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Global Feedback Stabilization of Quantum Noiseless Subsystems

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***Abstract*— In this paper, we investigate the state preparation problem of quantum noiselsss subsystems for the quantum Markovian systems via quantum feedback control. The controlled dynamics we consider are given by the so-called stochastic master equation including the coupling terms with the environment. We formulate the problem as a stochastic stabilization problem of an invariant set. This formulation allows us to utilize the stochastic Lyapunov technique and derive a globally stabilizing controller. The effectiveness of this method is evaluated by applying it to the 3-qubit systems subject to the collective noise.**

***Index Terms*— Quantum noiseless subsystems, Stochastic master equation, Stochastic stabilization problem**

I. INTRODUCTION

For realizing the quantum information processing, one must protect the information encoded in the system against errors and decoherence due to the interaction of the system and its surrounding environment. One of the most general protection methods is the noiseless subsystem encoding [2], [3]. In order for quantum information to be isolated from the noisy environment, one represents the information not directly by the total quantum system, but by the properly chosen quantum subsystem. If completely noise-protected subsystem exists, it is called noiseless subsystem. In this paper, we consider Quantum Information Processing (QIP) with the noiseless subsystem being the information carrier.

In general, it is necessary for implementing QIP protocols that one can control the system into an intended pure state and manipulate its state universally [5]. Similarly, QIP with noiseless subsystems requires control methodologies to manipulate the subsystem state [9], [12]. However, this topic has not been tackled intensively yet by the control theoretic way in contrast to the control of the physical degrees of freedom such as spin systems [4], [6], [10]. In this paper, we consider the state preparation problem for the noiseless subsystem of the quantum Markovian systems via quantum feedback control, where the controlled dynamics are given by the stochastic master equation. In particular, we formulate the control problem as a stochastic stabilization problem of an invariant set.

While state preparation problems for spin systems have been intensively studied, the problem here is not trivial because of the complexity of the dynamics including the interaction with the environment. The underlying idea of the main result is that *the noiseless subsystem can be regarded as an independent noise-free quantum system*. Such a viewpoint makes the apparently complex problem relatively simple and allows us to derive a globally stabilizing controller by the similar discussion to the control problem for the spin systems.

This paper is organised as follows. In Section II, we introduce the notion of noiseless subsystems and the definition of the stochastic stability used in this paper. Section III is the main part of this paper: we first give the controlled dynamics. Then, we state our control problem and provide its solution. Section IV is devote to the verification of the effectiveness of the main result, where we consider the control problem of 3-qubit systems. Section V concludes the paper.

We use the following notation: for matrices $A = (a_{ij})$ and B, the symbol A^* represents its conjugate transpose, i.e., $A^* = (a^*_{ji})$, and $[A, B] = AB - BA$. The symbol $\otimes$ stands for the tensor product or the Kronecker product, $\oplus$ represents the direct sum of vector spaces, $\mathrm{i} = \sqrt{-1}$ and $\mathbb{I}_\bullet$ is the identity operator on a Hilbert space $\mathcal{H}_\bullet$ (if $\bullet$ is a number, $\mathbb{I}_n$ represents the $n \times n$ identity matrix). The eigenstate of a matrix A corresponding to a eigenvalue λ is defined by $\rho_\lambda = v_\lambda v^*_\lambda$, where v_λ is a normalilzed eigenvector corresponding to λ.

II. PRELIMINARY

A. *Quantum Markovian Dynamics, Noiseless Subsystems*

Consider the system $\mathcal{I}$ defined on Hilbert space $\mathcal{H}_I = \mathbb{C}^n$ and provide its dynamics by the master equation

$$\dot{\rho}_t = -\mathrm{i}[H, \rho_t] + \sum_k \gamma_k \mathcal{D}(L_k, \rho_t), \tag{1}$$

where ρ_t is a density matrix of the system $\mathcal{I}$ at time t, i.e. an element of the set

$$\mathfrak{D} = \{\rho \in \mathbb{C}^{n\times n} : \rho = \rho^* \geq 0, \mathrm{tr}(\rho) = 1\}. \tag{2}$$

The Hermitian matrix H and the complex matrix L_k denote the effective Hamiltonian and the Lindblad operator, respectively. The superoperator $\mathcal{D}(c, \rho)$ is defined by $\mathcal{D}(c, \rho) = c\rho c^* - \frac{1}{2}(c^*c\rho + \rho c^*c)$. Assume that the second term in the right-hand side of (1) is zero. Then, the system dynamics are given by the Schrödinger equation

$$\dot{\rho}_t = -\mathrm{i}[H, \rho_t].$$

The density matrix at time t is represented by $\rho_t = U_t\rho_0U^*_t$ with an unitary matrix $U_t = \exp(-\mathrm{i}Ht)$. As a result, we can ensure that the purity of the system $\mathrm{tr}(\rho^2)$, which explains how the system is useful for QIP, is preserved:

$$\mathrm{tr}(\rho^2_t) = \mathrm{tr}(U_t\rho_0U^*_tU_t\rho_0U^*_t) = \mathrm{tr}(\rho^2_0).$$

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On the other hand, unless the second term in the right-hand side of (1) is zero, the purity monotonically decreases without exception: for $t \geq s$, $0 \leq \mathrm{tr}(\rho_t^2) \leq \mathrm{tr}(\rho_s^2) \leq 1$. Relaxation rate of the coupling is specified by positive constants γ_k. Here, the important thing is that the dissipation of the purity is an irreversible process. Hence, for the QIP purpose, we must use the physical system whose decoherence time is as long as possible, or consider the methodologies in order to avoid the dissipation.

One of the approaches for these requests is the noiseless subsystem encoding. While the purity of the system $\mathcal{I}$ dissipates due to the action of L_k, that of the subsystem is invariant for the time evolution if H and L_k safisfy some conditions.

