer decibel values between 1 and 20 (Table I) and rational decibel approximations corresponding to integer factors between 1 and 10 (Table II) that are within 1.5% of the actual values appeared in [2]. That same paper also showed the utility of these approximation with a back-of-the-envelope calculation of the average compound growth rate of the Dow Jones industrial average from inception to 2006.

In this follow-on paper, I present three *entertainment problems* further illustrating the utility of these approximations that may be of use to educators, and I derive by hand the little known fact that e corresponds to approximately $8\frac{2}{3}$ dB. Inspired by the book by James Lawrence Powell of the same name [3], "Night Comes to the Cretaceous" estimates the size the meteor believed to have caused the extinction of the dinosaurs from the diameter of the Chicxulub crater in the Yucatan, the crater excavate by the detonation of the first hydrogen bomb, and the radius of the Earth's orbit around the Sun. "Euler Would Be Proud" shows how to easily determine powers of e without a calculator and how to determine natural logarithms without a calculator. "The Amazing Number 1.6" demonstrates that the doubling of the number of transistors on a chip every 18 months of Moore's Law corresponds to an annual compound growth rate of 60% and then shows the implications of this growth rate on computational power over periods of decades.

Section V contains concluding remarks.

II. "NIGHT COMES TO THE CRETACEOUS"

In "Night Comes to the Cretaceous: Dinosaur Extinction and the Transformation of Modern Biology" by James Lawrence Powell provides an excellent account of the development of and controversy surrounding the hypothesis that the impact of a meteor in the Yucatan 65 million years ago led to the extinction of the dinosaurs. [3] This hypothesis is now is widely accepted. [4]

The objective of this entertainment problem is to estimate the size of the meteor. The procedure to as follows.

1) Determine the energy released by the impact.
2) Estimate the velocity of the meteor.
3) Use the relation for kinetic energy $E = \frac{1}{2}mv^2$ to determine the mass of the meteor.
4) Estimate the density ρ of the meteor and use this value to determine its volume V.
5) Determine the diameter d of meteor assuming that it was spherical from the relation $V = \frac{\pi}{6}d^3$.

In addition to the assumption the meteor was spherical, the following assumptions are employed.

1) The energy released by an explosion at ground level is proportional to the volume of the crater excavated by the explosion.

W. Messner is with Department of Mechanical Engineering, Carnegie Mellon University, 5000 Forbes Avenue, Pittsburgh, Pennsylvania USA bmessner@andrew.cmu.edu

