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Approximation Methods and Spatial Interpolation in Distributed Control Systems†

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Abstract— **We propose an approximation method to solve large-scale optimal control problems for spatially distributed systems. The finite-section method is employed to construct finite-dimensional approximations to the large-scale optimal control problem. Then, we study the limit behavior of the approximation method and show that the solution of the approximate problems converge strongly to the solution of the large-scale problem. These techniques are applied to design finite-dimensional local optimal controllers. Finally, a spatial interpolation method is proposed that can patch all locally designed controllers to construct a parameterized family of stabilizing controller for the spatially distributed system. Furthermore, we characterize the class of stabilizing controllers which have finite supports.**

I. Introduction

Analysis and synthesis of distributed control systems has become a resonant part of control theory research in recent years [1]–[7] . Researchers have been interested in development and analysis of control protocols that are localized and spatially distributed and designed to achieve a global objective using only local interactions.

For all practical purposes, in a spatially distributed system each subsystem can only have a partial access to the state of the entire network. In other words, depending on the system's communication infrastructure each subsystem can only communicate with local neighboring subsystems. The structural properties of the optimal control of spatially distributed systems have been studied in [7] and [1]. In [1], we studied the spatial structure of the optimal control of spatially distributed dynamical systems with linear quadratic (LQ) performance criteria and arbitrary interconnection topologies. The importance of these results is that a significant drop-off in complexity can be achieved by localization. In Section III, we show that one can determine the communication requirements between sensors and actuators and the distance to which local information needs to be passed to achieve certain levels of performance. This is specifically useful in designing near-optimal distributed control algorithms.

The goal of this paper is to propose a synthesis method that can patch locally designed controllers to construct a global state-feedback control law K that

(1) stabilizes the closed-loop large-scale system.
(2) can be computed locally.
(3) has a finite support.
(4) is optimal with respect to some global cost function.

In a spatially distributed system, a local controller designer can only have access to local information about the dynamics of the entire system. Intuitively, the more information the local controller designer gets about the system, the closer (in some topology norm) the locally designed optimal controller gets to the centralized optimal controller. This simple observation motivates us to study the asymptotic behavior of the approximation skims for distributed control systems. In Section IV, we will introduce the finite-section approximation method [8]. The finite-section method is a suitable tool for approximation of certain operator equations (e.g., Lyapunov and Riccati equations) using finite-dimensional matrix techniques. Roughly speaking, a finite-section approximation with width n of a large-scale matrix at row with index zero is the $(2n+1)\times(2n+1)$-dimensional matrix obtained by clipping out from the large-scale matrix the window of width $2n+1$ which is centered at row with index zero on the main diagonal. We apply the finite-section method to large-scale spatially distributed dynamical systems to derive approximate finite-dimensional dynamical systems. Then, we show that the solutions of the corresponding finite-dimensional Lyapunov and Riccati equations to these approximate systems converge strongly to the unique solutions of the operator Lyapunov and Riccati equations. These results are used to conclude that the finite-dimensional LQR state-feedback control law converges strongly to the corresponding infinite-dimensional LQR state-feedback control law. This implies that sufficiently accurate approximations of the LQR state-feedback control law can be obtained as the size of the finite-section window becomes larger.

In Section V, we employ the approximation methods developed in Section IV to design local optimal state-feedback control laws. Furthermore, an interpolation method is proposed to patch these local controller to build a stabilizing state-feedback control law for the infinite-dimensional spatially distributed system. In fact, we characterize a parameterized family of such stabilizing state-feedback control laws. Then, we identify the subclass of stabilizing state-feedback control laws that have finite supports. It is shown that such stabilizing controllers can be computed in a distributed fashion, i.e., by using only local information.

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II. Notations and Preliminaries

$\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^+$ the set of nonnegative real numbers, $\mathbb{Z}$ the set of integer numbers, and $\mathbb{N}$ the set of natural numbers. A subset $\mathbb{G}$ of $\mathbb{Z}^d$ is referred to as the *spatial domain* if it consists of countably many d-tuples $i = (i_1, \ldots, i_d)$. In this paper, we assume that $d = 1$.