Definition 1: [9] A quantum subsystem $\mathcal{S}$ of a system $\mathcal{I}$ is a quantum system whose state space is a tensor factor $\mathcal{H}_S$ of a subspace $\mathcal{H}_{SF}$ of $\mathcal{H}_I$,

$$\mathcal{H}_I = \mathcal{H}_{SF} \oplus \mathcal{H}_R = (\mathcal{H}_S \otimes \mathcal{H}_F) \oplus \mathcal{H}_R, \tag{3}$$

for some co-factor $\mathcal{H}_F$ and remainder space $\mathcal{H}_R$.

The condition for the purity of the subsystem to be invariant for the time evolution is given by the matrix representation of the Hamiltonian and the Lindblad operators. In order to prepare for introducing such conditions, we define a block decomposition of matrices acting on $\mathcal{H}_I$ as in [9]. Let $m = \dim(\mathcal{H}_S)$, $f = \dim(\mathcal{H}_F)$, $r = \dim(\mathcal{H}_R)$, and let $\{|\phi_j^S\rangle\}_{j=1}^m$, $\{|\phi_k^F\rangle\}_{k=1}^f$, $\{|\phi_l^R\rangle\}_{l=1}^r$ denote orthonormal bases for $\mathcal{H}_S$, $\mathcal{H}_F$, $\mathcal{H}_R$, respectively. The basis for $\mathcal{H}_I$ is determined by (3) as follows.

$$\{|\varphi_i\rangle\} = \{|\phi_j^S\rangle \otimes |\phi_k^F\rangle\}_{j,k=1}^{m,f} \cup \{|\phi_l^R\rangle\}_{l=1}^r$$

The block decomposition of matrices acting on $\mathcal{H}_I$ is naturally induced by the basis $\{|\varphi\rangle_i\}$:

$$X = \left(\begin{array}{c|c} X_{SF} & X_P \\ \hline X_Q & X_R \end{array}\right). \tag{4}$$

For the later discussion, we define the notation

$$\bar{X}_S := \mathrm{tr}_F(X_{SF}) \tag{5}$$

for a matrix X acting on $\mathcal{H}_I$. Here, $\mathrm{tr}_F(\cdot)$ is the partial trace, which is the unique linear operator defined by $\mathrm{tr}_F(X_S \otimes X_F) = X_S \,\mathrm{tr}(X_F)$ for matrices X_S and X_F acting on $\mathcal{H}_S$ and $\mathcal{H}_F$. Hence, it follows that $\mathrm{tr}(X_{SF}) = \mathrm{tr}(\bar{X}_S)$ [9]. In order to avoid the notational complexity, we standardize by $\mathrm{tr}(\cdot)$ the notation of the trace for matrices acting on each Hilbert space $\mathcal{H}_\bullet$.

With the definition of (4), (5) for $X = \rho$, $\bar{\rho}_S$ represents the (virtual) quantum state of the subsystem. However, as clear from the relation $\mathrm{tr}(\rho) = \mathrm{tr}(\bar{\rho}_S) + \mathrm{tr}(\rho_R) = 1$, $\bar{\rho}_S$ does not satisfy the definition of the density matrix unless $\mathrm{tr}(\rho_R) = 0$ (the density matrix $\bar{\rho}_S$ satisfies $\bar{\rho}_S \geq 0$ and $\mathrm{tr}(\bar{\rho}_S) = 1$). This motivates us the following definition.

Definition 2: The system $\mathcal{I}$ with the state ρ is said to be initialized in $\mathcal{H}_{SF}$ with the reduced state ρ_S if the blocks of the state ρ at $t = 0$ satisfy

(i) $\mathrm{tr}_F(\rho_{SF}) = \rho_S$,
(ii) $\rho_P = 0, \rho_R = 0$.

Then, under the following assumption, the purity $\mathrm{tr}(\bar{\rho}_S)$ is invariant, i.e., the subsystem $\mathcal{H}_S$ becomes noiseless.

Assumption 1: The effective Hamiltonian H and the Lindblad operators L_k satisfy

$$H = 0, \tag{6}$$

$$L_k = \left(\begin{array}{c|c} \mathbb{I}_S \otimes L_{F,k} & L_{P,k} \\ \hline 0 & L_{R,k} \end{array}\right), \tag{7}$$

$$\sum_k \gamma_k (\mathbb{I}_S \otimes L_{F,k}^*) L_{P,k} = 0. \tag{8}$$

Theorem 1: Let the system $\mathcal{I}$ evolve under the dynamics (1). Then, under Assumption 1, $\mathcal{S}$ is a *noiseless subsystem*: for arbitrarily given $\bar{\rho}_{S,0}$, the system $\mathcal{I}$ initialized in $\mathcal{H}_{SF}$ with reduced state $\bar{\rho}_{S,0}$ satisfies

$$\bar{\rho}_{S,t} = \bar{\rho}_{S,0}, \quad \forall t \geq 0. \tag{9}$$

Note that (9) does not hold in the case of $\mathrm{tr}(\bar{\rho}_{S,0}) = \mathrm{tr}(\rho_{SF,0}) \neq 1$, unless $L_{P,k} = 0$ for any k. Thus, if the initial state satisfies the conditions in Definition 2, we can create the noise-protected quantum information on the subsystem.

Remark 1: We here imposed the assumption of $H = 0$, so that the result (9), which is stronger than the invariance of the purity, is derived. For more general results with $H \neq 0$, see [9].

B. Stochastic Stability

As the controlled dynamics introduced in the next section have the stochastic behavior, we first define the stochastic stability.

