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An LMI Approach to Mixed H_2/H_∞ Robust Fault-tolerant Control Design with Uncertainties

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Abstract— **This paper studies mixed H_2/H_∞ robust fault-tolerant control for a class of uncertain systems and its application to flight tracking control. A sufficient condition is derived by introducing some important auxiliary variables, which guarantees that the uncertain closed-loop system is robustly stable and satisfies the mixed H_2/H_∞ constraint in both normal and fault cases. In the framework of Linear Matrix Inequality (LMI) approach, a multi-objective optimization problem is solved with much less conservative via an iterative algorithm. Simulation results obtained with a nonlinear fighter aircraft model show the effectiveness of the proposed method.**

I. Introduction

To improve the performance of modern fighter aircraft, the associated flight control systems become more complex and thus faults happen more frequently. In general, control component faults such as actuator and/or sensor faults are the most fatal failures, namely, these faults often result in performance degradation, even instability. Therefore, researches on Fault-Tolerant Control (FTC) have increased progressively over the last three decades. Among them, robust FTC [1]-[3] is one of the popular methods to design Fault-Tolerant Control System (FTCS) for aircraft.

It is well known that H_2 control is adapted to deal with transient performance while H_∞ control guarantees robust stability in the face of uncertainties and disturbances. To manage the trade-off between the system performance and robustness, the mixed H_2/H_∞ control was first introduced by Bernstein and Haddad [4]. In recent years, Linear Matrix Inequality (LMI)-based methods [5]-[10] have become one of the most effective tools to solve the mixed H_2/H_∞ control problem due to the development of the interior point algorithm for convex optimization. However, to the best knowledge of authors, robust FTC against actuator faults has never been considered in the mixed H_2/H_∞ control, also, its application to flight control system has not been reported in the literature.

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In this paper, a new synthesis approach is developed for the mixed H_2/H_∞ robust fault-tolerant control with application to flight tracking control. We first introduce some auxiliary variables [11] to the LMI characterizations regarding to the mixed H_2/H_∞ performance index, which separate Lyapunov function variables from controller gain variable. Then, a sufficient condition is derived from the above transformation, which guarantees that the uncertain closed-loop system is robustly stable and satisfies the mixed H_2/H_∞ constraint in both normal and fault cases. Subsequently, an Iterative LMI (ILMI) algorithm is developed from the proposed sufficient condition to solve the multi-objective optimization problem with much less conservativeness. The proposed approach is applied to flight control system of a nonlinear fighter aircraft model known as ADMIRE to ensure tracking performance.

Notation: For a matrix *X*, X^T , $tr(X)$ and $\sigma_{\max}(X)$ denote its transposition, trace and largest singular value, respectively. The Symbol "*" within a matrix presents the symmetric entries. $X = diag(x_1,\ldots,x_n)$ denotes that *X* is a diagonal matrix and diagonal element is $x_i\,(i=1,\ldots,n)$. $I_{n\times n}$ denotes an identity matrix with *n* dimension.

II. Preliminaries and Problem Formulation

Consider an uncertain Linear Time-Invariant (LTI) system described by

$$\begin{cases}\dot{x}(t) = [A+\Delta A(t)]x(t)+[B+\Delta B(t)]u(t)+[G+\Delta G(t)]w(t)\\ y(t) = Cx(t)\end{cases} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state, $u(t)\in\mathbb{R}^m$ is the control input, $w(t)\in\mathbb{R}^h$ is the disturbance input and $y(t)\in\mathbb{R}^p$ is the measured output. *A, B, C* and *G* are known real constant matrices with appropriate dimensions, which describe the nominal system. ΔA, ΔB and ΔG are real-valued time-varying matrix functions representing the norm-bounded parameter uncertainties:

$$\Delta A = E_a\Delta(t)F_a,\ \Delta B = E_b\Delta(t)F_b,\ \Delta G = E_g\Delta(t)F_g \tag{2}$$

where $\Delta^T(t)\Delta(t)\le I$ [13].

The system (1) with actuator faults can then be rewritten as:

$$\begin{cases}\dot{x}(t) = [A+\Delta A(t)]x(t)+[B+\Delta B(t)]\omega_L u(t)+[G+\Delta G(t)]w(t)\\ y(t) = Cx(t)\end{cases} \tag{3}$$

where ω_L are the matrices of actuator effectiveness factors and satisfy

$$\omega_L \in \Theta \ \{\omega_L = diag[\omega_{L1},\ \omega_{L2},\ \ldots,\ \omega_{Lm}],$$
$$\omega_{Li} \in [\underline{\omega}_{Li},\ \overline{\omega}_{Li}],\ \ i = 1, 2,\ \ldots,\ m\} \tag{4}$$
$$L = 0,\ 1,\ \ldots,l_p,\ l_p \le 2^m - 1$$

with $\underline{\omega}_{Li}$ and $\overline{\omega}_{Li}$ representing the lower and upper bounds of ω_{Li} respectively. Partial loss in control effectiveness is given by $0 \le \omega_{Li} \le 1$. It is worth mentioning that $\omega_{Li} = 0$ means total outage in the *i*th actuator and $\omega_{Li} = 1$ denotes a healthy actuator.

