2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Discounted Cost Infinite Time Horizon Cumulant Control

Ronald W. Diersing, Chang-Hee Won, and Michael K. Sain

***Abstract*— Cumulants are gaining in popularity for use in stochastic control and game theory. They also have been effective in application to building and vibration control problems. Much of the work has been done for the finite time horizon case. In this paper, cost cumulants will be used on a discounted cost function. The control will be concerned with the first two cumulants, the mean and variance. The approach will initially be done for a nonlinear system with non-quadratic costs and sufficient conditions are determined. With the sufficient conditions in place, attention will be turned to the linear quadratic special case. A coupled Riccati equation will be seen to give an optimal cumulant control law.**

I. Introduction

Recently, the k cost cumulant (kCC) and minimum cost variance (MCV) have received attention. [4]-[6],[7]. The most well known cumulant control, of course, is the linear quadratic Gaussian (LQG) method. LQG can be seen as a cumulant control because it is a control method in which the control wishes to minimize the first cumulant, the mean, of a cost. Another well known control method that could also be seen as cumulant control is risk sensitive control. In risk sensitive control, the controls tries to minimize a series of cost cumulants. The cumulants are related to the more well known moments. It is well known that the moments can be determined from the first characteristic function. Furthermore, if all of the moments are known, then the first characteristic function is completely characterized, and thereby so is the probability distribution. The same can be said of cumulants, with the major difference being that cumulants can be determined from the second characteristic function (which is simply the natural logarithm of the first). What has been found from the finite time horizon case is that for the linear quadratic special case, cumulants yield quadratic cost function and linear controllers, whereas moments may not. What this paper contains is a treatment of cumulants for the discounted cost infinite time horizon problem. There have been some infinite time horizon cumulant work, namely [6] and [8]. The main difference here is the use of the discounted cost.

The paper begins with a formulation of the problem and then moves on to discuss some preliminaries. With those out of the way, the discussion turns to sufficient conditions for the nonlinear system, non-quadratic cost case. Once these conditions are determined, they are used in the linear quadratic special case.

R. Diersing is an Assistant Professor with the Department of Engineering, University of Southern Indiana, Evansville, IN 47712. rwdiersing@usi.edu

C. Won is an Assistant Professor of Electrical Engineering, Temple University, Philadelphia, PA 19122 cwon@temple.edu

M. Sain is Freimann Professor of Electrical Engineering, University of Notre Dame, Notre Dame, IN 46556 avemaria@nd.edu

II. Problem Formulation

For the infinite time horizon problem, we will consider the system

$$dx(t) = f(x(t), u(t))dt + \sigma(x(t))d\xi(t) \qquad (1)$$

where $x(0) = x_0$ is a random variable independent of ξ, $x \in \mathbb{R}^n$ is the state, $u \in U \subset \mathbb{R}^p$ is the control, and ξ is a d-dimensional Brownian motion with variance W. The functions f, u_i will be assumed to satisfy both linear growth and Lipschitz conditions. That is, f and σ satisfy the following conditions.

(i) There exists a constant C such that

$$\|f(x,u)\| \leq C\,(1 + \|x\| + ||u||)$$
$$\|\sigma(x)\| \leq C(1 + \|x\|)$$

for all $(x,u) \in \mathbb{R}^n \times \mathcal{U}$, $x \in \mathbb{R}^n$, and $\|\cdot\|$ is the Euclidean norm.

(ii) There is a constant K so that

$$\|f(\tilde{x},\tilde{u}) - f(x,u)\| \leq K\,(\|\tilde{x} - x\| + \|\tilde{u} - u\|)$$
$$\|\sigma(\tilde{x}) - \sigma(x)\| \leq K\|\tilde{x} - x\|$$

for all $x, \tilde{x} \in \mathbb{R}^n$; $u, \tilde{u} \in \mathcal{U}$.

Furthermore, the control strategy $u(t) = \mu(x(t))$ satisfies the following conditions:

(i) for some constant $\tilde{C}$

$$\|\mu(x)\| \leq \tilde{C}(1 + \|x\|)$$

(ii) there exists a constant $\tilde{K}$ such that

$$\|\mu(\tilde{x}) - \mu(x)\| \leq \tilde{K}(\|\tilde{x} - x\|)$$

where $x, \tilde{x} \in \mathbb{R}^n$. Often we will suppress the dependence on t and x and refer to the strategies as simply μ.

If the strategy μ satisfies these conditions, then they are admissible strategies. We can rewrite the stochastic differential equation as

$$dx(t) = \tilde{f}(x(t))dt + \sigma(x(t))d\xi(t) \qquad x(t_0) = x_0 \qquad (2)$$

where the strategy μ has been substituted into f, called $\tilde{f}$. The conditions of Theorem V4.1 of [2] are now satisfied and we see that if $E\|x(t_0)\|^2 < \infty$, then a solution of (1) exists. Furthermore the solution $x(t)$ is unique in the sense that if there exists another solution $\tilde{x}(t)$ with $\tilde{x}(t_0) = x_0$, then the two solutions have the same sample paths with probability 1. The resulting process is a Markov diffusion process ([2] pg. 123) and the moments of $x(t)$ are bounded.