Definition 3: Let x_t^z be a diffusion process on the metric space $\mathcal{X}$, started at $x_0 = z$, and let $\mathcal{M}$ denote an invariant set of the diffusion, i.e., for $z \in \mathcal{M}$, $x_t^z \in \mathcal{M}, \forall t \geq 0$. Then

1) the invariant set $\mathcal{M}$ is said to be stable in probability if

$$\lim_{\mathfrak{d}(z,\mathcal{M})\to 0} \mathbb{P}\left(\sup_{0\leq t<\infty} \mathfrak{d}(x_t^z, \mathcal{M}) \geq \varepsilon\right) = 0 \quad \forall \varepsilon > 0, \tag{10}$$

2) the invariant set $\mathcal{M}$ is globally stable if it is stable in probability and additionally

$$\mathbb{P}\left(\lim_{t\to\infty} \mathfrak{d}(x_t^z, \mathcal{M}) = 0\right) = 1 \quad \forall z \in \mathcal{X}, \tag{11}$$

where $\mathfrak{d}(x, \mathcal{M})$ denotes the distance from a point to a set and is defined by

$$\mathfrak{d}(x, \mathcal{M}) := \inf_{q\in\mathcal{M}} \|x - q\|. \tag{12}$$

The stochastic stability of the dynamics is probed by employing a stochastic version of the Lyapunov theorem and the LaSalle invariance theorem. Let $V(\cdot)$ be a $\mathbb{R}$-valued nonnegative continuous function on the metric space $\mathcal{X}$ and define a level set Q_m by $Q_m = \{x \in \mathcal{X} : V(x) < m\}$. Moreover, let $\tau_m = \inf\{t : x_t^z \notin Q_m\}$, $t \wedge s := \min\{t, s\}$ and define the infinitesimal operator $\mathscr{A}$ of the process x_s. Then, we obtain the following propositions.

Lemma 1: [8] Let $\mathscr{A}V(x) \leq 0$ in Q_m. If x_t^z is a right continuous strong Markov process defined untill at least some $\tau' > \tau_m$ a.s., then, for $z \in Q_m$ and $\lambda \leq m$, we have

$$\mathbb{P}\left(\sup_{0\leq t<\infty} V(x^z_{t\wedge\tau_m}) \geq \lambda\right) \leq \frac{V(z)}{\lambda}. \tag{13}$$

Proposition 1: Let $\mathscr{A}V(x) \leq 0$ in Q_m and x_t^z be a right continuous strong Markov process defined untill at least some $\tau' > \tau_m$ a.s.. If $V(x) = 0$ for $x \in \mathcal{M}$ and $V(x) \neq 0$ for $x \notin \mathcal{M}$, then $\mathcal{M}$ is stable in probability.

Proof: The statement is proved by using Lemma 1. ■

Proposition 2: [4] Let $\mathscr{A}V(x) \leq 0$ in Q_m. Suppose

1) Q_m has a compact closure,
2) $x^z_{t\wedge\tau_m}$ is Feller continuous,
3) $\mathbb{P}(\|x^z_{t\wedge\tau_m} - z\| > \varepsilon) \to 0$ as $t \to 0$, for any $\varepsilon > 0$, uniformly for $z \in Q_m$.

Then $x^z_{t\wedge\tau_m}$ converges in probability to the largest invariant set contained in $C_m = \{x \in Q_m : \mathscr{A}V(x) = 0\}$. Hence, x_t^z converges in probability to the largest invariant set contained in C_m for almost all paths which never leave Q_m.

III. FEEDBACK STABILIZATION PROBLEM

A. State Preparation via Quantum Feedback Control

In this section, we suppose Assumption 1 holds. In this paper, we investigate the state preparation problem of the noiseless subsystem introduced in the last section by the feedback control scheme as in [4], [6], [10]. In order to perform the feedback control, we first input a probe to the system whose dynamics are given by (1), and monitor the system's observable $M \in \mathbb{C}^{n\times n}$ continuously by homodyne detection. Further, we control the monitored system by the Hamiltonian F, where F is a Hermitian matrix. Then, the quantum state $\rho_t \in \mathfrak{D}$ conditioned on the measurement record $\{y_t\}$ is given by the following Itô type stochastic differential equation called *Stochastic Master Equation* (SME):

$$\begin{aligned} d\rho_t = &-\mathrm{i}[H,\rho_t]dt + \sum_{k=1}^{p}\gamma_k\mathcal{D}(L_k,\rho_t)dt \\ &-\mathrm{i}u_t[F,\rho_t]dt + \mathcal{D}(M,\rho_t)dt + \sqrt{\eta}\mathcal{G}(M,\rho_t)dW_t, \end{aligned} \tag{14}$$

where $u_t \in \mathbb{R}$ is a control input and $\eta \in (0,1]$ represents the detector efficiency. The Wiener process $W_t \in \mathbb{R}$ is the innovation and is associated with the measurement output y_t as follows: $dW_t = dy_t - \sqrt{\eta}\,\mathrm{tr}((M+M^*)\rho_t)dt$. For the derivation of the SME and its solution properties as a stochastic process, see [1] and [4], respectively.

The mathematical description of the problem in this paper is given as follows.

Problem 1: Let $M_S \in \mathbb{C}^{m\times m}$ be the Hermitian matrix in which the maximum eigenvalue $\lambda_{(1)}$ is nondegenerate, and $\bar{\rho}_{(1)}$ be the eigenstate of M_S corresponding to the eigenvalue $\lambda_{(1)}$. Further, under the dynamics (14) and Assumption 1. let

$$\Lambda_{(1)} = \{\rho \in \mathfrak{D} : \bar{\rho}_S = \bar{\rho}_{(1)}\} \tag{15}$$

be an invariant set. Then, find the control Hamiltonian F, the measurement operator M and the controller $u_t = u(\rho_t), u : \mathfrak{D} \to \mathbb{R}$ which globally stabilize the invariant set $\Lambda_{(1)}$. Moreover, show that $\mathbb{E}[\bar{\rho}_{S,t}] \to \bar{\rho}_{(1)}$ as $t \to \infty$.