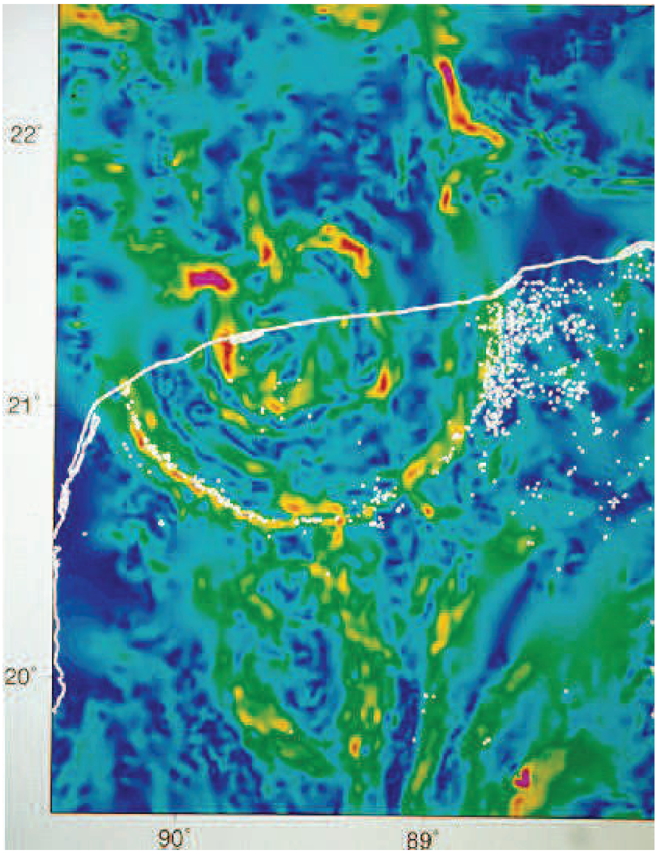
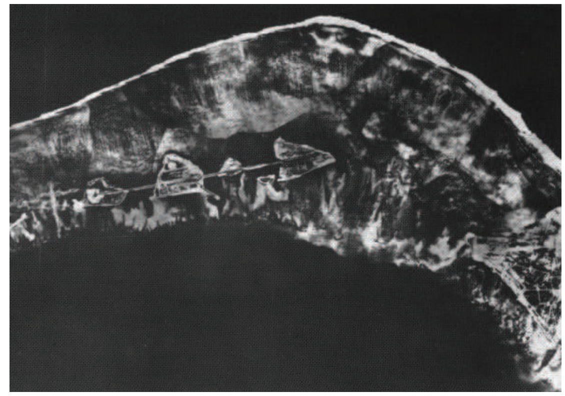
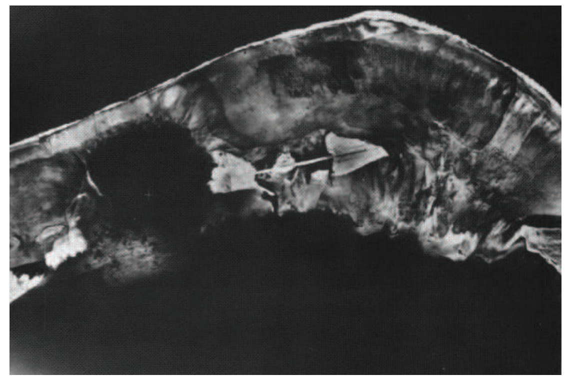


Fig. 1. Gravity anomoly map of the Chicxulub Crater. The white line indicates the coastline of the Yucatan with water to the north (up). The white dots indicate cenotes. The diameter of the arc of cenotes is approximately 190 km. (Courtesy of Dr. Alan Hildebrand)



(a)



(b)

Fig. 2. (a) Before and (b) after photographs of Elugelab show the crater excavated by the detonation of the first thermonuclear device–the Mike Shot. The crater diameter is approximately 1.9 km. (http://nuclearweponsarchive.org)

2) The excavated volume is proportional to the third power of the diameter of the crater.
3) The meteor was traveling at the velocity of Earth as it orbits the Sun.

It is clear that the calculation will require extraction of integer roots. This is where decibels will be employed.

A. Energy calculation

Fig. 1 shows a map indicating the size and location of the Chicxulub Crater on the northern coast of the Yucatan Penninsula. The diameter of Chicxulub is approximately $d_C = 190$ kilometers. [4]

To estimate the energy released by the impact, we compare it to a crater excavated by an explosion of known energy. The so-called "'Mike Shot"' was the detonation of the first thermonuclear device on 1 November 1952 on the island of Elugelab in the Eniwetok Atoll in the Pacific Ocean. [5] The energy of the Mike Shot was 10 MT (1 MT is the energy of 10^6 metric tons of TNT), and it excavated a crater approximately $d_M = 1.9$ km in diameter, obliterating Elugelab. [6] (See Fig. 2) Designating E_C as the energy of the impact that created the Chicxulub crater, and E_M as the energy of the Mike Shot, the energy of the Chicxulub impact was approximately

$$\begin{aligned} E_C &= \left(\frac{E_C}{E_M}\right) E_M = \left(\frac{d_C}{d_M}\right)^3 E_M \\ &\approx \left(\frac{190\,km}{1.9\,km}\right)^3 \times 10\,MT \\ &= 100^3 \times 10\,MT = 10^7\,MT \end{aligned} \tag{1}$$

The energy of 1 gram of TNT (trinitrotoluene) is approximately 4,000 J (Joules). [7] One MT is therefore

$$\begin{aligned} 1\,MT &\approx \left(\frac{4000\,J}{gram}\right)\left(\frac{10^3\,gram}{kg}\right)\left(\frac{10^3\,kg}{ton}\right)\left(\frac{10^6\,ton}{megaton}\right) \\ &= 4 \times 10^{15}\,J \end{aligned} \tag{2}$$