Since this is a stochastic optimal control problem, it makes sense that we are concerned with some sort of cost. In this case, the cost to be considered will be a discounted non-quadratic cost. The discounted cost function will be given as

$$J = \int_0^\infty e^{-\beta t} l(x(t), u(t)) dt \tag{3}$$

where $\beta > 0$ is a constant and l is a positive semidefinite continuous function that satisfies a polynomial growth condition. The control's objective will then be to minimize in some way the mean and variance of this cost function. Before moving on to that, however, it is important to state an important formula to be used.

Dynkin Formula

The Dynkin formula is the mechanism that allows us to determine the Hamilton-Jacobi-Bellman equation for the variance. Before moving on, however, it is important to state a notation, namely that the expectation $E_{tx}\{\cdot\}$ is simply $E\{\cdot|x(t)\}$. Much of these results may be found in [3], but we are restating them here for completeness. To begin, we let Q be a subset of $\mathbb{R}^n$ and $C_p1, 2(Q)$ denote a class of functions that have continuous first and second partial derivatives with respect to x. Commonly the Dynkin formula for a function $\Phi(t, x) \in C_p^{1,2}(Q)$ be given as

$$\begin{aligned} \Phi(t,x) =& E_{tx}\left\{\int_t^{t_1} -\mathcal{O}^\mu \Phi(s, x(s)) ds\right\} \\ &+ E_{tx}\left\{\Phi(t_1, x(t_1))\right\} \end{aligned} \tag{4}$$

where the operator $\mathcal{O}^\mu$ is given by

$$\mathcal{O}^\mu = \frac{\partial}{\partial t} - G^\mu$$

with

$$\begin{aligned} -G^\mu =& f'(x(t), \mu(x(t)))\frac{\partial}{\partial x} \\ &+ \frac{1}{2} tr\left(\sigma(x(t)) W \sigma'(x(t)) \frac{\partial^2}{\partial x^2}\right). \end{aligned}$$

For the discounted cost case, we may be concerned more with the case when we let $\Phi(t, x) = e^{-\beta s}\phi(x)$. Then by substitution and some manipulations, we obtain

$$\begin{aligned} e^{-\beta t}\phi(x) =& E_{tx}\left\{\int_t^{t_1} -\left(\frac{\partial}{\partial s} - G^\mu\right)\left(e^{-\beta s}\phi(x(s))\right) ds\right\} \\ &+ E_{tx}\left\{e^{-\beta t_1}\phi(x(t_1))\right\} \\ =& E_{tx}\left\{\int_t^{t_1} -\left(\frac{\partial}{\partial s}\right)\left(e^{-\beta s}\phi(x(s)) ds\right)\right\} \\ &- E_{tx}\left\{\int_t^{t_1} G^\mu(e^{-\beta s}\phi(x(s))) ds\right\} \\ &+ E_{tx}\left\{e^{-\beta t_1}\phi(x(t_1))\right\} \\ =& E_{tx}\left\{\int_t^{t_1} \left(\beta e^{-\beta s}\phi(x(s)) ds\right)\right\} \\ & E_{tx}\left\{e^{-\beta s}\left(G^\mu \phi(x(s)\right) ds\right\} \\ &+ E_{tx}\left\{e^{-\beta t_1}\phi(x(t_1))\right\}. \end{aligned}$$

This then reduces, with some manipulation, to

$$\begin{aligned} & e^{-\beta t_1} E_x\left\{\phi(x(t_1))\right\} - e^{-\beta t}\phi(x) \\ &\quad = E_{tx}\left\{\int_t^{t_1} e^{-\beta s}\left[\beta\phi + G^\mu\phi\right](x(s)) ds\right\}, \end{aligned}$$

and letting $t = 0$, gives the new Dynkin formula for the discounted cost case,

$$\begin{aligned} & e^{-\beta t_1} E_x\left\{\phi(x(t_1))\right\} - \phi(x) \\ &\quad = E_{tx}\left\{\int_0^{t_1} e^{-\beta s}\left[\beta\phi + G^\mu\phi\right](x(s)) ds\right\}. \end{aligned}$$

III. Problem Definitions

Before moving on to the development of the Hamilton Jacobi Bellman equations, it is worthwhile to state several definitions to be used later on.

Definition 1: A function $\mathcal{M} : \mathbb{R}^n \to \mathbb{R}^+$ is an admissible mean cost function if there exists an admissible control μ such that $\mathcal{M}(x) = E_x\{J\}$.