Consider the measured output *Sy(t)* tracks the reference signal *r(t)* without steady-state error in both normal and fault cases, that is

$$\lim_{t\to\infty} e(t) = 0, \qquad e(t) = r(t) - Sy(t) \tag{5}$$

where *e(t)* is tracking error and $S \in \mathbb{R}^{l\times P}$ is a known constant matrix used to select the measured output. Based on the standard tracking optimization problem setting, the reference signal *r(t)* is taken as a disturbance. It is well known that the tracking error integral action of a controller can effectively eliminate the steady-state tracking error [1]. In order to obtain a mixed H_2/H_∞ robust fault-tolerant controller with state feedback plus integral action of the tracking error, we restructure the system (3) as an augmented system and the state-space description is given by

$$\begin{bmatrix} e(t) \\ \dot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & -SC \\ 0 & A+E_a\Delta(t)F_a \end{bmatrix}\begin{bmatrix} \int_0^t e(t)dt \\ x(t) \end{bmatrix} + \begin{bmatrix} 0 \\ B+E_b\Delta(t)F_b \end{bmatrix}\omega_L u(t) + \begin{bmatrix} I & 0 \\ 0 & G+E_g\Delta(t)F_g \end{bmatrix}\begin{bmatrix} r(t) \\ w(t) \end{bmatrix} \quad L = 0,1,\ldots,2^m-1 \tag{6}$$

Define the augmented state $\overline{x}(t) = [(\int_0^t e(t)dt)^T,\ x^T(t)]^T$, the augmented disturbance $v(t) = [(r^T(t),\ w^T(t)]^T$ and the augmented output $\overline{y}(t) = [(\int_0^t e(t)dt)^T,\ y^T(t)]^T$, and then the augmented system can be represented as:

$$\begin{cases} \dot{\overline{x}}(t) = [\overline{A}+\overline{E}_a\overline{\Delta}(t)\overline{F}_a] + [\overline{B}+\overline{E}_b\overline{\Delta}(t)\overline{F}_b]\omega_L u(t) + [\overline{G}+\overline{E}_g\overline{\Delta}(t)\overline{F}_g]v(t) \\ \overline{y}(t) = \overline{C}\overline{x}(t) \\ z_\infty(t) = [C_\infty + D_\infty\omega_L K\overline{C}]\overline{x}(t) \\ z_2(t) = [C_2 + D_2\omega_L K\overline{C}]\overline{x}(t) \end{cases} \tag{7}$$

where $z_\infty(t) \in \mathbb{R}^{p_1}$ and $z_2(t) \in \mathbb{R}^{p_2}$ are the controlled outputs, C_∞, C_2, D_∞, D_2 are known real constant matrices and

$$\overline{A} = \begin{bmatrix} 0 & -SC \\ 0 & A \end{bmatrix},\ \overline{B} = \begin{bmatrix} 0 \\ B \end{bmatrix},\ \overline{G} = \begin{bmatrix} I & 0 \\ 0 & G \end{bmatrix},$$
$$\overline{E}_a = \begin{bmatrix} 0 \\ E_a \end{bmatrix},\ \overline{F}_a = [0 \quad F_a],\ \overline{E}_b = \begin{bmatrix} 0 \\ E_b \end{bmatrix},\ \overline{F}_b = F_b,$$
$$\overline{E}_g = \begin{bmatrix} 0 \\ E_g \end{bmatrix},\ \overline{F}_g = [0 \quad F_g],\ \overline{\Delta}(t) = \Delta(t).$$

Obviously, if a controller stabilizes the augmented system (7), it also stabilizes the original system (3).

III. Mixed H_2/H_∞ Controller Design

In this section, we develop a new ILMI method to achieve a mixed H_2/H_∞ robust fault-tolerant controller such that:

-First, the controller stabilizes the uncertain closed-loop augmented system in both normal and fault cases.

-Second, the controller satisfies the mixed H_2/H_∞ performance constraint.

Consider the following state feedback controller for the augmented system (7):

$$u(t) = K\overline{x}(t) = K_e\int_0^t e(t)dt + K_x x(t) \tag{8}$$

where $K = [K_e \quad K_x] \in \mathbb{R}^{m\times(l+p)}$ is the state feedback controller gain to be determined.

Assumption 1: The state of the system is available at every time instant.

The uncertain closed-loop augmented system is represented as follows by substituting (8) into (7)

$$\begin{cases} \dot{\overline{x}}(t) = \{[\overline{A}+\overline{E}_a\overline{\Delta}(t)\overline{F}_a] + [\overline{B}+\overline{E}_b\overline{\Delta}(t)\overline{F}_b]\omega_L K\}\overline{x}(t) + [\overline{G}+\overline{E}_g\overline{\Delta}(t)\overline{F}_g]v(t) \\ \overline{y}(t) = \overline{C}\overline{x}(t) \\ z_\infty(t) = [C_\infty + D_\infty\omega_L K]\overline{x}(t) \\ z_2(t) = [C_2 + D_2\omega_L K]\overline{x}(t) \end{cases} \tag{9}$$

Let $T_{z_\infty v}$ and $T_{z_2 v}$ denote the transfer function from *v(t)* to $z_\infty(t)$ and $z_2(t)$, respectively. Based on the presentations of H_2 and H_∞ norm [12], we first present the following definition and lemmas, which play important roles to the development of our main results.