The Banach spaces $\ell_p(\mathbb{G})$ for $1 \leq p \leq \infty$ is defined as the set of all sequences $x = (x_i)_{i \in \mathbb{G}}$ satisfying

$$\sum_{i \in \mathbb{G}} |x_i|^p \; < \; \infty,$$

and endowed with the norm

$$\|x\|_p := \left(\sum_{i \in \mathbb{G}} |x_i|^p \right)^{\frac{1}{p}}.$$

Throughout the paper, we will use the shorthand notation ℓ_p for $\ell_p(\mathbb{G})$. An operator $A : \ell_p \to \ell_p$ is bounded if it has a finite induced norm, i.e., the following quantity

$$\|A\|_{p,p} := \sup_{\|x\|_p = 1} \|Ax\|_p, \tag{1}$$

is bounded. The set of all bounded linear operators of ℓ_p into ℓ_p is denoted by $\mathscr{B}(\ell_p)$. The identity operator is denoted by I. An operator A has an *algebraic* inverse on a Banach space $\mathcal{X}$ if it has an inverse A^{-1} in $\mathcal{X}$.

Every bounded linear operator $A \in \mathscr{B}(\ell_p)$ can be represented as a matrix

$$A = \big[a_{ki}\big].$$

The orthogonal projection onto a $2n + 1$-dimensional subspace is defined as

$$P_{n,i}\, x = [\ \ldots, 0, x_{-n+i}, \ldots, x_i, \ldots, x_{n+i}, 0, \ldots\ ]^T.$$

For simplicity of notations, P_n stands for projection $P_{n,0}$. We denote

$$[\![A_{n,i}]\!]_i := P_{n,i}\, A\, P_{n,i}.$$

When $\mathbb{G} = \mathbb{Z}^d$, the range of $P_{n,i}$ is a subspace of ℓ_p of dimension $(2n+1)^d$. We emphasize that $[\![A_{n,i}]\!]_i$ is a finite rank operator acting on ℓ_p and $A_{n,i}$ is a finite $(2n+1)^d \times (2n+1)^d$ matrix acting on $\mathbb{R}^{(2n+1)^d}$.

We need to define the notation $[\![\,.\,]\!]_i$ formally, as we will use this notation extensively throughout the paper. Let $C_n = [-n, n] \times [-n, n] \cap \mathbb{G} \times \mathbb{G}$, the integer vectors in the cube of length $2n$ centered at the origin. Given a $(2n+1) \times (2n+1)$ matrix X, $[\![X]\!]_i$ is an operator acting on ℓ_p which is obtained by replacing the $(i,i) + C_n$ block of the zero operator by matrix X such that the $(n+1, n+1)$ entry of X coincides with entry (i,i) of $[\![X]\!]_i$.

The common elements of the operators $X = (x_{ij})$ and $Y = (y_{ij})$ can be extracted using the following operation

$$(X \wedge Y)_{ij} := \begin{cases} x_{ij} & \text{if} \quad x_{ij} = y_{ij} \\ 0 & \text{otherwise} \end{cases},$$

for all $i, j \in \mathbb{G}$. We can also superpose two given operators as follows

$$X \vee Y = X + Y - X \wedge Y.$$

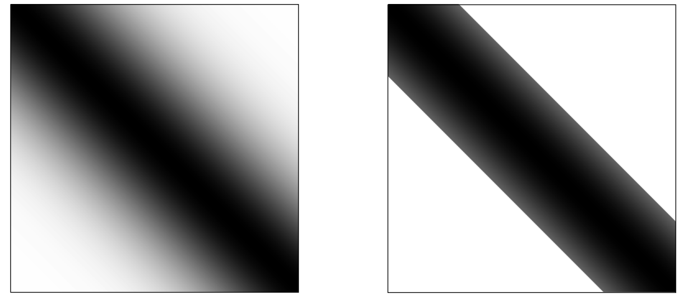

Fig. 1. The left picture depicts an SD state-feedback control law K for a SD system. The right picture describes the spatially truncated state-feedback control law K_N.