Using this value, energy released by the impact at Chicxulub was approximately

$$E_C \approx \left(10^7\right)\left(4 \times 10^{15}\,MT\right) = 4 \times 10^{22}\,J. \tag{3}$$

B. Velocity estimate

The meteor was most likely orbiting the Sun. Since the meteor crossed the Earth's orbital radius, we assume that its speed when it entered the atmosphere, v_C, was the same as the magnitude of the velocity of the Earth as it orbits the Sun. We use this value as the impact speed, since the atmosphere would have had little effect on an object with such a low surface to volume ratio.

The Earth's orbit is nearly circular with an average radius of about $r_E = 1.5 \times 10^8\ km$. [8] The circumference of this orbit is approximate $2\pi r_E = 3\pi \times 10^8\ km$. The period of the Earth's orbit is approximate $T_E = 365.25$ days or $365.25 \times 24 \times 60 \times 60 \approx 3.16 \times 10^7 \approx \pi \times 10^7$ seconds.

The estimate of the speed of the meteor at impact is

$$v_C \approx \frac{2\pi r_E}{T_E} \approx \frac{3\pi \times 10^8\, km}{\pi \times 10^7\, sec} = 3 \times 10^4\, m\, s^{-1}. \tag{4}$$

C. Mass estimate

The mass of the meteor, m_C was twice its kinetic energy divided by the square of its speed.

$$\begin{aligned} m_C &= \frac{2E_C}{v_C^2} \approx \frac{2 \times \left(4 \times 10^{22}\, J\right)}{(3 \times 10^4\, m\, s^{-1})^2} \\ &\approx \left(\frac{8 \times 10^{22}}{9 \times 10^8}\right) kg = \left(\frac{8}{9}\right) \times 10^{14}\, kg \end{aligned} \tag{5}$$

D. Volume and Diameter estimates

The volume of the meteor was its mass divided by its average density. If the meteor was primarily water ice, then its density would have been $\rho_{H_2O} \approx 1000\, kg\, m^{-3}$ or less. If the meteor was composed mostly of iron, then its density would have been $\rho_{Fe} \approx 8000\, kg\, m^{-3}$ or even higher. We will consider both cases.

1) ρ_{H_2O}: Using the density of water, the volume of the meteor would have been

$$V_{C_{ice}} \approx \frac{m_C}{\rho_{H_2O}} \approx \frac{\left(\frac{8}{9}\right) \times 10^{14}\, kg}{1000\, kg\, m^{-3}} = \left(\frac{8}{9}\right) \times 10^{11}\, m^3 \tag{6}$$

Using $V = \frac{\pi}{6} d^3$

$$\begin{aligned} d_{C_{ice}} &\approx \left(\frac{6 V_{C_{ice}}}{\pi}\right)^{\frac{1}{3}} \approx \left(\frac{6 \times \left(\frac{8}{9}\right) \times 10^{10}\, m^3}{\pi}\right)^{\frac{1}{3}} \\ &\approx \left(\frac{5.3 \times 10^{11}\, m^3}{\pi}\right)^{\frac{1}{3}} \end{aligned} \tag{7}$$

To calculate the third root it is convenient to convert to decibels, and to use linear interpolation to determine the approximation to $20\log(5.3)$

$$\begin{aligned} 20\log(5.3) &\approx 20\log(5) \\ &\quad + \left(\frac{5.3-5}{5.6-5}\right)(20\log(5.6) - 20\log(5)) \\ &\approx 14.5 dB. \end{aligned} \tag{8}$$

Applying decibels gives

$$\begin{aligned} 20\log\left(d_{C_{ice}}\right) &\approx 20\log\left[\left(\frac{5.3 \times 10^{11}\, m^3}{\pi}\right)^{\frac{1}{3}}\right] \\ &= \frac{1}{3}[20\log(5.3)] + \\ &\quad \frac{1}{3}\left[20\log\left(10^{11}\right) - 20\log(\pi)\right] \\ &\approx \frac{1}{3}[14.5 + 220 - 10] \\ &\approx \frac{224.5}{3} \approx 74.8\, dB. \end{aligned} \tag{9}$$

Here the approximation $\pi \approx \sqrt{10}$ was employed to give $20\log(\pi) \approx 10\, dB$.