Definition 1: Mixed H_2/H_∞ problem

Given a H_∞ level *γ*, find an admissible state feedback controller gain *K* stabilizes the closed-loop system with satisfying

$$\min \ \|T_{z_2 v}\|_2 \quad \text{subject to} \quad \|T_{z_\infty v}\|_\infty < \gamma \tag{10}$$

Lemma 1 [13]: Given matrices *Y, E* and *F* of appropriate dimensions where *Y* is symmetrical and $\Delta^T(t)\Delta(t) \le I$. Then

$$Y + E\Delta(t)F + F^T\Delta^T(t)E^T < 0 \tag{11}$$

holds if and only if there exists a scalar $\varepsilon > 0$ such that

$$Y + \varepsilon EE^T + \varepsilon^{-1}F^TF < 0 \tag{12}$$

Lemma 2: (H_2/H_∞ performance) [14] For a given positive scalar γ, if there exist symmetric positive definite matrices *P, Q* and state-feedback controller gain *K* satisfying the optimization problem: minimize$[tr(Q)]$ subject to

$$\begin{bmatrix} (A+BK)^TP + P(A+BK) & PG & (C_\infty + D_\infty K)^T \\ G^TP & -I & 0 \\ C_\infty + D_\infty K & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{13}$$

$$(A+BK)^T P+P(A+BK)+(C_2+D_2K)^T(C_2+D_2K)<0 \quad (14)$$

$$\begin{bmatrix} Q & G^T P \\ PG & P \end{bmatrix}>0 \quad (15)$$

The state-feedback controller satisfies the control objective described in Definition 1.

In Lemma 2, the common Lyapunov function matrices P and Q are used to solve the mixed H_2/H_∞ optimization problem. Hence, this method enlarges the conservativeness of the design procedure and cannot provide good solutions. In what follows, we will develop an ILMI method to obtain the mixed H_2/H_∞ controller gain with less conservativeness.

Theorem 1: For given positive scalars γ_L and λ_L, if there exist symmetric positive definite matrices $P_{\infty L}$, P_{2L}, Q_L, positive scalars ε_1, ε_2, ε_3 and controller gain K satisfying the optimization problem: minimize$[tr(Q_0)]$ subject to

$$\begin{bmatrix} (\bar A+\bar B\omega_L K)^T P_{\infty L}+P_{\infty L}(\bar A+\bar B\omega_L K)+\varepsilon_1^{-1}\bar F_a^T\bar F_a & * & * & * & * & * & * \\ \bar G^T P & -I+\varepsilon_3^{-1}\bar F_g^T\bar F_g & * & * & * & * & * \\ C_\infty+D_\infty\omega_L K & 0 & -\gamma_L^2 I & * & * & * & * \\ \bar E_a^T P_{\infty L} & 0 & 0 & -\varepsilon_1^{-1}I & * & * & * \\ \bar E_b^T P_{\infty L} & 0 & 0 & 0 & -\varepsilon_2^{-1}I & * & * \\ \bar F_b\omega_L K & 0 & 0 & 0 & 0 & -\varepsilon_2 I & * \\ \bar E_g^T P_{\infty L} & 0 & 0 & 0 & 0 & 0 & -\varepsilon_3^{-1}I \end{bmatrix}<0 \quad (16)$$

$$\begin{bmatrix} (\bar A+\bar B\omega_L K)^T P_{2L}+P_{2L}(\bar A+\bar B\omega_L K)+\varepsilon_1^{-1}\bar F_a^T\bar F_a & * & * & * & * \\ C_2+D_2\omega_L K & -I & * & * & * \\ \bar E_a^T P_{2L} & 0 & -\varepsilon_1^{-1}I & * & * \\ \bar E_b^T P_{2L} & 0 & 0 & -\varepsilon_2^{-1}I & * \\ \bar F_b\omega_L K & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix}<0 \quad (17)$$

$$\begin{bmatrix} Q_L+\varepsilon_3^{-1}\bar F_g^T\bar F & * & * \\ P_{2L}\bar G & P_{2L} & * \\ 0 & \bar E_g^T P_{2L} & -\varepsilon_3^{-1}I \end{bmatrix}>0 \quad (18)$$

$$tr(Q_L)<\lambda_L^2 \quad (19)$$

The obtained controller satisfies the control objective described in Definition 1.