III. Spatial Truncation and Stability of the Closed-Loop System

In this section, we show that for a given SD system with an SD stabilizing state-feedback law, there always exist a family of stabilizing state-feedback control laws which have finite supports. As an illustration, we consider the following linear system

$$\dot{x} = Ax + Bu, \tag{2}$$

where A and B are SD operators with respect to a given coupling function χ_α (see [1]). We assume that the standard LQR assumptions hold. Therefore, the optimal state-feedback control law K for system (2) that minimizes the following quadratic performance criterion

$$J = \int_0^\infty \left(x^T Q x + u^T R u \right) dt,$$

where Q and R are SD operators with respect to χ_α, is given by

$$K = -R^{-1} B^T P, \tag{3}$$

in which P satisfies the corresponding algebraic Riccati equation. In general, K does not have a finite support. If the optimal state-feedback control law $K = [k_{ij}]$ is spatially decaying with respect to χ_α, then we have

$$\sup_{k \in \mathbb{G}} \sum_{i \in \mathbb{G}} |k_{ij}|\; \chi_\alpha(|i-j|) \; < \; \infty.$$

This implies that

$$\lim_{|i-j| \to \infty} |k_{ij}|\; \chi_\alpha(|i-j|) = 0.$$

In this case, K can be *spatially truncated* to obtain a *localized* (i.e., finite-support) state-feedback law. We define the spatially truncated operator K_N as follows

$$\big(K_N\big)_{ij} = \begin{cases} k_{ij} & \text{if} \quad |i-j| \leq N \\ 0 & \text{if} \quad |i-j| > N \end{cases}, \tag{4}$$

where N is the truncation length. Fig. 1 graphically shows K and K_N.

Lemma 3.1: Suppose N is the spatial truncation length.

The truncation error between the state-feedback control laws (3) and (4) can be quantified as follows:

(i) For exponentially decaying coupling functions

$$\|K - K_N\|_{2,2} \;\leq\; C\, e^{-\alpha N},$$

(ii) For algebraically decaying coupling functions

$$\|K - K_N\|_{2,2} \;\leq\; C'\, N^{\tau-\alpha},$$

for all $\alpha > \tau$ where τ is the decay margin,

and $C, C' > 0$ are some numbers which depend on the value of α.

Proof: The proof is not provided due to space limitations. ■

The spatial decay of the truncation error implies that the error can be made arbitrarily small as N becomes large. In fact, this approximation property enables us to employ the small-gain theorem to show that the truncated state-feedback law is stabilizing. In small-gain stability argument, we describe the truncated closed-loop system as follows (cf. [7])

$$\dot{x} \;=\; (A + BK)x + u \tag{5}$$

$$u \;=\; B(K_N - K)x. \tag{6}$$

The mapping from u to x is bounded on ℓ_2 because K is exponentially stabilizing. The small-gain theorem now gives a sufficient condition for the stability of the closed-loop system as follows

$$\|B\|_{2,2}\; \|K - K_N\|_{2,2} \sup_{\mathrm{Re}(s)>0} \sigma_{\max}(G(s)) \;<\; 1,$$

where $G(s) = (sI - (A + BK))^{-1}$. Therefore, a stabilizing truncation length N can be computed as follows

(i) For exponentially decaying systems

$$N \;>\; \inf_{\alpha} \,\log\left(C\; \|B\|_{2,2} \sup_{\mathrm{Re}(s)>0} \sigma_{\max}(G(s))\right)^{\frac{1}{\alpha}}. \tag{7}$$

(ii) For algebraically decaying systems

$$N \;>\; \inf_{\alpha} \;\left(C\; \|B\|_{2,2} \sup_{\mathrm{Re}(s)>0} \sigma_{\max}(G(s))\right)^{\frac{1}{\tau-\alpha}}. \tag{8}$$

These results are important as it tells us that for spatially decaying systems whenever K is also SD, there is always a family of stabilizing state-feedback control laws which have finite supports. For example, if operators A and B in (2) have finite supports, one can expect to construct a stabilizing state-feedback control law with a finite support. Thus, the challenging is how to construct a global control law with finite support by patching locally designed finite-dimensional state-feedback control laws. In Section V, we will propose a spatial interpolation method to assemble local controllers to build a stabilizing state-feedback law with a finite support.