Rewriting $74.8\, dB = 14.8\, dB + 60\, dB$, use linear interpolation to determine the factor corresponding to $14.9\, dB$ from Table I.

$$10^{\frac{14.8}{20}} \approx 5 + 0.8\,(5.6 - 5) \approx 5.5. \tag{10}$$

Thus diameter of the meteor assuming a density of $1000\, kg\, m^3$ was

$$d_{C_{ice}} \approx 10^{\frac{74.8}{20}} = 10^{\frac{14.8}{20}} \times 10^{\frac{60}{20}} \approx 5.5 \times 1000 = 5500\, m \tag{11}$$

2) ρ_{Fe}: Since the ratio of ρ_{Fe} to ρ_{H_2O} is a perfect cube, it is easy to calculate the diameter using the density $\rho_{Fe} = 8000\, kg\, m^3$, from the results of the previous section. The volume of the meteor would have been

$$V_{C_{Fe}} = \frac{m_C}{\rho_{Fe}} = \frac{\rho_{H_2O}}{\rho_{Fe}} \frac{m_C}{\rho_{H_2O}} = \frac{1}{8} V_{C_{ice}}. \tag{12}$$

Again using $V = \frac{\pi}{6} d^3$

$$\begin{aligned} d_{C_{Fe}} &= \left(\frac{6 V_{C_{Fe}}}{\pi}\right)^{\frac{1}{3}} = \left(\frac{6\left(\frac{1}{8} V_{C_{ice}}\right)}{\pi}\right)^{\frac{1}{3}} \\ &= \left(\frac{1}{8}\right)^{\frac{1}{3}} \left(\frac{6 V_{C_{ice}}}{\pi}\right)^{\frac{1}{3}} \\ &= \frac{1}{2} d_{C_{ice}} \approx 2800\, m. \end{aligned} \tag{13}$$

III. EULER WOULD BE PROUD

Negative integer powers of e are useful identifying the time constants of dynamic systems from impulse response or step response data. In other instances, determining natural logarithms of specific numerical values is useful. This section shows how to easily calculate such values using the decibel value corresponding to e.

A. Decibel approximation corresponding to e

We first note that [9]

$$\begin{aligned} e \approx 2.718 &= \frac{27}{10} + \frac{18}{1000} \\ &= \frac{27}{10} + \frac{\left(\frac{2}{3}\right)(27)}{(100)(10)} \\ &= \frac{27}{10} + \left(\frac{2}{300}\right)\left(\frac{27}{10}\right) \qquad (14) \\ &= \frac{27}{10}\left[1 + \left(\frac{2}{300}\right)\right] \qquad (15) \end{aligned}$$

Next raise e to the third power employing this relation and the Binomial Theorem.

$$\begin{aligned}
e^3 &\approx \frac{27^3}{10^3}\left[1+\left(\frac{2}{300}\right)\right]^3 \\
&= \frac{3^9}{1000}\left[1+3\left(\frac{2}{300}\right)+3\left(\frac{2}{300}\right)^2+\left(\frac{2}{300}\right)^3\right] \\
&\approx \frac{3^9}{1000}\left[1+3\left(\frac{2}{300}\right)\right] \\
&= \frac{(3)(81)(81)}{1000}\left[1+\left(\frac{2}{100}\right)\right] \\
&= \frac{(3)(6,561)}{1000}(1.02) = (19.683)(1.02) \\
&= 19.683+0.39362 = 20.07662 \approx 20 \qquad (16)
\end{aligned}$$

Therefore we have the important approximation

$$e \approx 20^{\frac{1}{3}} \qquad (17)$$

Note that this approximation has an error of less than 0.2%.