B. Main Result

The stabilization problem of noise-free quantum systems has been studied intensively [4], [6], [10]. However, they are not directly applicable to Problem 1 for the following reasons:

- High dimensionality of the dynamics (14). To construct the (virtual) quantum system on the noiseless subsystem $\mathcal{H}_S = \mathbb{C}^2$, which is the lowest dimensional Hilbert space for representing the quantum system, we have to prepare at least 8 dimensional quantum system $\mathcal{I}$. For such high dimensional systems, it is not realistic to apply the numerical method proposed in [6].
- Effect of the noise. This changes the dynamical properties of the quantum systems significantly. As a result, it becomes difficult to apply the analytical method proposed in [4], [10] to our control problem.

Note here that we do not necessarily need to control the whole state of ρ_t. The central idea for the controller design of this paper is that we neglect the unconcerned part of ρ for the stabilization problem, i.e., the elements other than $\bar{\rho}_S$. That is, we design the control law based on the control Lyapunov function

$$V_1(\rho) = 1 - \mathrm{tr}(\bar{\rho}_S\bar{\rho}_{(1)}),$$

which is the natural distance between the system and the target state as in the existing result [4], [10].

To do this, we begin by specifying the class of the control Hamiltonian and the measurement operator. This is the natural extension of the spin control systems.

Assumption 2: 1) The measurement operator M is given by

$$M = \left(\begin{array}{c|c} M_S\otimes\mathbb{I}_F & 0 \\ \hline 0 & M_R \end{array}\right), \tag{16}$$

where $M_S \in \mathbb{C}^{m\times m}$ is the same matrix with that in Problem 1 and $M_R \in \mathbb{C}^{r\times r}$ is the Hermitian matrix whose eigenvalues are all less than $\lambda_{(1)}$.

2) The control Hamiltonian F satisfies the following conditions: there exist the constants $\beta, C \in \mathbb{R}$ such that all the eigenvalues of

$$-\mathrm{i}\beta F - M^2 - \frac{1}{2}\sum_k \gamma_k L_k^* L_k + CM \tag{17}$$

are nondegenerate, and that all the elements of $P^* v_{(1)}$ are nonzero, where $v_{(1)}$ is the eigenvector of M which corresponds to the eigenvalue $\lambda_{(1)}$ and P is the matrix consisting of all the eigenvalues of (17), i.e., $P = [u_1, \cdots, u_n]$ (each of $\{u_i\}_i$ is an eigenvector of (17)).

Remark 2: The assumption 1) is a necessary condition for Problem 1 in some sense. On the other hand, the assumption 2), which are used for the proof of Lemma 6, gurarantees that we can always draw the system out of the orthogonal space to $\Lambda_{(1)}$. Note that these assumptions are not very restrictive. In fact, 3-qubit systems suffering from the collective noise satisfy these assumptions; see Section IV. In the control problem of noise-free spin systems, these assumption are automatically satisfied.

We hereafter denote the eigenvalues of the measurement operator M by $\lambda_{(i)}, i = 1, 2, 3, \cdots$, where $\lambda_{(1)} > \lambda_{(2)} > \lambda_{(3)} > \cdots$, and the subset of $\mathfrak{D}$ spanned by the eigenstates corresponding to $\lambda_{(i)}$ by $\Lambda_{(i)}$. The following $\mathbb{R}$-valued non-negative continuous functions $V_1, V_2 : \mathfrak{D} \to \mathbb{R}$ are repeatedly used for the later discussion:

$$V_1(\rho) := 1 - \operatorname{tr}(\rho\rho_{(1)}), \tag{18}$$

$$V_2(\rho) := 1 - \operatorname{tr}(\rho\rho_{(1)})^2, \tag{19}$$

$$\rho_{(1)} := \left(\begin{array}{c|c} \bar{\rho}_{(1)} \otimes \mathbb{I}_F & 0 \\ \hline 0 & 0 \end{array}\right). \tag{20}$$

Further, we define the $\mathbb{R}$-valued continuous function $u_1 : \mathfrak{D} \to \mathbb{R}$,

$$u_1(\rho) = -\operatorname{tr}(\mathrm{i}[F, \rho]\rho_{(1)}).$$

The next theorem is the main result of this paper.

Theorem 2: Under Assumption 1, 2, suppose that the dynamics of the system $\mathcal{I}$ is given by (14). Then, for $\alpha, \beta > 0$ satisfying

$$\frac{\beta^2}{8\alpha\eta(\lambda_{(1)} - \lambda_{(2)})^2} < 1, \tag{21}$$

the control law

$$u_t = \alpha u_1(\rho_t) + \beta V_1(\rho_t) \tag{22}$$

globally stabilizes the invariant set $\Lambda_{(1)}$ and $\mathbb{E}[\bar{\rho}_{S,t}] \to \bar{\rho}_{(1)}$ as $t \to \infty$.

C. Proof of Theorem 2

For simplifying the notation, define the subsets of the state space $\mathfrak{D}$,

$$\mathfrak{D}_\gamma = \{\rho \in \mathfrak{D} : V_2(\rho) = \gamma\}, \tag{23}$$

$$\mathfrak{D}_{<\gamma} = \{\rho \in \mathfrak{D} : 0 \leq V_2(\rho) < \gamma\}, \tag{24}$$

$$\mathfrak{D}_{>\gamma} = \{\rho \in \mathfrak{D} : \gamma < V_2(\rho) \leq 1\}. \tag{25}$$

Here, note that the following holds.

Lemma 2: The set $\mathfrak{D}_0$ is equal to the target set $\Lambda_{(1)}$.