Proof: Inequalities (17), (18) and (19) are equivalent to:

$$\begin{bmatrix} (\bar A+\bar B\omega_L K)^T P_{\infty L}+P_{\infty L}(\bar A+\bar B\omega_L K) & * & * \\ \bar G^T P_{\infty L} & -I & * \\ C_\infty+D_\infty\omega_L K & 0 & -\gamma_L^2 I \end{bmatrix}$$
$$+\varepsilon_1\begin{bmatrix} P_{\infty L}\bar E_a \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \bar E_a^T P_{\infty L} & 0 & 0 \end{bmatrix}+\varepsilon_1^{-1}\begin{bmatrix} \bar F_a^T \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \bar F_a & 0 & 0 \end{bmatrix}$$
$$+\varepsilon_2\begin{bmatrix} P_{\infty L}\bar E_b \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \bar E_b^T P_{\infty L} & 0 & 0 \end{bmatrix}+\varepsilon_2^{-1}\begin{bmatrix} K^T\omega_L\bar F_b^T \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \bar F_b\omega_L K & 0 & 0 \end{bmatrix}$$
$$+\varepsilon_3\begin{bmatrix} P_{\infty L}\bar E_g \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} \bar E_g^T P_{\infty L} & 0 & 0 \end{bmatrix}+\varepsilon_3^{-1}\begin{bmatrix} 0 \\ \bar F_g^T \\ 0 \end{bmatrix}\begin{bmatrix} 0 & \bar F_g & 0 \end{bmatrix}<0 \quad (20)$$

$$\begin{bmatrix} (\bar A+\bar B\omega_L K)^T P_{2L}+P_{2L}(\bar A+\bar B\omega_L K) & * \\ C_2+D_2\omega_L K & -I \end{bmatrix}$$
$$+\varepsilon_1\begin{bmatrix} P_{2L}\bar E_a \\ 0 \end{bmatrix}\begin{bmatrix} \bar E_a^T P_{2L} & 0 \end{bmatrix}+\varepsilon_1^{-1}\begin{bmatrix} \bar F_a^T \\ 0 \end{bmatrix}\begin{bmatrix} \bar F_a & 0 \end{bmatrix} \quad (21)$$
$$+\varepsilon_2\begin{bmatrix} P_{2L}\bar E_b \\ 0 \end{bmatrix}\begin{bmatrix} \bar E_b^T P_{2L} & 0 \end{bmatrix}+\varepsilon_2^{-1}\begin{bmatrix} K^T\omega_L\bar F_b^T \\ 0 \end{bmatrix}\begin{bmatrix} \bar F_b\omega_L K & 0 \end{bmatrix}<0$$

$$\begin{bmatrix} Q_L & * \\ P_{2L}\bar G & P_{2L} \end{bmatrix}+\varepsilon_3\begin{bmatrix} 0 \\ P_{2L}\bar E_g \end{bmatrix}\begin{bmatrix} 0 & \bar E_g^T P_{2L} \end{bmatrix}$$
$$+\varepsilon_3^{-1}\begin{bmatrix} \bar F_g^T \\ 0 \end{bmatrix}\begin{bmatrix} \bar F_g & 0 \end{bmatrix}>0 \quad (22)$$

Then, the following expressions are derived from Lemma 1:

$$\begin{bmatrix} \{[\bar A+\bar E_a\bar\Delta(t)\bar F_a]+[\bar B+\bar E_b\bar\Delta(t)\bar F_b]\omega_L K\}^T P_{\infty L} \\ +P_{\infty L}\{[\bar A+\bar E_a\bar\Delta(t)\bar F_a]+[\bar B+\bar E_b\bar\Delta(t)\bar F_b]\omega_L K\} & * & * \\ [\bar G+\bar E_g\bar\Delta(t)\bar F_g]^T P_{\infty L} & -I & * \\ C_\infty+D_\infty\omega_L K & 0 & -\gamma_L^2 I \end{bmatrix}<0 \quad (23)$$

$$\begin{bmatrix} \{[\bar A+\bar E_a\bar\Delta(t)\bar F_a]+[\bar B+\bar E_b\bar\Delta(t)\bar F_b]\omega_L K\}^T P_{2L} \\ +P_{2L}\{[\bar A+\bar E_a\bar\Delta(t)\bar F_a]+[\bar B+\bar E_b\bar\Delta(t)\bar F_b]\omega_L K\} & * \\ C_2+D_2\omega_L K & -I \end{bmatrix}<0 \quad (24)$$

$$\begin{bmatrix} Q_L & * \\ P_{2L}[\bar G+\bar E_g\bar\Delta(t)\bar F_g] & P_{2L} \end{bmatrix}>0 \quad (25)$$

Based on Lemma 2, it is obvious that the controller gain K satisfies the control objective described in Definition 1. This completes the proof. ■

Remark 1: An optimal mixed H_2/H_∞ controller can be achieved via formulation minimize$[tr(Q)]$ in Lemma 2. Theorem 1 calculate the optimal solution in normal case and sub-optimal solution in fault cases, i.e. minimize$[tr(Q_0)]$ and guarantee the upper bound of H_2 and H_∞ performance index are smaller than $\lambda_L (L = 1, \ldots, l_p)$ and $\gamma_L (L = 0, 1, \ldots, l_p)$.