IV. Approximation Methods in Distributed Control Problems

In general, solving infinite-dimensional operator equations, if not impossible, is a tedious task even when they are linear. In some special cases where the structure of the operator equation enjoys some kind of spatial symmetry, e.g., spatially invariance [7], Fourier transform can be used to convert an infinite-dimensional problem into a family of finite-dimensional problems in the frequency domain. In such cases, one only needs to solve a parameterized family of finite-dimensional problems in the frequency domain. Then the inverse transform can be used to find the solution in the spatial domain. In this section, our aim is to propose a way to construct local finite-dimensional approximations of the infinite-dimensional LQR problem.

Functional analysis solves equations in infinitely many variables, linear algebra solve equations in finitely many variables, and numerical analysis build the bridge between these two branch of mathematics and approximation methods are how it is done [8]. In the following subsection, we present one of the most natural and most important approximation methods for bounded linear operators: the finite-section method. By means of this method, we will propose approximations to the unique solutions of the operator Lyapunov and Riccati equations, and specifically, we will show that the approximate solutions converge strongly to the unique solutions of the operator Lyapunov and Riccati equations.

Without loss of generality and for the sake of simplicity of the notations, we perform our analysis only for node $i = 0$.

A. The Finite-Section Method

Consider the orthogonal projection operator onto a $2n+1$-dimensional subspace:

$$P_n\, x = [\ \ldots, 0, x_{-n}, \ldots, x_0, \ldots, x_n, 0, \ldots\ ]^T.$$

One can see that P_n converges strongly (i.e. pointwise) to I as $n \to \infty$ and Im P_n (i.e. image of P_n) can be identified with $\mathbb{R}^{2n+1}$.

Definition 4.1: The *finite-section approximation method* of a bounded linear operator A is a sequence $(\widetilde{A}_n)$ of operators $\widetilde{A}_n \in \mathscr{B}(\mathrm{Im}\ P_n)$ defined as

$$\widetilde{A}_n = P_n A P_n,$$

for all $n \in \mathbb{N}$.

It is straightforward to show that $\widetilde{A}_n$ converges strongly to A as $n \to \infty$ (see the proof of Theorem 4.1).

By definition, $\widetilde{A}_n$ can be identified with a finite-dimensional matrix $A_n \in \mathbb{R}^{(2n+1)\times(2n+1)}$. In that case, we use the following notation

$$[\![A_n]\!] := \widetilde{A}_n.$$

The goals of this section is to provide a solution for the following abstract problem. Suppose that $\mathcal{L} : \mathscr{B}(\ell_2) \to \mathscr{B}(\ell_2)$ and $Q \in \mathscr{B}(\ell_2)$ are given and that the operator equation

$$\mathcal{L}(X) = Q,$$

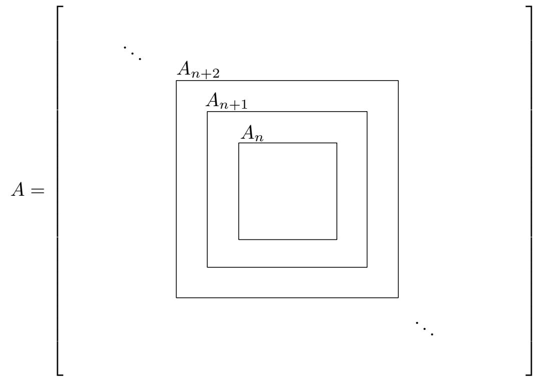

Fig. 2. The finite-section windows.

has a unique solution X. Let $(Q_n)_{n\in\mathbb{N}}$ be a sequence of finite-section approximation for Q and a sequence $(\mathcal{L}_n)_{n\in\mathbb{N}}$ of maps be given. We investigate under what conditions $(X_n)_{n\in\mathbb{N}}$, the solution of the following equations

$$\mathcal{L}_n(X_n) = Q_n,$$

converges in the topology norm to X as $n \to \infty$. In the following subsections, we study this problem for operator Lyapunov and Riccati equations.

B. Approximation Methods for Lyapunov Equation

In the following, we propose an approximation method to the unique solution of a Lyapunov equation. One of the important applications of such approximation is that it can be used to compute a sufficiently accurate estimate of the $\mathcal{H}_2$-norm [9], [10] of an infinite-dimensional system by means of numerical analysis.