Definition 2: An admissible mean cost function $\mathcal{M}$ defines a class of control laws $\mathcal{U}_M$ such that $\mu \in \mathcal{U}_M$ if and only if the control law μ is admissible and satisfies Definition 1.

Definition 3: An MCV control strategy $\mu^* \in \mathcal{U}_M$ is one that minimizes the second moment, i.e. $E_x\left\{J^2\right\} = \mathcal{M}_2(x)$ for $x \in \mathbb{R}^n$.Furthermore the variance is then determined from $\mathcal{V} = \mathcal{M}_2(x) - \mathcal{M}^2(x)$.

IV. Nonlinear Solution

We will begin by providing several lemmas to be used in the proof of the control's minimum cost variance law. First we will consider a necessary condition for the discounted cost's mean value and then proceed to a sufficient condition for the mean.

Lemma 1: Let $\mathcal{M} \in C_p^{1,2}(Q)$ be an admissible mean cost function and μ be an admissible control law that satisfies Definition 1. Then $\mathcal{M}$ satisfies

$$\beta\mathcal{M}(x) = -G^\mu\mathcal{M}(x) + l(x, \mu(x)). \tag{5}$$

Lemma 2 (Verification Lemma): Let $\mathcal{M} \in C_p^{1,2}(Q)$ be a solution to

$$\beta\mathcal{M}(x) = -G^\mu\mathcal{M}(x) + l(x, \mu(x)). \tag{6}$$

Then

$$\mathcal{M}(x) = E\left\{\int_0^\infty e^{-\beta s} l(x(s), \mu(x(s)))\right\}$$

for all $\mu \in \mathcal{U}_M$.

Proof. From the Dynkin formula, if we let $\phi(x) = \mathcal{M}(x)$, we obtain

$$\begin{aligned} & e^{-\beta t_1} E_x\{\mathcal{M}(x(t_1)\} - \mathcal{M}(x) = \\ &\quad E_x\left\{\int_0^{t_1} e^{-\beta s}\left[-G^\mu\mathcal{M} - \beta\mathcal{M}\right](x(s)) ds\right\}, \end{aligned}$$

which by some manipulation becomes

$$\begin{aligned} \mathcal{M}(x) =& E_x\left\{\int_0^{t_1} e^{-\beta s}\left[G^\mu\mathcal{M} + \beta\mathcal{M}\right](x(s)) ds\right\} \\ &+ e^{-\beta t_1} E_x\{\mathcal{M}(x(t_1)\}. \end{aligned} \tag{7}$$

However, notice that from (6) we have

$$\beta\mathcal{M}(x) = -G^{\mu}\mathcal{M}(x) + l(x,\mu), \tag{8}$$

which with a slight manipulation yields

$$\beta\mathcal{M}(x) + G^{\mu}\mathcal{M}(x) = l(x,\mu). \tag{9}$$

So by substituting this into (7), we obtain

$$\begin{aligned}\mathcal{M}(x) =& E_x\left\{\int_0^{t_1} e^{-\beta s} l(x(s),\mu(x(s))ds\right\} \\ &+ e^{-\beta t_1}E_x\{\mathcal{M}(x(t_1)\},\end{aligned}$$

and by letting $t_1 \to \infty$,

$$\mathcal{M}(x) = E_x\left\{\int_0^{\infty} e^{-\beta s} l(x(s),\mu(x(s))ds\right\}.$$

□

With the mean case given, we now have a second verification lemma, this time for the second moment. However, before we begin, we will give a lemma that will be useful in the proof of the second moment's verification lemma.

Lemma 3: Consider the running cost function $L(t,x,\mu) = e^{-\beta t}l(x,\mu)$, which is denoted by L_t. then the equality

$$(j+1)\int_t^{t_f} L_s\left[\int_s^{t_f} L_r dr\right]^j ds = \left[\int_t^{t_f} L_r dr\right]^{j+1} \tag{10}$$

holds.

Proof. First we should change the limits of integration:

$$\int_t^{t_f} L_s\left[\int_s^{t_f} L_r dr\right]^j ds = (-1)^j\int_{t_f}^{t} L_s\left[\int_{t_f}^{s} L_r dr\right]^j ds$$