Remark 2: Theorem 1 gives a sufficient condition that guarantees the uncertain closed-loop augmented system is robustly stable, and satisfies the mixed H_2 / H_∞ constraint in both normal and fault cases. The presumed actuator faults are introduced into matrix inequalities through the different Lyapunov function variables $P_{\infty L}$ and P_{2L}. Therefore, the conservativeness of Theorem 1 is smaller than that of Lemma 2. Nevertheless, the matrix inequalities in Theorem 1 are not jointly convex. To solve this difficulty, some important auxiliary variables are introduced into the following theorem, which separate the Lyapunov function variables from the controller gain variable.

Theorem 2: For given positive scalars γ_L and λ_L as well as initial controller gain K_0, initial Lyapunov function variables $P_{\infty L0}$ and P_{2L0}, if there exist symmetric positive definite matrices $P_{\infty L}$, P_{2L}, Q_L, positive scalars ε_1, ε_2, ε_3 and controller gain K satisfying the optimization problem: minimize$[tr(Q_0)]$ subject to (18), (19) and

$$\begin{bmatrix} N_{\infty L} & * & * & * & * & * & * & * \\ \bar{B}^T P_{\infty L} + \omega_L K & -I & * & * & * & * & * & * \\ \bar{G}^T P & 0 & -I + \varepsilon_3^{-1} \bar{F}_g^T \bar{F}_g & * & * & * & * & * \\ C_\infty + D_\infty \omega_L K & 0 & 0 & -\gamma_L^2 I & * & * & * & * \\ \bar{E}_a^T P_{\infty L} & 0 & 0 & 0 & -\varepsilon_1^{-1} I & * & * & * \\ \bar{E}_b^T P_{\infty L} & 0 & 0 & 0 & 0 & -\varepsilon_2^{-1} I & * & * \\ \bar{F}_b \omega_L K & 0 & 0 & 0 & 0 & 0 & -\varepsilon_2 I & * \\ \bar{E}_g^T P_{\infty L} & 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon_3^{-1} I \end{bmatrix} < 0 \quad (26)$$

$$\begin{bmatrix} N_{2L} & * & * & * & * & * \\ \bar{B}^T P_{2L} + \omega_L K & -I & * & * & * & * \\ C_2 + D_2 \omega_L K & 0 & -I & * & * & * \\ \bar{E}_a^T P_{2L} & 0 & 0 & -\varepsilon_1^{-1} I & * & * \\ \bar{E}_b^T P_{2L} & 0 & 0 & 0 & -\varepsilon_2^{-1} I & * \\ \bar{F}_b \omega_L K & 0 & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0 \quad (27)$$

where

$$\begin{aligned} N_{\infty L} = {} & \bar{A}^T P_{\infty L} + P_{\infty L} \bar{A} + \varepsilon_1^{-1} \bar{F}_a^T \bar{F}_a \\ & - P_{\infty L} \bar{B}\bar{B}^T P_{\infty L0} - P_{\infty L0} \bar{B}\bar{B}^T P_{\infty L} + P_{\infty L0} \bar{B}\bar{B}^T P_{\infty L0} \\ & - K^T \omega_L \omega_L K_0 - K_0^T \omega_L \omega_L K + K_0^T \omega_L \omega_L K_0 \end{aligned}$$

$$\begin{aligned} N_{2L} = {} & \bar{A}^T P_{2L} + P_{2L} \bar{A} + \varepsilon_1^{-1} \bar{F}_a^T \bar{F}_a \\ & - P_{2L} \bar{B}\bar{B}^T P_{2L0} - P_{2L0} \bar{B}\bar{B}^T P_{2L} + P_{2L0} \bar{B}\bar{B}^T P_{2L0} \\ & - K^T \omega_L \omega_L K_0 - K_0^T \omega_L \omega_L K + K_0^T \omega_L \omega_L K_0 \end{aligned}$$

The obtained state feedback controller guarantees that the uncertain closed-loop augmented system (9) is robustly stable and satisfies the mixed H_2 / H_∞ constraint in both normal and fault cases.

Proof: Inequalities (26) and (27) are equivalent to

$$(16) + \begin{bmatrix} (P_{\infty L} - P_{\infty L0})\bar{B} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} (P_{\infty L} - P_{\infty L0})\bar{B} \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} (K - K_0)\omega_L \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} (K - K_0)\omega_L \\ 0 \\ 0 \end{bmatrix}^T < 0 \quad (28)$$

$$(17) + \begin{bmatrix} (P_{2L} - P_{2L0})\bar{B} \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} (P_{2L} - P_{2L0})\bar{B} \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} (K - K_0)\omega_L \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} (K - K_0)\omega_L \\ 0 \\ 0 \end{bmatrix}^T < 0 \quad (29)$$

respectively. It is obvious that the formulations (16) and (17) are satisfied if the formulations (28) and (29) are satisfied. The subsequent proof can be readily obtained via Theorem 1, and so is omitted here for brevity. ■

Remark 3: The matrix inequalities (26) and (27) can be transformed into LMIs by giving the initial controller gain K_0 and initial Lyapunov function variables $P_{\infty L0}$ and P_{2L0}. The conservativeness of this transformation lies in the differences between $K - K_0$, $P_{\infty L} - P_{\infty L0}$ and $P_{2L} - P_{2L0}$. Thus, the following iterative algorithm will be developed to minimize the proposed conservativeness.