Theorem 4.1: Let A be Hurwitz and Q a positive-definite matrix on ℓ_2. Suppose that A_n and Q_n are the finite-section approximations of A and Q, respectively, and that an integer number $n_0 > 0$ exists such that A_n is Hurwitz. Then the following Lyapunov equation

$$A_n^T X_n + X_n A_n + Q_n = 0,$$

has a unique positive-definite solution X_n for all $n \geq n_0$. Furthermore, we have

$$\lim_{n\to\infty} [\![X_n]\!]\phi = X\phi,$$

for all $\phi \in \ell_2$ and X is the unique positive-definite solution of

$$A^T X + XA + Q = 0.$$

Proof: The proof is not included due to space limitations. ■

Intuitively, the result of Theorem 4.1 implies that if the size of the finite-section window is increased (see Fig. 2), the approximation X_n become more accurate.

Remark 4.1: In Theorem 4.1, we assumed that a number $n_0 > 0$ exists such that A_n is Hurwitz for all $n \geq n_0$. This assumption can be relaxed by assuming that A_n and $-A_n$ do not possess any common eigenvalues. The latter assumption means that the Lyapunov equation has a unique solution but it may not have a closed-form. The proof for this case is beyond the scope of this paper and we do not outline it here. We refer to [11] for more details.

C. Approximation Methods for Algebraic Riccati Equation

We consider the following finite-dimensional LQR problem

$$\min_{K_n} \; J_n = \int_0^\infty \left(x_n^T Q_n x_n + u_n^T R_n u_n \right) dt$$

$$\text{subject to:} \;\; \dot{x}_n = A_n x_n + B_n u_n$$
$$u_n = K_n x_n,$$

in which A_n, B_n, Q_n and R_n are the finite-section approximations for linear operators A, B, Q and R, respectively. We know that the optimal state-feedback law is given by

$$K_n = -R_n^{-1} B_n^T P_n,$$

in which P_n satisfies the corresponding algebraic Riccati equation. We will use the following theorem to show that the sequence (K_n) converges strongly to (3).

Theorem 4.2: Let A_n, B_n, Q_n and R_n be the finite-section approximations for A, B, Q and R, respectively. Suppose that a number n_0 exists that the following algebraic Riccati equation

$$A_n^T X_n + X_n A_n - X_n B_n R_n^{-1} B_n^T X_n + Q_n = 0,$$

has a unique positive definite solution X_n for every $n \geq n_0$. Then,

$$\lim_{n\to\infty} [\![X_n]\!]\phi = X\phi,$$

for all $\phi \in \ell_2$ and X is the unique positive definite solution of

$$A^T X + XA - XBR^{-1}B^T X + Q = 0.$$

Proof: We refer the reader to [11] for a proof. ■

One of the consequences of Theorem 4.2 is that

$$\lim_{n\to\infty} [\![K_n]\!]\phi = K\phi.$$

for all $\phi \in \ell_2$.

Remark 4.2: The finite section approximation method is naturally the best choice when we are dealing with SD linear operators. It can be shown that the finite-section approximation for Lyapunov and Riccati equations converge exponentially (or algebraically) fast to the solution of the operator Lyapunov or Riccati equations [11].

V. Spatial Interpolation of Local State-Feedback Control Laws

We focus our attention to the following class of spatially distributed systems

$$\dot{x} \;=\; Ax + Bu, \tag{9}$$

where $A = [a_{ij}]$ is a banded (i.e., finite support) matrix with bandwidth ω, i.e.,

$$a_{ij} = 0 \quad \text{for all} \quad |i-j| > \frac{\omega}{2},$$

and $B = I$. In [1], it is shown that a banded matrix is SD with respect to all coupling characteristic functions.

In this section, we investigate a relationship between the results of Sections III and IV. In Section III by using spatial truncation technique, we implicitly characterized a family of stabilizing state-feedback control laws which have finite supports. In Section IV, we showed that arbitrarily accurate approximations of K can be computed by using the finite-section approximation method. More specifically, we are interested in finding conditions under which finite-support state-feedback control laws can be constructed by using approximate state-feedback control laws K_n.