Now recall that for two differential functions F and G we can integrate by parts

$$\int_{t_f}^{t} F(s)g(s)ds = F(t)G(t) - F(t_f)G(t_f) - \int_{t_f}^{t} f(s)G(s)ds$$

where $f(s) = \frac{dF(s)}{ds}, G(s) = \int_{t_f}^{s} g(r)dr$. Let $g(s) = L_s$ and

$$F(s) = \left[\int_{t_f}^{s} L_r dr\right]^j.$$

With these definitions we see that

$$\begin{aligned}f(s) &= jL_s\left[\int_{t_f}^{s} L_r dr\right]^{j-1} \\ G(s) &= \int_{t_f}^{s} L_r dr\end{aligned}$$

which then yields

$$\begin{aligned}(-1)^j\int_{t_f}^{t} L_s\left[\int_{t_f}^{s} L_r dr\right]^j ds &= (-1)^j\left[\int_{t_f}^{t} L_s ds\right]^{(j+1)} \\ &- (-1)^j\int_{t_f}^{t} jL_s\left[\int_{t_f}^{s} L_r dr\right]^j ds\end{aligned}$$

which is

$$(j+1)\int_{t_f}^{t} L_s\left[\int_{t_f}^{s} L_r dr\right]^j ds = \left[\int_{t_f}^{t} L_s ds\right]^{(j+1)}$$

and the lemma is proved. □

Lemma 4 (Verification Lemma): Let $\mathcal{M}$ be an admissible mean cost function and $\mathcal{M}_2 \in C_p^{1,2}(Q)$ be a nonnegative solution to

$$\begin{aligned}2\beta\mathcal{M}_2(x) = \\ \min_{\mu\in\mathcal{U}_M}\left\{-G^{\mu}\mathcal{M}_2(x) + 2\mathcal{M}(x)l(x,\mu)\right\}.\end{aligned} \tag{11}$$

Then

$$\mathcal{M}_2(x) \le E\left\{\left[\int_0^{\infty} e^{-\beta s} l(x(s),\mu(x(s)))ds\right]^2\right\}$$

for every $\mu \in \mathcal{U}_M$. If, say, $\bar{\mu}$ is also the minimizing argument of (11), then

$$\mathcal{M}_2(x) = E\left\{\left[\int_0^{\infty} e^{-\beta s} l(x(s),\bar{\mu}(x(s)))ds\right]^2\right\}.$$

Proof. Consider the following HJB equation given in (11). By the definition of the Dynkin formula for $\mathcal{M}_2(x)$, we have

$$\begin{aligned}&e^{-2\beta t}\mathcal{M}_2(x) = \\ &E_{tx}\bigg\{\int_t^{t_1} e^{-2\beta s}\left[-\frac{\partial e^{-(2)\beta s}\mathcal{M}_2}{\partial s} + G^{\mu}\mathcal{M}_2\right] \\ &\cdot (x(s))ds\bigg\} + e^{-2\beta t_1}E_x\{\mathcal{M}_2(x(t_1)\} \\ &= E_{tx}\bigg\{\int_t^{t_1} e^{-2\beta s}\left[2\beta\mathcal{M}_2 + G^{\mu}\mathcal{M}_2\right] \\ &\cdot (x(s))ds\bigg\} + e^{-2\beta t_1}E_x\{\mathcal{M}_2(x(t_1)\},\end{aligned} \tag{12}$$

but in a similar way as in the mean value case, we have

$$2\beta\mathcal{M}_2(x) + G^{\mu}\mathcal{M}_2(x) \le 2\mathcal{M}(x)l(x,\mu). \tag{13}$$

Using this gives

$$\begin{aligned}e^{-2\beta t}\mathcal{M}_2(x) \le& e^{-2\beta t_1}E_x\{\mathcal{M}_2(x(t_1)\} \\ &+ E_{tx}\left\{\int_t^{t_1} e^{-2\beta s}\mathcal{M}(x(s))l(x(s),\mu(s))ds\right\},\end{aligned}$$

but

$$e^{-\beta t}\mathcal{M}(x) = E_{tx}\left\{\int_t^{\infty} e^{-\beta s} l(x(s),\mu(x(s)))ds\right\}.$$

So, by substitution, we have

$$\begin{aligned}
&e^{-2\beta t}\mathcal{M}_2(x) \le +e^{-\beta t_1}E_x\{\mathcal{M}_2(x(t_1)\} \\
&+ E_{tx}\Bigg\{\int_t^{t_1} 2e^{-\beta s} \\
&\cdot E_{sx}\left\{\int_s^{\infty} e^{-\beta\tau} l(x(\tau),\mu(x(\tau)))d\tau\right\} l(x(s),\mu(x(s)))ds\Bigg\} \\
&\le e^{-\beta t_1}E_x\{\mathcal{M}_2(x(t_1)\} \\
&+ E_{tx}\Bigg\{E_{sx}\Bigg\{\int_t^{t_1} 2e^{-\beta s} l(x(s),\mu(x(s)) \\
&\cdot \int_s^{\infty} e^{-\beta t} l(x(t),\mu(x(t)))dtds\Bigg\}\Bigg\} \\
&+ e^{-\beta t_1}E_x\{\mathcal{M}_2(x(t_1)\}.
\end{aligned}$$