Proof: We omit the proof. ■

To prove Theorem 2, it is required to show the stochastic properties of the quantum dynamics. As in the reference [4], the following proposition holds.

Proposition 3: Let the control law u_t be given by $u_t = u(\rho_t)$ with $u \in \mathbb{C}^1(\mathfrak{D}, \mathbb{R})$ and denote by $\varphi_t(\rho, u)$ the solution of (14) whose initial state and control law are given by ρ and u. Then, the following properties hold.

1) For any $\varepsilon > 0$, $\mathbb{P}(\|\varphi_t(\rho, u) - \rho\| > \varepsilon) \to 0$ as $t \to 0$, uniformly for $\rho \in \mathfrak{D}$.
2) $\varphi_t(\rho, u)$ is Feller continuous: if $V : \mathfrak{D} \to \mathbb{R}$ is continuous, $\mathbb{E}[V(\varphi_t(\rho, u))]$ is continuous on ρ.
3) $\varphi_t(\rho, u)$ is a right continuous strong Markov process in $\mathfrak{D}$.

The proof of Theorem 2 consists of the following steps.

1. $\Lambda_{(1)}$ is stable in probability.
2. There exists $0 < \gamma < 1$ such that for almost all paths which never leave the set $\mathfrak{D}_{<1-\gamma^2}$, they converge into the invariant set $\Lambda_{(1)}$, i.e., $\mathfrak{d}(\rho_t, \Lambda_{(1)}) \to 0$ as $t \to \infty$.
3. For almost all paths there exist a finite time T, and after it they never leave $\mathfrak{D}_{<1-\gamma^2}$.

We hereafter denote the infinitesimal operator associated with the dynamics (14) by $\mathscr{A}$.

Step 1. The statement of Step 1. is proved by finding the Lyapunov function which satisfies the conditions of Proposition 1 for the invariant set $\Lambda_{(1)}$.

Lemma 3: With the control input (22), $\mathscr{A}V_2(\rho) \leq 0$ is satisfied for $\forall\rho \in \mathfrak{D}_{<1-(\gamma_0/2)^2}$, where

$$\frac{\gamma_0}{2} = \frac{\beta^2}{8\alpha\eta(\lambda_{(1)} - \lambda_{(2)})^2} < 1. \tag{26}$$

Proof: The direct calculation of $\mathscr{A}V_2(\rho_t)$ yields

$$\begin{aligned}\mathscr{A}V_2(\rho_t) = -2\operatorname{tr}(\rho_t\rho_{(1)})\{&u_t u_1(\rho_t) + \Phi(\rho_t) \\ &+2\eta(\lambda_{(1)} - \operatorname{tr}(M\rho_t))^2 \operatorname{tr}(\rho_t\rho_{(1)})\},\end{aligned} \tag{27}$$

$$\Phi(\rho) := \operatorname{tr}\left(\sum_k \gamma_k L_{k,P}\rho_R L_{k,P}^*(\bar{\rho}_{(1)} \otimes \mathbb{I}_F)\right).$$

Since $\operatorname{tr}(\rho_t\rho_{(1)})$ is nonnegative in $\mathfrak{D}$, to prove the statement of the lemma, it is sufficient to show the nonnegativity of the terms in the curly brackets in (27) for $\rho_t \in \mathfrak{D}_{<1-(\gamma_0/2)^2}$. By noting that the nonnegativity of density matrices leads to the nonnegativity of $\Phi(\rho_t)$, we observe that the statement above is proved by showing the nonnegativity of (28).

$$\begin{aligned}&u_t u_1(\rho_t) + 2\eta(\lambda_{(1)} - \operatorname{tr}(M\rho_t))^2 \operatorname{tr}(\rho_t\rho_{(1)}) \\ &= \alpha\left(u_1(\rho_t) + \frac{\beta}{\alpha}\frac{V_1(\rho_t)}{2}\right)^2 + \Psi(\rho_t)\end{aligned} \tag{28}$$

$$\Psi(\rho) := 2\eta(\lambda_{(1)} - \operatorname{tr}(M\rho))^2 \operatorname{tr}(\rho\rho_{(1)}) - \frac{\beta^2}{\alpha}\frac{V_1(\rho)^2}{4}$$

Let us show the nonnegativity of $\Psi(\rho)$ in $\mathfrak{D}_{<1-(\gamma_0/2)^2}$. To this end, choose a constant $\gamma \in (\gamma_0/2, 1]$ and suppose $\rho \in \mathfrak{D}_{1-\gamma^2}$ $(\subset \mathfrak{D}_{<1-(\gamma_0/2)^2})$. By noting $V_1(\rho) = 1 - \gamma$ and $\operatorname{tr}(\rho\rho_{(1)}) = \gamma$ from the definition of $\mathfrak{D}_{1-\gamma^2}$, we can see that $\Psi(\rho_t)$ is monotonically increasing with respect to $\lambda_{(1)} - \operatorname{tr}(M\rho_t) \geq 0$. Next, consider the spectral decomposition of the Hermitian matrix M. When we denote by $\{P_j^{\lambda_{(i)}}\}$ the set of the (mutually orthogonal) eigenstates corresponding to the eigenvalues $\lambda_{(i)}$, M is expressed as

$$M = \sum_{i,j} \lambda_{(i)} P_j^{\lambda_{(i)}}.$$

Moreover, by defining $p_i = \operatorname{tr}(\sum_j P_j\rho_t)$, it follows that $\operatorname{tr}(M\rho_t) = \sum_i \lambda_{(i)}p_i$. Here, note $\{p_i\}$ satisfies $\sum_i p_i = 1, p_i \geq 0$. This fact and $\lambda_{(1)} > \lambda_{(2)} > \lambda_{(3)} > \cdots$ implies that the minimum value of $\lambda_{(1)} - \operatorname{tr}(M\rho_t)$ in $\mathfrak{D}_{1-\gamma^2}$ is $\lambda_{(1)} - \gamma\lambda_{(1)} - (1-\gamma)\lambda_{(2)}$: the minimum value is given by $\rho = \gamma\rho^{(1)} + (1-\gamma)\rho^{(2)}$, $\rho^{(1)} \in \Lambda_{(1)}$, $\rho^{(2)} \in \Lambda_{(2)}$. Hence, we obtain the following inequality.