Now use this approximation to find the decibel value corresponding to a factor of e.

$$\begin{aligned}
20\log(e) &\approx 20\log\left(20^{\frac{1}{3}}\right) = \frac{1}{3}(20\log(20)) \\
&= \frac{1}{3}(20\log(2)+20\log(10)) \\
&\approx \frac{1}{3}(6+20) = 8\frac{2}{3}\ dB \qquad (18)
\end{aligned}$$

We can use this fact, Table I, and linear interpolation to determine the negative integer powers of e without a calculator.

B. Approximate value of powers of e

1) e^{-1}: This subsection shows in detail the method for using the decibel equivalent to determine a negative integer power.

$$20\log\left(e^{-1}\right) = -20\log(e) \approx -8\frac{2}{3}\ dB \qquad (19)$$

To make conversion to decimals easier, it is preferred to write the decibel value as a positive number between 0 and 20 dB plus a multiple of -20 dB. Thus

$$20\log\left(e^{-1}\right) \approx 11\frac{1}{3}\ dB\ -20\ dB \qquad (20)$$

Since 11 dB corresponds to a factor of 3.6 and 12 dB corresponds to a factor of 4, linear interpolation gives that $11\frac{1}{3}$ dB corresponds to a factor of

$$10^{\left(11\frac{1}{3}\right)\left(\frac{1}{20}\right)} \approx 3.6+\frac{1}{3}(4-3.6) = 3.6+\frac{0.4}{3} \approx 3.73 \qquad (21)$$

Then

$$e^{-1} \approx 10^{\left(11\frac{1}{3}\right)\left(\frac{1}{20}\right)} \div 10^{\left(\frac{20}{20}\right)} = \frac{3.73}{10} = 0.373 \qquad (22)$$

This approximation is within +1.4% of the actual value.

2) Approximate values of e^{-2}, e^{-3}, and e^{-4}: Using the method above we have the following approximations

$$\begin{aligned}
20\log\left(e^{-2}\right) &\approx -17\frac{1}{3}\ dB = 2\frac{2}{3}\ dB\ -20\ dB \\
20\log\left(e^{-3}\right) &\approx -26\ dB = 14\ dB\ -40\ dB \\
20\log\left(e^{-4}\right) &\approx -34\frac{2}{3}\ dB = 5\frac{1}{3}\ dB\ -40\ dB \qquad (23)
\end{aligned}$$

Using Table I, and linear interpolation for the fractional decibel values, we obtain the following approximations.

$$\begin{aligned}
10^{\left(2\frac{2}{3}\right)\left(\frac{1}{20}\right)} &\approx 1.25+\frac{2}{3}(1.4-1.25) = 1.35 \\
10^{\left(5\frac{1}{3}\right)\left(\frac{1}{20}\right)} &\approx 1.8+\frac{1}{3}(2-1.8) \approx 1.87 \qquad (24)
\end{aligned}$$

which lead to the following estimates for powers of e.

$$\begin{aligned}
e^{-2} &\approx 1.35 \div 10 = 0.135 \\
e^{-3} &\approx 5 \div 100 = 0.05 \\
e^{-4} &\approx 1.87 \div 100 = 0.0187 \qquad (25)
\end{aligned}$$

The errors in these approximations are -0.25%, +0.4%, and +2.1% respectively.

C. Natural logarithm values from the decibel

The relation (16) allows easy determination of $\ln(10)$ and subsequently leads to an easy method for determining natural logarithm values from the decibel.

$$\begin{aligned}
e^{30} &= \left(e^3\right)^{10} \approx 20.08^{10} = (1.004)^{10} \times 20^{10} \\
&\approx (1+10(0.004)) \times 2^{10} \times 10^{10} \\
&= 1.04 \times 1024 \times 10^{10} \\
&= 1.04 \times 1.024 \times 10^3 \times 10^{10} \\
&\approx 1.065 \times 10^{13} \qquad (26)
\end{aligned}$$

Now take natural logarthms of both sides

$$\begin{aligned}
30 &\approx \ln\left(1.065 \times 10^{13}\right) = \ln(1.065)+\ln\left(10^{13}\right) \\
&\approx 0.065+13\ln(10) \qquad (27)
\end{aligned}$$

Solving for $\ln(10)$ gives

$$\ln(10) \approx \frac{(30-0.065)}{13} = \frac{29.965}{13} = 2.305 \qquad (28)$$

where the final step is obtained by long division.