However, since $t \le s$, we know that $E_{tx}\{E_{sx}\{\cdot\}\} = E_{tx}\{\cdot\}$, which gives

$$\begin{aligned}
e^{-2\beta t}\mathcal{M}_2(x) \le & E_{tx}\Bigg\{\int_t^{t_1} 2e^{-\beta s} l(x(s),\mu(x(s)) \\
&\cdot \int_s^{\infty} e^{-\beta t} l(x(t),\mu(x(t)))dtds\Bigg\}
\end{aligned} \tag{14}$$

but by Lemma3, we know that

$$\begin{aligned}
&\left[\int_t^{t_1} e^{-\beta t} l(x(t),\mu(x(t)))dt\right]^2 = \\
&\int_t^{t_1} 2e^{-\beta s} l(x(s),\mu(x(s))) \\
&\cdot \int_s^{t_1} e^{-\beta\tau} l(x(\tau),\mu(x(\tau)))d\tau ds.
\end{aligned} \tag{15}$$

Using (14), (15), and by letting $t = 0$; we obtain

$$\begin{aligned}
\mathcal{M}_2(x) \le & E_x\left\{\left[\int_0^{t_1} e^{-\beta s} l(x(s),\mu(x(s)))ds\right]^2\right\} \\
&+ e^{-\beta t_1}E_x\{\mathcal{M}_2(x(t_1)\}
\end{aligned} \tag{16}$$

which, as $t_1 \to \infty$, becomes

$$\mathcal{M}_2(x) \le E_x\left\{\left[\int_0^{\infty} e^{-\beta s} l(x(s),\mu(x(s)))ds\right]^2\right\}.$$

For the case when the μ is optimal, i.e. μ^*, the inequality becomes an equality. □

With the second moment verification lemma in place, we can move on to the sufficient condition for the variance of the discounted cost function.

Theorem 1: Let $\mathcal{M}$ be an admissible mean cost function, $\mathcal{M} \in C_p^{1,2}(Q)$ with an associated class of control laws $\mathcal{U}_M$. Also consider a function $\mathcal{V} \in C_p^{1,2}$ that is a solution to

$$\begin{aligned}
2\beta\mathcal{V}(x) = \min_{\mu\in\mathcal{U}_M} \Bigg\{ & - G^{\mu}\mathcal{V}(x) \\
&+ \left(\frac{\partial\mathcal{M}}{\partial x}(x)\right)' \sigma(x)W\sigma'(x)\left(\frac{\partial\mathcal{M}}{\partial x}(x)\right)\Bigg\}.
\end{aligned} \tag{17}$$

If μ^* is the minimizing argument of (17), then μ^* is the minimum cost variance control law for the discounted cost case.

Proof. For the minimum cost variance (MCV) case, we will consider the infinite time horizon HJB equation

$$\begin{aligned}
2\beta\mathcal{V}(x) = \min_{\mu\in\mathcal{U}_M} \Bigg\{ & - G^{\mu}\mathcal{V}(x)\left(\frac{\partial\mathcal{M}}{\partial x}(x)\right)' \sigma(x) \\
&\cdot W\sigma'(x)\left(\frac{\partial\mathcal{M}}{\partial x}(x)\right)\Bigg\}.
\end{aligned} \tag{18}$$

To show this, consider the second moment HJB equation

$$2\beta\Updownarrow_2(x) = \min_{\mu\in\mathcal{U}_M} \{-G^{\mu}\mathcal{M}_2(x) + 2\mathcal{M}(x)l(x,\mu)\} \tag{19}$$

where $\mathcal{M}(x)$ is from the first moment. But recall that $\mathcal{M}_2 = \mathcal{V} + \mathcal{M}^2$, so by substitution we have

$$\begin{aligned}
2\beta(\mathcal{V}(x) + \mathcal{M}^2(x)) = \min_{\mu\in\mathcal{U}_M} \Bigg\{ & - G^{\mu}[\mathcal{V}(x) + \mathcal{M}^2(x)] \\
&+ 2\mathcal{M}(x)l(x,\mu)\Bigg\}.
\end{aligned} \tag{20}$$

By the definition of $-G^{\mu}$ and use of the chain rule, we can see that

$$\begin{aligned}
-G^{\mu}[\mathcal{M}^2] =& 2\mathcal{M}f'\frac{\partial\mathcal{M}}{\partial x} + \frac{1}{2}tr\left(\sigma W\sigma'\frac{\partial}{\partial x}\left(2\mathcal{M}\frac{\partial\mathcal{M}}{\partial x}\right)\right) \\
=& 2\mathcal{M}f'\frac{\partial\mathcal{M}}{\partial x} + \frac{1}{2}tr\Bigg(\sigma W\sigma' \\
&\cdot\frac{\partial}{\partial x}\left[2\left(\frac{\partial\mathcal{M}}{\partial x}\right)\left(\frac{\partial\mathcal{M}}{\partial x}\right)' + 2\mathcal{M}\frac{\partial^2\mathcal{M}}{\partial x^2}\right]\Bigg) \\
=& -2\mathcal{M}G^{\mu}\mathcal{M} + \left(\frac{\partial\mathcal{M}}{\partial x}\right)'\sigma W\sigma'\left(\frac{\partial\mathcal{M}}{\partial x}\right)
\end{aligned}$$