Algorithm 1:

Step 1 Select some small scalars $\gamma_L > 0$ and $\lambda_L > 0$, then obtain the initial controller gain $K_{opt}^0 = Z_{opt} X_{opt}^{-1}$ via the optimization problem: minimize$[tr(W_0)]$ subject to $X > 0$, $W_L > 0$ and LMIs (19),

$$\begin{bmatrix} X\bar{A}^T + \bar{A}X + Z^T \bar{B}^T + \bar{B}Z + \varepsilon_1 \bar{E}_a^T \bar{E}_a + \varepsilon_2 \bar{E}_b^T \bar{E}_b + \varepsilon_3 \bar{E}_g^T \bar{E}_g & * & * & * & * \\ & -I + \varepsilon_3^{-1} \bar{F}_g^T \bar{F}_g & * & * & * \\ C_\infty X + D_\infty \omega_L Z & 0 & -\gamma_L^2 I & * & * \\ \bar{F}_a X & 0 & 0 & -\varepsilon_1 I & * \\ \bar{F}_b \omega_L Z & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0 \quad (30)$$

$$\begin{bmatrix} W_L & * \\ X C_2^T + Z^T \omega_L D_2 & X \end{bmatrix} > 0 \quad (31)$$

$$tr(W_L) < \lambda_L^2 \quad (32)$$

Step 2 Let $K = K_{opt}^0$, minimize$[tr(Q_0)]$ subject to $P_{\infty L} > 0$, $P_{2L} > 0$, $Q_L > 0$ and the LMI constrains described in Theorem 1, then we get the initial Lyapunov function variables $P_{\infty L}^0$ and P_{2L}^0.

Step 3 At the ith iteration ($i > 0$), let $P_{\infty L0}^i = P_{\infty L}^{i-1}$,

$P_{2L0}^i = P_{2L}^{i-1}$ and $K_0 = K_{opt}^{i-1}$, minimize$[tr(Q_0)]$ subject to $P_{\infty L}^i > 0$, $P_{2L}^i > 0$, $Q_L^i > 0$ and the LMI constrains described in Theorem 2, then we get the Lyapunov function variables $P_{\infty L}^i$ and P_{2L}^i as well as controller gain K_{opt}^i.

Step 4 If $\left|tr(P_{20}^i - P_{20}^{i-1})\right| < \delta$ where δ is a given error tolerance, the calculated $K = K_{opt}^i$ is the optimal mixed H_2/H_∞ controller gain, stop. Otherwise, let $i = i+1$ and return to Step 3.

Remark4: In Algorithm 1 and 2, the sequence $tr(P_{20}^{(i)})_{i=1}^{\infty}$ is convergent since $P_{20}^{(i-1)}$ is a feasible solution and $P_{20}^{(i)}$ is an optimal solution for the *i*th iteration in Step 3). Hence, Algorithm 1 and 2 must be convergent.

IV. Flight Control Example

In this section, the simulation results of flight tracking control for nonlinear ADMIRE aircraft model are presented to demonstrate the advantage of the proposed method.

The ADMIRE model describes a single seated, single engine small fighter aircraft with a delta-canard configuration [15]. The linear aircraft model is described by expression (1) in this paper, where $x(t) = [\alpha\ \beta\ p\ q\ r]^T$ is the state, $u(t) = [\delta_{rc}\ \delta_{lc}\ \delta_{roe}\ \delta_{rie}\ \delta_{lie}\ \delta_{loe}\ \delta_r]^T$ is the control surface deflection and *W(t)* is disturbance. The tracking signal is $r(t) = [\alpha\ \beta\ p]^T$ since these variables are close to the maneuver ability of aircraft. For the considered flight case,

$$A=\begin{bmatrix} -0.5432 & 0.0137 & 0 & 0.9778 & 0 \\ 0 & -0.1179 & 0.2215 & 0 & -0.9661 \\ 0 & -10.5130 & -0.9967 & 0 & 0.6176 \\ 2.6221 & -0.0030 & 0 & -0.5057 & 0 \\ 0 & 0.7075 & -0.0939 & 0 & -0.2127 \end{bmatrix},$$

$$B=\begin{bmatrix} 0.0035 & 0.0035 & -0.0318 & -0.0548 & -0.0548 & -0.0318 & 0.0004 \\ -0.0063 & 0.0063 & 0.0024 & 0.0095 & -0.0095 & 0.0024 & 0.0287 \\ 0.6013 & -0.6013 & -2.2849 & -1.9574 & 1.9574 & 2.2849 & 1.4871 \\ 0.8266 & 0.8266 & -0.4628 & -0.8107 & -0.8107 & -0.4628 & 0.0024 \\ -0.2615 & 0.2615 & -0.0944 & -0.1861 & 0.1861 & 0.0944 & -0.8823 \end{bmatrix},$$

$$G=[0.00541\ \ 0\ \ 0\ \ 0.00682\ \ 0]^T.$$

We introduce seven kinds of actuator faults into the design procedure of the mixed H_2/H_∞ controller. Considering the worst situation for actuator failures, assume that the actuator is total outage, namely, the effectiveness factor of the actuator decreases to zero. Thus, every fault holds the following condition: one effectiveness factor $\omega_{Li} = 0$ and other six effectiveness factors $\omega_{Lj} = 1,\ j = 1,\ 2\cdots,7, j \neq i$.