Having the value of $\ln(10)$ enables determination of natural logarthims from decibel values. Let y be a positive number and let x be the corresponding decibel value, i.e. $10^{\left(\frac{x}{20}\right)} = y$. Using the relation

$$\begin{aligned}
\ln(y) &= \ln\left[10^{\left(\frac{x}{20}\right)}\right] = \ln\left[10^{\left(\frac{x}{20}\right)}\right] \\
&= \ln\left[\left(e^{\ln(10)}\right)^{\left(\frac{x}{20}\right)}\right] = \ln\left[e^{\ln(10)\left(\frac{x}{20}\right)}\right] \\
&= \frac{\ln(10)}{20}x \approx 0.115x \qquad (29)
\end{aligned}$$

where the last result is obtained by inspection or long division.

Given a decibel value x can be obtained by hand (or in one's head) by the following procedure.

1) Divide x by 10.
2) Divide the result of (1) by 10. This is $0.01x$.
3) Add the result of (2) to the result of (1). This is $0.11x$.
4) Divide the result of (2) by 2. This is $0.005x$
5) Add the result of (4) to the result of (3). This is $0.115x$.

IV. THE AMAZING NUMBER 1.6

Moore's Law is the empirical observation that from 1966 to 2006 that number of transistors on a chip increased by approximately factor of 2 every 18 months or 1.5 years. Assuming that the number of transistors on a microprocessor is linearly related to the speed of computation of a computer, Moore's Law implies a doubling of computer speed every 18 months over that period of 40 years.

A doubling every 18 months correponds to an annual compount growth rate given by

$$r_{Moore} = 2^{\frac{2}{3}} \tag{30}$$

Computing this fractional power is easy with the decibel.

$$\begin{aligned} 20\log(r_{Moore}) &= 20\log\left(2^{\frac{2}{3}}\right) = \left(\frac{2}{3}\right) \times 20\log(2) \\ &\approx \left(\frac{2}{3}\right) \times 6 = 4 \end{aligned} \tag{31}$$

Using Table I

$$r_{Moore} \approx 10^{\frac{4}{20}} \approx 1.6. \tag{32}$$

The annual compound growth rate of Moore's Law is 1.6 or 60% per year.

Why is this amazing? Consider the implications of Moore's Law over longer periods of time. For 5 years the increase in the number of transitors will be

$$20\log\left(1.6^5\right) \approx 5 \times 20\log(1.6) = 5 \times 4 = 20, \tag{33}$$

which implies

$$1.6^5 = 10^{\frac{20}{20}} = 10. \tag{34}$$

Therefore, Moore's Law corresponds to a factor 10 increase in computation speed every 5 years. For 10, 15, and 20 years, the increases in computation speed are approximately

$$\begin{aligned} 1.6^{10} &= \left(1.6^5\right)^2 \approx 10^2 = 100, \\ 1.6^{15} &= \left(1.6^5\right)^3 \approx 10^3 = 1000, \\ 1.6^{20} &= \left(1.6^5\right)^4 \approx 10^4 = 10,000 \end{aligned} \tag{35}$$

respectively.

The number 10,000 is another amazing number, because there are approximately 10,000 hours in a year ($8,760 = 365 \times 24$). Now imagine the following scenario: it is 1985, the year of the author's graduation from college, and he devises a computational problem which he estimates would take 25 *years* run on his PC. Given that Moore's Law showed no signs of abating, he realizes that in 2005, computers will be about 10,000 times faster, and his computation problem will take only 25 *hours* on a new PC. **Therefore, he is better off waiting 20 years, buying a new computer, running the problem overnight, and finishing 5 years early.** This is why 1.6 is amazing!

V. CONCLUSIONS

This paper presented three *entertainment problems* that use usedeveloped rational factor approximations corresponding to integer decibel values between 1 and 20 and rational decibel approximations for integer factors between 1 and 10. By using these approximations, sophisticated calculations involving quotients, products, powers, and logarithms can be performed without a calculator or computer. The methods for determining powers of e and natural logarithms may be of particular use to practitioners for back of the envelope calculations.

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