where arguments have been suppressed. Substituting the expression for $-G^{\mu}M^2(x)$ and with some manipulation (20) becomes

$$\begin{aligned}
2\beta\mathcal{V}(x) = \min_{\mu\in\mathcal{U}_M} \Bigg\{ & - G^{\mu}\mathcal{V}(x) \\
&+ \left(\frac{\partial\mathcal{M}}{\partial x}(x)\right)' \sigma(x)W\sigma'(x)\left(\frac{\partial\mathcal{M}}{\partial x}(x)\right) \\
&+ 2\mathcal{M}(x)l(x,\mu) - 2\mathcal{M}(x)[\beta\mathcal{M}(x) + G^{\mu}\mathcal{M}(x)]\Bigg\}
\end{aligned}$$

but $\mathcal{M}$ is an admissible mean cost function, so we have

$$\beta\mathcal{M}(x) + G^{\mu}\mathcal{M}(x) = l(x,\mu).$$

So substituting this in yields

$$\begin{aligned}
2\beta\mathcal{V}(x) = \min_{\mu\in\mathcal{U}_M} \Bigg\{ & - G^{\mu}\mathcal{V}(x) \\
&+ 2\mathcal{M}(x)l(x,\mu) - 2\mathcal{M}(x)l(x,\mu) \\
&+ \left(\frac{\partial\mathcal{M}}{\partial x}(x)\right)' \sigma(x)W\sigma'(x)\left(\frac{\partial\mathcal{M}}{\partial x}(x)\right)\Bigg\}
\end{aligned} \tag{21}$$

which gives the desired results

$$2\beta\mathcal{V}(x) = \min_{\mu\in\mathcal{U}_M}\Bigg\{ - G^{\mu}\mathcal{V}(x) + \left(\frac{\partial\mathcal{M}}{\partial x}(x)\right)'\sigma(x)W\sigma'(x)\left(\frac{\partial\mathcal{M}}{\partial x}(x)\right)\Bigg\}. \tag{22}$$

□

V. Linear Quadratic Case

With some results for the non-linear system with discounted non-quadratic costs at hand, it makes sense to apply these results to the linear quadratic case. Furthermore, we will determine an infinite time horizon minimum cost variance control law. But to begin, we define our autonomous linear system as

$$dx(t) = [Ax(t) + Bu(t)]dt + Ed\xi(t) \tag{23}$$

where A, B, and E are respectively $n \times n$, $n \times m$, and $n \times d$ matrices whose elements are constant real values, and $x(0) = x_0$ is the initial condition. With the linear system is the quadratic discounted cost, given as

$$J = \int_0^{\infty} e^{-\beta t}\left[x'(t)Qx(t) + u'(t)Ru(t)\right]dt \tag{24}$$

where Q is a $n \times n$ positive semidefinite matrix and R is a $m \times m$ positive definite matrix. Let us assume that the cost functions for the mean and variance are quadratic, that is

$$\begin{aligned}\mathcal{M}(x) =& x'Mx + m\\ \mathcal{V}(x) =& x'Vx + v\end{aligned}$$

where M, V are $n \times n$ matrices and m, v, are scalars. The control wants to minimize the variance while holding the mean to a constraint. So using Lemma 2 and Theorem 1, we can write

$$\begin{aligned}\min_{\mu\in\mathcal{U}_M}\Bigg\{ & - \beta m - \beta x'Mx + (Ax + B\mu)'Mx\\ & + x'M(Ax + B\mu) + tr(EWEM)\\ & + x'Qx + \mu'R\mu + \gamma\Big[- \beta v - \beta x'Vx\\ & + (Ax + B\mu)'Vx + x'V(Ax + B\mu)\\ & + tr(EWEV) + 4MEWE'M\Big]\Bigg\} = 0\end{aligned}$$

and minimizing gives an optimal control law of

$$u^*(t) = \mu^*(x(t)) = -R^{-1}B'[M + \gamma V]x(t). \tag{25}$$

Using this control law and Lemma 2, we obtain an algebraic equation for the mean of

$$\begin{aligned}\beta M =& A'M + MA - MBR^{-1}B'M\\ & + Q + \gamma^2 VBR^{-1}B'V.\end{aligned} \tag{26}$$

Similarly for the variance we obtain

$$\begin{aligned}2\beta V =& A'V + VA - 2\gamma VBR^{-1}B'V\\ & - MBR^{-1}B'V - VBR^{-1}B'M\\ & + 4MEWE'M\end{aligned} \tag{27}$$

from the control law, (25) and Theorem 1. Now let us state these results in terms of a theorem.