Select the following weighting matrices

$$C_\infty = \begin{bmatrix} diag(4,\ 4,\ 4,\ 3,\ 3,\ 2) & 0_{6\times2} \\ 0_{7\times6} & 0_{7\times2} \end{bmatrix}, D_\infty = \begin{bmatrix} 0_{6\times7} \\ 3*I_{7\times7} \end{bmatrix},$$

$$C_2 = \begin{bmatrix} I_{3\times3} & 0_{3\times5} \\ 0_{7\times3} & 0_{7\times5} \end{bmatrix}, D_2 = \begin{bmatrix} 0_{3\times7} \\ 0.15*I_{5\times5} \end{bmatrix}.$$

Then, the mixed H_2/H_∞ controller K_{ILMI} can be obtained through Algorithm 1. For comparison purposes, the standard mixed H_2/H_∞ controller K_{SLMI} and the standard LQR controller K_{LQR} are also achieved, respectively.

$$K_{ILMI} = \begin{bmatrix} 5.2005 & 10.0045 & -0.7048 & -5.0724 & -8.9905 & -0.5272 & -1.9465 & 7.2874 \\ 5.2008 & -10.0345 & 0.6919 & -5.0788 & 8.9548 & 0.5260 & -1.9600 & -7.2816 \\ -3.4769 & -1.7279 & -5.7080 & 3.0718 & -0.1784 & 1.3764 & 1.2158 & 0.0990 \\ -5.0835 & 2.4920 & -5.4232 & 4.3884 & -3.7686 & 0.9878 & 1.7219 & 2.8513 \\ -5.0917 & -2.4795 & 5.4319 & 4.3938 & 3.8122 & -0.9893 & 1.7271 & -2.8648 \\ -3.4604 & 1.7425 & 5.7043 & 3.0583 & 0.2019 & -1.3747 & 1.2128 & -0.1013 \\ 0.0604 & 14.0909 & -0.5367 & -0.0429 & -12.2743 & -0.6893 & -0.0220 & 9.2019 \end{bmatrix}$$

Figs. 1-3 show the comparisons of the three proposed controllers using the nonlinear aircraft model. Considering the convenience of comparisons, we assume that all of the actuator faults occur at 40 seconds. In Fig. 2, when the actuator of the right canard is a total outage, the transient behaviors of the controller K_{ILMI} are similar to that of the controllers K_{SLMI} and K_{LQR}. However, as the number of actuator failures increases, for example, the actuators of the right canard and left inner elevon lose simultaneously, it is easy to see that the controller K_{ILMI} results in superior tracking performance than the controller K_{SLMI}, and the controller K_{LQR} from Fig. 3. Furthermore, in Fig. 4, when the actuators of right and left canards and left inner elevon are total outage simultaneously, the controller K_{LQR} cannot stabilize the aircraft and controller K_{SLMI} has unacceptable peak value and overshoot. The controller K_{ILMI} obtained by our method just suffers from slight performance degradation. In summary, the controller K_{ILMI} yields better performance than other controllers in the event of actuator faults, without any sacrifices of performance in fault free case.

V. Conclusion

In this paper, we have proposed a mixed H_2/H_∞ robust fault-tolerant controller design method to a class of uncertain systems. Based on the concept of auxiliary variable, a sufficient condition is derived to guarantee that the closed-loop system is robustly stable and satisfies the mixed H_2/H_∞ constraint in both normal and fault cases. An ILMI algorithm developed from the sufficient condition yields less conservativeness than the previous methods. Nonlinear simulations are also presented to illustrate the advantage of the proposed method.

There remains a future work to discuss rigorously about the case: Dynamic Output Feedback (DOF) control. Moreover, the obtained results in this paper may be extended to polytopic uncertain systems due to the decoupling of Lyapunov function variables and controller gain.

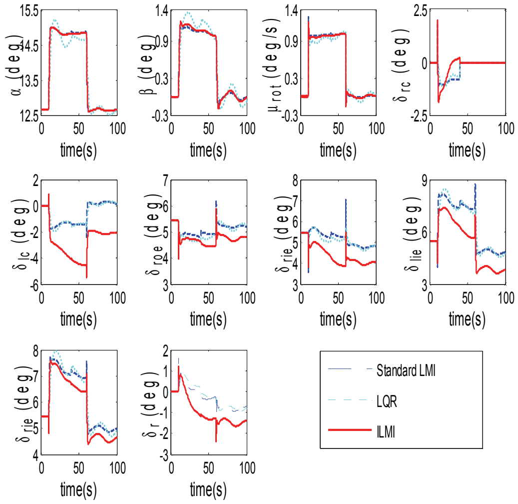

Fig. 1. Nonlinear simulation in the fault case: right canard outage.