$$\begin{aligned}\Psi(\rho) &\geq 2\eta\left\{\lambda_{(1)} - \gamma\lambda_{(1)} - (1-\gamma)\lambda_{(2)}\right\}^2\gamma - \frac{\beta^2}{\alpha}\frac{V_1(\rho_t)^2}{4} \\ &\geq 2\eta(\lambda_{(1)} - \lambda_{(2)})^2(1-\gamma)^2\gamma - \frac{\beta^2}{\alpha}\frac{V_1(\rho_t)^2}{4} \\ &\geq 2\eta V_1(\rho_t)^2\left\{\gamma(\lambda_{(1)} - \lambda_{(2)})^2 - \frac{\beta^2}{8\eta\alpha}\right\}\end{aligned} \tag{29}$$

Under the condition (26) in $\mathfrak{D}_{<1-(\gamma_0/2)^2}$, we obtain $\Psi(\rho) \geq 0$. This completes the proof. ■

Step 2. We first prove the following lemma.

Lemma 4: For almost all paths which never exist $\mathfrak{D}_{<1-(\gamma_0/2)^2}$, ρ_t converges in probability into $\Lambda_{(1)}$ as $t \to \infty$.

Proof: Consider the Lyapunov function $V_2(\rho)$. From the discussion in Lemma 3, $\mathscr{A}V_2(\rho) \leq 0$ in $\mathfrak{D}_{<1-(\gamma_0/2)^2}$ and all the conditions of Proposition 2 are satisfied. Hence, by showing that the largest invariant set contained in $\{\rho \in \mathfrak{D}_{<1-(\gamma_0/2)^2} : \mathscr{A}V_2(\rho) = 0\}$ is equal to $\Lambda_{(1)}$, the statement is proved.

As the state space is restricted to $\mathfrak{D}_{<1-(\gamma_0/2)^2}$, $\mathrm{tr}(\rho\rho_{(1)}) > 0$ is obtained. In order to satisfy $\mathscr{A}V_2(\rho) = 0$, $\Psi(\rho) = 0$ is necessary from (27) and (28). This implies $V_1(\rho) = 0$ from (29). On the contrary, if $\lambda_{(1)} - \mathrm{tr}(M\rho) \neq 0$, then it follows that $V_1(\rho) \neq 0$. Therefore, we have

$$\lambda_{(1)} - \mathrm{tr}(M\rho) = 0. \tag{30}$$

We compute the set of the initial states in which $\mathrm{tr}(M\rho_t)$ is constant for $\forall t \geq 0$. Using Itô rule, we obtain

$$\begin{aligned} d\,\mathrm{tr}(M\rho_t) &= -\mathrm{i}u_t\,\mathrm{tr}(M[F,\rho_t])dt \\ &\quad + \sum_k \mathrm{tr}\left(M \cdot \gamma_k \mathcal{D}(L_k,\rho_t)\right)dt \\ &\quad + 2\sqrt{\eta}\{\mathrm{tr}(M^2\rho_t) - \mathrm{tr}(M\rho_t)^2\}dW_t. \end{aligned}$$

Hence, in order for $\mathrm{tr}(M\rho_t)$ to be constant for $\forall t \geq 0$, we must have

$$\mathrm{tr}(M^2\rho_0) - \mathrm{tr}(M\rho_0)^2 = 0. \tag{31}$$

By noting $\rho_t \in \mathfrak{D}_{<1-(\gamma_0/2)^2}$, the only subset in which the condition (31) is satisfied is $\Lambda_{(1)}$.

Further, we can verify that (30) holds in $\Lambda_{(1)}$. Thus, from the discussion above, the assertion is proved. ■

By using Lemma 4, the statement of Step 2. is proved by the same discussion in Lemma 4.9 in [4].

Step 3. We first prepare two lemmas.

Lemma 5: Suppose that the initial value of (14) is $\rho_0 \in \mathfrak{D}_{<1-\gamma_0^2}$. Then,

$$\mathbb{P}\left[\sup_{0\leq t<\infty} V_2(\rho_t) \geq \left(\frac{\gamma_0}{2}\right)^2\right] \leq 1 - p = \frac{1-\gamma_0^2}{1-(\gamma_0/2)^2} < 1. \tag{32}$$

Proof: From Lemma 1 and $\mathscr{A}V_2(\rho) \leq 0$ in $\mathfrak{D}_{<1-\gamma_0^2}$, the assertion is proved. ■

Lemma 6: Let $\tau_{\rho_0}(\mathfrak{D}_{>1-\gamma_0^2})$ be the first exit time of ρ_t from $\mathfrak{D}_{>1-\gamma_0^2}$ where ρ_0 denotes the initial value of ρ_t. Then,

$$\sup_{\rho_0 \in \mathfrak{D}_{>1-\gamma_0^2}} \mathbb{E}[\tau_{\rho_0}(\mathfrak{D}_{>1-\gamma_0^2})] < \infty. \tag{33}$$

Proof: We only sketch the proof. First, we prove that $\min_{t\in[0,T]} \mathbb{E}[V_2(\rho_t)] < 1 - \gamma_0^2$ for some $0 < T < \infty$. Then, by using Proposition 3.2 in [10], i.e.,

$$\mathbb{E}[\tau_{\rho_0}(\mathfrak{D}_{>1-\gamma_0^2})] \leq \frac{T}{1 - \sup_{\rho_0\in\mathfrak{D}} \mathbb{P}[\tau_{\rho_0}(\mathfrak{D}_{>1-\gamma_0^2}) > T]}$$

for all $T \geq 0$ and $\rho_0 \in \mathfrak{D}$, we can obtain (33).