The state-space matrix A can be decomposed as follows

$$A = \sum_{i \in \mathbb{G}} [\![A_i]\!]_i - [\![A_i^-]\!]_i, \qquad (10)$$

in which

$$[\![A_i]\!]_i = P_{n,i} A P_{n,i},$$

and

$$[\![A_i^-]\!]_i = \sum_{j \in \mathcal{N}_i} [\![A_i]\!]_i \wedge [\![A_j]\!]_j, \qquad (11)$$

for all $i \in \mathbb{G}$. It is reasonable to presume that each subsystem i can only have access to the block elements A_i and A_i^-. This is because in a spatially distributed system, there are physical limitations on each subsystem to observe the dynamics of the entire network. This means that the model of the entire system is locally available to each subsystem. Let $N = 2n+1$ be the width of the finite-section approximation window. In the following, we show that one can slide the approximation window (see Fig. 3) up and down along the main diagonal of A to obtain N-dimensional matrices A_i.

We assume that a stabilizing state-feedback control law K_i can be designed for state-space matrix A_i for all $i \in \mathbb{G}$. Thus, there exist positive definite matrices P_i such that

$$P_i \; (A_i + K_i)^T + (A_i + K_i) \; P_i + Q_i \leq 0, \qquad (12)$$

for every $Q_i > 0$ and all $i \in \mathbb{G}$. The inequality (12) can be also represented equivalently in ℓ_2 as follows

$$[\![P_i]\!]_i \; ([\![A_i]\!]_i + [\![K_i]\!]_i)^T + ([\![A_i]\!]_i + [\![K_i]\!]_i) \; [\![P_i]\!]_i + [\![Q_i]\!]_i \; \leq 0.$$

In the next subsection, we show that under some mild conditions (namely existence of covering Lyapunov matrices) a stabilizing state-feedback control law can be interpolated using local state-feedback control laws K_i for all $i \in \mathbb{G}$.

A. Spatial Interpolation of Local Controllers

In the following, we introduce the notion of covering Lyapunov matrix. Then, it is shown that under this assumption we can characterize a parameterized family of stabilizing state-feedback controllers for system (9).

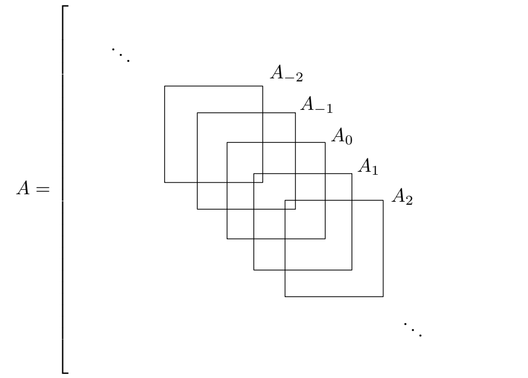

Fig. 3. The state-space matrix representation in terms of its finite-section approximations.

Definition 5.1: A positive-definite matrix P_i is called *covering Lyapunov matrix* if a positive-definite matrix $Q_{ij} \in \mathbb{R}^{n_{ij}}$ exists such that the following inequality holds

$$[\![P_i]\!]_i [\![A_j+K_j]\!]_j^T + [\![A_j+K_j]\!]_j [\![P_i]\!]_i + [\![Q_{ij}]\!]_{\frac{n_{ij}-1}{2}} \; \leq 0, \qquad (13)$$

for all $j \in \mathcal{N}_i$ where $n_{ij} = N + |i-j|$.

Note that $\frac{n_{ij}-1}{2}$ is not an integer number when n_{ij} is an even number. In this situation, for notational purposes we identify the center of the window (which captures Q_{ij}) by the mid-point of the line passing through the points representing indices $n_{ij}-1$ and $n_{ij}+1$.

The following theorem asserts that under the existence of covering Lyapunov matrices assumption, we can identify a parameterized family of stabilizing state-feedback control laws for system (9).