Theorem 2: Consider the linear quadratic case. Suppose that M and V are solutions to the algebraic Riccati equations (26) and (27). Then the minimum cost variance control law is given as in (25) and $\mathcal{M}(x)$ and $\mathcal{V}(x)$ are constructed with the aid of

$$\begin{aligned}\beta m =& tr(EWEM)\\ \beta v =& tr(EWEV).\end{aligned} \tag{28}$$

There is one thing that is left to discuss. This is the important issue of stability. It remains to be shown that the control law given in the previous theorem is in fact stable. To do this we will consider the pair (A, B) to be stabilizable and $(\sqrt{Q}, A)$ detectable. Consider the algebraic Riccati equations (26) and (27). With some simple manipulation, it is seen that these can be given as

$$\begin{aligned}(A - \frac{\beta}{2}I)'M + M(A - \frac{\beta}{2}I) - MBR^{-1}B'M&\\ +Q\gamma^2 VBR^{-1}B'V = 0&\end{aligned} \tag{29}$$

and

$$\begin{aligned}(A - \beta I)'V + V(A - \beta I) - 2\gamma VBR^{-1}B'V&\\ -MBR^{-1}B'V - VBR^{-1}B'M&\\ +4MEWE'M = 0.&\end{aligned} \tag{30}$$

However, it can be shown that (for a constant α) if $(A - \alpha I, B)$ can be shown to be stabilizable and $(\sqrt{Q}, A - \alpha I)$ detectable, then the control law is stable. To show this assume (A, B) stabilizable and $(\sqrt{Q}, A)$ detectable. If (A, B) is stabilizable, then there exists a matrix K such that roots of

$$|\lambda I - A + BK| = 0 \tag{31}$$

are such that $Re\{\lambda\} < 0$. However, in (29), instead of A, we have $(A - \frac{\beta}{2}I)$ and for (30), we have $(A - \beta I)$, with $\beta > 0$. So in general, we have $(A - \alpha I)$. So substituting $A - \alpha I$ in for A in (31) gives,

$$|(\bar{\lambda} + \alpha)I + -A + BK| = 0$$

where we use $\bar{\lambda}$ for the eigenvalues, instead λ. However, this is the same as

$$|\lambda I - A + BK| = 0$$

where $\lambda = \bar{\lambda} + \alpha$. Because the pair (A, B) is completely stabilizable, then

$$Re\{\lambda\} < 0$$

which by substitution yields

$$Re\{\bar{\lambda} + \alpha\} < 0.$$

But because α is both real and positive, we know then that

$$Re\{\bar{\lambda}\} < 0$$

and the pair $(A - \alpha I, B)$ is stabilizable.

Now we must show the same process with the detectability of $(\sqrt{Q}, A - \alpha I)$. This is dual for stabilizability. We assume that the pair $(\sqrt{Q}, A)$ is detectable. We must shown that there exists a K such that the eigenvalues of $\sqrt{Q}K + A'$ are on the left hand side of the complex plane. But we know that $(\sqrt{Q}, A)$ is detectable, so that means that eigenvalues found through

$$|\lambda I - (A' + \sqrt{Q}K)| = 0$$

are such that $Re\{\lambda\} < 0$. However, for $(\sqrt{Q}, A - \alpha I)$, we have

$$|(\bar{\lambda} + \alpha)I - (A' + \sqrt{Q}K)| = 0$$

which means that $\lambda = \bar{\lambda} + \alpha$, and if $Re\{\lambda\} < 0$, then $Re\{\bar{\lambda}\} < 0$. Now consider Theorem 6.6.3 of [8]. It states the following.

Theorem 3: Assume $(\bar{A}, B)$ is stabilizable and $(\sqrt{Q}, \bar{A})$ is detectable, $Q \geq 0, R > 0$, and γ is a nonnegative constant. Then the coupled algebraic Riccati equations

$$\begin{aligned} \bar{A}'M + M\bar{A} - MBR^{-1}B'M \\ +Q\gamma^2 VBR^{-1}B'V = 0 \end{aligned} \tag{32}$$

and

$$\begin{aligned} \bar{A}'V + V\bar{A} - 2\gamma VBR^{-1}B'V \\ -MBR^{-1}B'V - VBR^{-1}B'M \\ +4MEWE'M = 0 \end{aligned} \tag{33}$$

have unique solutions M^* and V^* in the class of symmetric, positive definite maps. Then, $\bar{A} - BR^{-1}B'(M^* + \gamma V^*) = \bar{A} + BK^*$ is stable where

$$K^* = -R^{-1}B'[M + \gamma V].$$

Proof. For the proof, see [8]. □

So, if we let $\bar{A} = A - \beta I$ for the variance and $\bar{A} = A - \frac{\beta}{2}I$ for the mean Riccati equations, we can see that this theorem then yields that the control given in (25) is stable.