The first claim is proved by the support theorem of Theorem 5.7.6 in [7] (or Proposition 3.1 in [10]). We first compute the stratonovich form of (14), and derive the corresponding deterministic equation by substituting the Wiener increment of the diffusion part with the increment of some piecewise smooth function ξ. By using this solution, we obtain

$$\begin{aligned} \frac{d}{dt}V_2(\rho_t) = 2\,\mathrm{tr}(\rho_t\rho_{(1)}) \times \Bigg\{ & iu_t\,\mathrm{tr}([F,\rho_t]\rho_{(1)}) - \sum_k \gamma_k\,\mathrm{tr}\left(\mathcal{D}(L_k,\rho_t)\rho_{(1)}\right) \\ & -4\lambda_{(1)}\eta\,\mathrm{tr}(\rho_t\rho_{(1)})\,\mathrm{tr}(M\rho_t) + 8\,\mathrm{tr}(M\rho_t)^2 + 4\lambda_{(1)}^2 \\ & -4\,\mathrm{tr}(M^2\rho_t) - 2\sqrt{\eta}\,\mathrm{tr}(\rho_t\rho_{(1)})(\lambda_{(1)} - \mathrm{tr}(M\rho_t))\xi_t \Bigg\}. \end{aligned}$$

Then, according to the support theorem, the closure $\{V_2(\rho_t) : \xi \in \Xi\}$ is equal to the support of $V_2(\rho_t)$ with respect to (14) in the finite time interval $[0,T]$, where Ξ is the set of all deterministic piecewise smooth functions. Hence, it is sufficient to show that the closure of $\{V_2(\rho_t) : \xi \in \Xi\}$ contains $[0,\gamma]$ when $V_2(\rho_0) = \gamma \leq 1$.

In order to prove this statement, we analyze the behavior of the deterministic function $V_2(\rho_t)$. That is, we show that there does not exist the invariant set of ρ_t except for $\mathfrak{D}_0$ when $\xi_t \neq 0$. Suppose $\xi_t \neq 0$, then we must at least have $\mathrm{tr}(\rho_t\rho_{(1)})(\lambda_{(1)} - \mathrm{tr}(M\rho_t))\xi_t = 0$ for $\dot{V}_2(\rho_t) = 0$. The state set under which $\mathrm{tr}(\rho_t\rho_{(1)})(\lambda_{(1)} - \mathrm{tr}(M\rho_t)) = 0$ is devided into three sets.

1) $\mathfrak{D}_0 = \{\rho \in \mathfrak{D} : \mathrm{tr}(\rho\rho_{(1)}) = 1\}$
2) $\{\rho \in \mathfrak{D} : \mathrm{tr}(M\rho) \neq \lambda_{(1)}\} \cap \mathfrak{D}_1$
3) $\{\rho \in \mathfrak{D} : \mathrm{tr}(M\rho) = \lambda_{(1)}\} \setminus \mathfrak{D}_0$

Clearly, $\mathfrak{D}_0$ is an invariant set of ρ_t. That the set 2) is not invariant is proved by the similar discussion with Lemma 4.3 and 4.4 in [4], where we use the conditions in Assumption 2. Finally, by investigating the value of $d\,\mathrm{tr}(M\rho_t)/dt$, it is shown that the set of 3) is not invariant. ■

Intuitively, Lemma 5 indicates that the sample paths never leave the set $\mathfrak{D}_{<1-(\gamma_0/2)^2}$ with nonzero probability if they once arrive $\mathfrak{D}_{<1-\gamma_0^2}$. On the other hand, Lemma 6 implies that all the sample paths starting from the outside of $\mathfrak{D}_{<1-\gamma_0^2}$ arrive $\mathfrak{D}_{<1-\gamma_0^2}$ in a finite time. Thus, by combining these two lemmas, we obtain the following.

Lemma 7: For almost all paths there exist a finite time T, and after T they never exit the set $\mathfrak{D}_{<1-(\gamma_0/2)^2}$.

Proof: The statement is proved by the similar discussion in Lemma 4.10 in [4]. ■

Proof of Theorem 2 The stability in probability follows Step 1., and the a.s. convergence to the target set is proved by unifying the remaing steps. Then, it follows that

$$\mathbb{E}\left[\lim_{t\to\infty} V_1(\rho_t)\right] = 0.$$

Since $V_1(\rho)$ is uniformly bounded, we obtain

$$V_1\left(\lim_{t\to\infty}\mathbb{E}[\rho_t]\right) = \lim_{t\to\infty}\mathbb{E}[V_1(\rho_t)] = \mathbb{E}\left[\lim_{t\to\infty} V_1(\rho_t)\right] = 0$$

by dominated convergence and linearity and continuity of $V_1(\rho)$. Hence, we obtain that $\mathbb{E}[\bar{\rho}_{S,t}] \to \bar{\rho}_{(1)}$ as $t \to \infty$.