Theorem 5.1: Suppose that P_i satisfies (12) where A_i is defined by (10) for all $i \in \mathbb{G}$. Assume that P_i is a covering Lyapunov matrix for all $i \in \mathbb{G}$. Then the following parameterized family of state-feedback control laws

$$K_N(\lambda) = Y(\lambda) X(\lambda)^{-1}, \qquad (14)$$

in which

$$X(\lambda) = \sum_{i \in \mathbb{G}} \lambda_i \; [\![P_i]\!]_i$$

$$Y(\lambda) = \sum_{i \in \mathbb{G}} \sum_{j \in \mathcal{N}_i} \lambda_j \; \left([\![K_i]\!]_i + [\![A_i^-]\!]_i\right) \; [\![P_j]\!]_j,$$

stabilize the dynamical system (9) for all $\lambda = (\lambda_i) \geq 0$ and $\lambda \in \ell_1$. Moreover, we have

$$X(\lambda)\Big(A + K_N(\lambda)\Big)^T + \Big(A + K_N(\lambda)\Big) X(\lambda) + Q(\lambda) \leq 0, \qquad (15)$$

where

$$Q(\lambda) = \sum_{i \in \mathbb{G}} \sum_{j \in \mathcal{N}_i} \lambda_j \; [\![Q_{ij}]\!]_{\frac{n_{ij}-1}{2}} \; \geq \; 0,$$

Proof: The proof is not included due to space limitations. ■

The above theorem makes an interesting connection between the results of Sections III and IV. This theorem provides us with a procedure with which we can construct stabilizing state-feedback control laws which have finite supports. One can see that $X(\lambda)$ is an invertible banded matrix with bandwidth $4n+1$. The inverse operator $X(\lambda)^{-1}$ is an exponentially decaying operator (see [2] for more details). However, the coefficients λ_i can be chosen in a way that $X(\lambda)^{-1}$ is a matrix with a finite-support.

It is straightforward to check that

$$[\![P_i]\!]_i \; [\![P_j]\!]_j = 0 \quad \text{for all} \quad |i-j| > 2n.$$

For a given $i \in \mathbb{G}$, consider the situation where only the following set of coefficients are nonzero

$$\lambda_k > 0,$$

for all $k \in \mathcal{I}_i$ in which

$$\mathcal{I}_i = \{k \in \mathbb{G} \mid k = i + jN \;\; \text{for all} \;\; j \in \mathbb{G}\},$$

and the rest of the coefficients are set to be zero, i.e.,

$$\lambda_k = 0,$$

for all $k \in \mathbb{G}$ and $k \notin \mathcal{I}_i$. We represent the set of all such coefficient as follow

$$\Lambda_i = \Big\{ \lambda = (\lambda_k) \mid \lambda_k > 0 \;\text{ for }\; k \in \mathcal{I}_i \text{ and } \lambda_k = 0 \;\text{ for }\; k \notin \mathcal{I}_i \Big\}.$$

For every $\lambda \in \Lambda_i$, $X(\lambda)$ is a block-diagonal operator and its inverse is also a block-diagonal operator. Let define

$$\Lambda = \bigcup_{i \in \mathbb{G}} \Lambda_i. \qquad (16)$$

The following result characterizes a parameterized family of stabilizing state-feedback control law for system (9) which have finite supports.

Lemma 5.1: A parameterized family of stabilizing state-feedback control laws for system (9) which have finite supports is characterized by

$$\Xi = \Big\{ K_N(\lambda) \mid \lambda \in \Lambda \Big\},$$

in which $K_N(\lambda)$ is defined by (14).

In Section III, we studied the limit behavior of the sequence of approximate state-feedback control laws, i.e.,

$$\lim_{n \to \infty} [\![K_i]\!]_i \phi = K_{\text{centralized}} \phi$$

for all $\phi \in \ell_2$. We have the following result on the limit behavior of the parameterized controller (14).

Lemma 5.2: The state-feedback control law (14) converges to $K_{\text{centralized}}$ strongly, i.e.

$$\lim_{N \to \infty} K_N(\lambda)\phi = K_{\text{centralized}} \phi,$$

for all $\phi \in \ell_2$ and all $\lambda \geq 0$ and $\lambda \in \ell_1$.

Proof: We refer to [11] for a proof. ■

VI. Conclusion

The finite-section approximation method was introduced. We showed that one can construct a series of approximate finite-dimensional LQR problems that their solutions converge strongly to the infinite-dimensional LQR problem. We proved that under the existence of covering Lyapunov matrices, a parameterized family of stabilizing state-feedback operators can be characterized. Furthermore, we identified the subclass of such stabilizing controllers that have finite supports.

VII. Acknowledgment

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