VI. SDOF Building Example

Consider the single degree of freedom (SDOF) building given in Fig. 1. This building model was given in [1] and

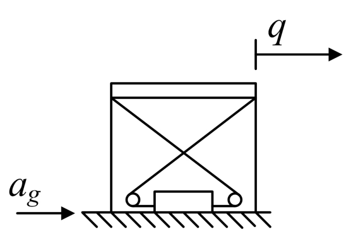

Fig. 1. SDOF Building

can be found to be

$$dx = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix} xdt + \begin{bmatrix} 0 \\ -\frac{4k_c\cos(\alpha)}{m} \end{bmatrix} udt + \begin{bmatrix} 0 \\ -1 \end{bmatrix} dw. \tag{34}$$

where $k = 7934lb/in$, $m = 16.69lb - s^2/in$, $c = 9.020lb - s/in$, $k_c = 2124lb/in$, and $\alpha = 36^o$. Furthermore, the weighting matrices Q and R given in (24) are defined as

$$Q = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}$$

and $R = kc$. The state is given as $x = (q, \dot{q})'$ where q is the displacement of the floor and $w = a_g$ is the earthquake ground acceleration. Both an LQG and MCV controller was designed for this system. The controllers were then simulated with the 1940 El Centro earthquake date history. The parameters for the MCV controller were set to $\gamma = 3$ and $\beta = 10^{-6}$. The simulation results are given in Table I. From the results, the MCV controller has a 20% decrease in peak displacement. Furthermore, the peak control effort was less than a 1% increase. This is a significant reduction in vibration, for a small cost in terms of control effort.

	LQG	MCV
Peak Displacement	0.1186	0.0891
Peak Velocity	3.4259	2.4319
Peak Control	0.2441	0.2454

TABLE I
Simulation Results

VII. Conclusion

In this paper, the discounted cost minimum cost variance control problem was examined. To begin, the nonlinear autonomous system with a non-quadratic discounted cost problem was discussed. Furthermore, sufficient conditions for an optimal control were determined. Then these results we applied to the linear autonomous system with a quadratic discounted cost. Here a set of coupled Riccati equations were determined and a optimal control law was found. The treatment was somewhat general and will be applied to the game theoretic case in the future.

References

[1] L. L. Chung, A. M. Reinhorn, and T. T. Soong, "Experiments on Active Control of Seismic Structures," *Journal of Engineering Mechanics,* vol. 114, pp. 241-256, 1988.

[2] W. H. Fleming, R. W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer-Verlag, New York, 1975.

[3] W. H. Fleming, H. M. Soner, *Controlled Markov Processes and Viscosity Solutions, 1st ed.*, Springer-Verlag, New York, 1993.

[4] K. D. Pham, M. K. Sain, and S. R. Liberty, "Finite Horizon Full-State Feedback kCC Control in Civil Structures Protection," *Stochastic Theory and Adaptive Control, Lecture Notes in Control and Information Sciences*, Proceedings of a Workshop held in Lawrence, Kansas, Edited by B. Pasik-Duncan, Springer-Verlag, Berlin Heidelberg, Germany, Vol. 280, pp. 369-383, September 2002.

[5] K. D. Pham, M. K. Sain, and S. R. Liberty, " Cost Cumulant Control: State-Feedback, Finite-Horizon Paradigm with Application to Seismic Protection," *Special Issue of Journal of Optimization Theory and Applications*, Edited by A. Miele, Kluwer Academic/Plenum Publishers, New York, Vol. 115, No. 3, pp. 685-710, December 2002.

[6] K. D. Pham, M. K. Sain, and S. R. Liberty, "Infinite Horizon Robustly Stable Seismic Protection of Cable-Stayed Bridges Using Cost Cumulants, " *Proceedings of American Control Conference*, pp. 691-696, Boston, Massachusetts, June 30, 2004.

[7] M. K. Sain, C. H. Won, B. F. Spencer Jr., S. R. Liberty, "Cumulants and Risk Sensitive Control: A Cost Mean and Variance Theory with Applications to Seismic Protection of Structures," *Advances in Dynamic Games and Applications, Annals of the International Society of Dynamic Games*, vol. 5, J. A. Filor, V. Gaisgory, K. Mizukami (Eds), Birkhauser, Boston, 2000.

[8] C. H. Won, "Cost Cumulants in Risk Sensitive and Minimal Cost Variance Control," Ph. D. Dissertation, University of Notre Dame, Notre Dame, IN, July 1994.