IV. CONTROL OF 3-QUBIT SYSTEMS

As an application, we consider the control of 3-qubit system subject to the collective noise, i.e., the system Hilbert space is given by $\mathcal{H}_I = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and the Lindblad operator is given by

$$L_k = \sum_{j=1}^{3} \mathbb{I}_2^{(1)} \otimes \sigma_k^{(j)} \otimes \mathbb{I}_2^{(3)} \quad k = x, y, z, \tag{34}$$

where σ_k denotes the Pauli matrix, i.e.,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, from [11], when we choose the bases of $\mathcal{H}_S$ and $\mathcal{H}_F$ as satisfying

$$|\phi_1^S\rangle \otimes |\phi_1^F\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)|0\rangle, \tag{35}$$

$$|\phi_2^S\rangle \otimes |\phi_1^F\rangle = \frac{1}{\sqrt{6}}(2|001\rangle - |010\rangle - |100\rangle), \tag{36}$$

$$|\phi_1^S\rangle \otimes |\phi_2^F\rangle = \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)|1\rangle, \tag{37}$$

$$|\phi_2^S\rangle \otimes |\phi_2^F\rangle = \frac{1}{\sqrt{6}}(2|110\rangle - |101\rangle - |011\rangle), \tag{38}$$

the subsystem $\mathcal{S}$ is noiseless, where $|0\rangle = (1,0)^\mathsf{T}, |1\rangle = (0,1)^\mathsf{T}$ are the standard basis of $\mathbb{C}^2$ and $|i\rangle \otimes |j\rangle \otimes |k\rangle$ is simply denoted by $|ijk\rangle$. Further, we determine the bases of $\mathcal{H}_R$ as follows:

$$|\phi_1^R\rangle = |000\rangle, \tag{39}$$

$$|\phi_2^R\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle), \tag{40}$$

$$|\phi_3^R\rangle = \frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle), \tag{41}$$

$$|\phi_4^R\rangle = |111\rangle. \tag{42}$$

By transforming the base consisting of (35)-(42) to the standard basis $\{e_i\}_{i=1}^8$, respectively, we can confirm that L_k, $k = x, y, z$ satisfy Assumption 1. In the following, we discuss on the latter basis.

Consider to make the state of the subsystem $\mathcal{H}_S$, $\bar{\rho}_S$, converge to $|\phi_1^S\rangle\langle\phi_1^S|$. That is, the control objective is to globally stabilize the set $\{\rho \in \mathfrak{D} : \bar{\rho}_S = \mathrm{diag}\{1, 0\} \equiv \bar{\rho}_{(1)}\}$. We employ the following control Hamiltonian and measurement operator:

$$F = \sigma_x \otimes \sigma_x \otimes \mathbb{I}_2 + 2 \cdot \mathbb{I}_2 \otimes \sigma_x \otimes \sigma_x + \sigma_x \otimes \mathbb{I}_2 \otimes \sigma_x,$$
$$M = -\sum_{k=x,y,z} \sigma_k \otimes \sigma_k \otimes \mathbb{I}_2.$$

Then, the measurement operator M is given by

$$M = \left(\begin{array}{c|c} M_S \otimes \mathbb{I}_2 & 0 \\ \hline 0 & -\mathbb{I}_4 \end{array}\right), \ M_S = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, $\lambda_{(1)} = 3$ is the maximum eigenvalue of M. By the direct calculation, we can check Assumption 2 is satisfied.

With this system and the control law (22), we perform the control simulation. In the simulation, we take $\gamma_k = \eta = \alpha = \beta = 1$ and the initial quantum state is given by

$$\rho = \left(\begin{array}{c|c} 0_{4\times4} & 0_{4\times4} \\ \hline 0_{4\times4} & \frac{1}{4}\mathbb{I}_4 \end{array}\right).$$

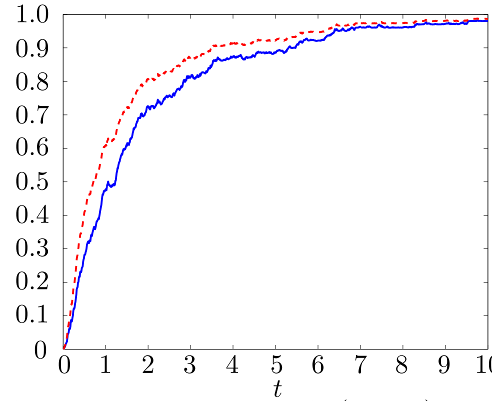

Fig. 1. Averaged time response of $\mathrm{tr}(\bar{\rho}_{S,t}\bar{\rho}_{(1)})$ and $\mathrm{tr}(\bar{\rho}_{S,t})$.

Figure 1 depicts the averaged time response of the function $\mathrm{tr}(\bar{\rho}_{S,t}\bar{\rho}_{(1)})$ and $\mathrm{tr}(\bar{\rho}_{S,t})$, where we used 100 sample paths to derive the curves. The blue solid line and the red dased line represent the fidelity between $\bar{\rho}_{S,t}$ and the target subsystem state $\mathrm{tr}(\bar{\rho}_{S,t}\bar{\rho}_{(1)})$ and the trace of $\bar{\rho}_S$, $\mathrm{tr}(\bar{\rho}_{S,t})$, respectively.

Note that the blue solid and red dashed lines do not cross, and that these converge in a similar way. This indicates that the control law (22) makes $\bar{\rho}_{S,t}$ converge to the target state, moving $\mathrm{tr}(\bar{\rho}_S)$ to 1: from the relation $\mathrm{tr}(\bar{\rho}_{S,t}\bar{\rho}_{(1)}) \le \mathrm{tr}(\bar{\rho}_{S,t})$, we have $\mathrm{tr}(\bar{\rho}_S\bar{\rho}_{(1)}) = 1$ only if $\mathrm{tr}(\bar{\rho}_{S,t}) = 1$.

V. CONCLUSION

In this paper, we proposed a new method for the state preparation of noiseless subsystems. This is done based on the idea that the noiseless subsystem can be regarded as an independent noise-free quantum system. The obtained result suggests the general framework to apply existing reuslts for noise-free quantum systems to the control of noiseless subsystems. It is expected that our framework is also effective for the general control problem for the noiseless subsystem.

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