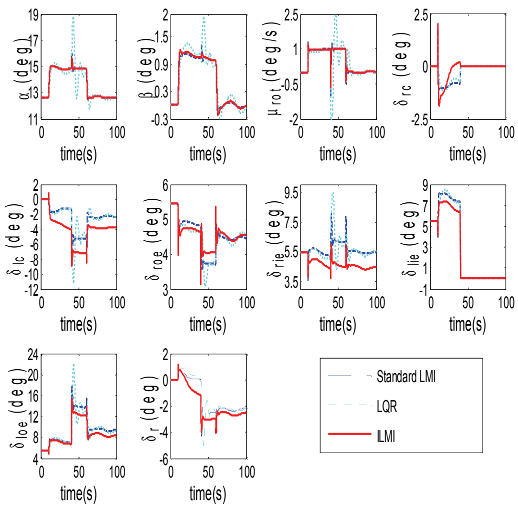

Fig. 2. Nonlinear simulation in the fault case: right canard and left inner elevon outages.

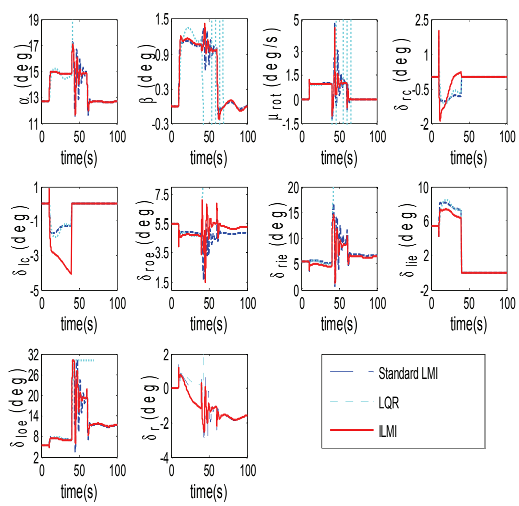

Fig. 3. Nonlinear simulation in the fault case: right and left canards and left inner elevon outages.

REFERENCES

[1] F. Liao, J. L. Wang, and G. H. Yang, "Reliable robust flight tracking control: An LMI approach," *IEEE Trans. Control Systems Technology*, vol. 10, no. 1, pp. 76–89, 2002.

[2] G. H. Yang, J. L. Wang, and Y. C. Soh, "Reliable H_∞ controller design for linear systems," *Automatica,* vol. 37, pp. 717–725, 2001

[3] F. Liao, J. L. Wang, E. K. Poh, and D. Li, "Fault-tolerant robust automatic landing control design," *Journal of Guidance, Control and Dynamics*, vol. 28, no. 5, pp. 854-871, 2005.

[4] D. S. Bernstein and W. M.Haddad, "LQG control with an H_∞ performance bound: A Riccati equation approach," *IEEE Trans. Automat. Control*, vol. 34, no. 3, pp. 293-305, 1989.

[5] X. Chen and K. M. Zhou, "Multiobjective H_2 and H_∞ control design," *SIAM J. Control Optim.*, vol. 40, no. 2, pp. 628–660, 2001.

[6] M. D. S. Aliyu, "Robust mixed H_2 / H_∞ control for linear systems with norm-bounded time-varying uncertainty," in *Proc. Amer. Control Conf.,* San Diego, California, pp. 3367-3371, 1999.

[7] F. Leibfritz, "An LMI-based algorithm for designing suboptimal static H_2 / H_∞ output feedback controllers," *SIAM J. Control Optim.*, vol. 39, no. 6, pp. 1711-1735, 2001.

[8] C. Scherer, P. Gahient, and M. Chilali, "Multiobjective output-feedback control via LMI optimization," *IEEE Trans. Automat. Control,* vol. 42, no. 7, pp. 896–911, 1997.

[9] T. Shimomura and T. Fujii, "An iterative method for mixed H_2 / H_∞ control design with uncommon LMI solutions," in *Proc. Amer. Control Conf.,* San Diego, California, pp. 3292-3296, 1999.

[10] D. Arzelier and D. Peaucelle, "An iterative method for mixed H_2 / H_∞ synthesis via static output-feedback," in *Proc. IEEE conf. on Decision and Control,* Las Vegas, Nevada USA, pp. 3464-3469, 2002.

[11] Y. Y. Cao, J. Lam, and Y. X. Sun, "Static output feedback stabilization: An LMI approach," *Automatica*, vol. 34, no. 12, pp. 1641-1645, 1998.

[12] K. M. Zhou, J. C. Doyle, and K. Glover, *Robust and optimal control*. Englewood Cliffs, NJ: Prentice-Hall, 1996.

[13] L. H. Xie and C. E. de Souza, "Robust H_∞ control for linear systems with norm-bounded time-varying uncertainty," *IEEE Trans. Automat. Control*, vol. 37, no. 8, pp. 1188-1191, 1992.

[14] S. P. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory*, SIAM Stud. Appl. Math. 15, SIAM, Philadelphia, 1994.

[15] L. Forssell and U. Nilsson, "ADMIRE, the aero-data model in a research environment version 4.0, model description," FOI, Stockholm, Sweden, Technical Rep. FOI-R–1624–SE, 2005.