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Robust Stability of Linear Interval Parameter Matrix Family Problem Revisited with Accurate Representation and Solution

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***Abstract*— This paper revisits the problem of checking the robust stability of matrix families generated by interval parameters in a matrix. Previous research on this topic (including that of this author) *erroneously assumed* that this family can be represented as a standard convex combination of vertex matrices (matrices evaluated at the end points of the interval parameters). Solutions offered to this important problem with this *erroneous assumption* suffered various set backs in the form of counterexamples which caused considerable disillusionment in the research community, especially for this author, warranting continued research on this problem. As a result of this new research, for the first time in the literature, in this paper, explicit expressions for the convex combination coefficients in terms of the interval parameters are derived. These expressions help to clarify and explain the misconceptions that currently exist in the research community about the nature of the convex combination coefficients induced by the interval parameters and shed significant insight into the 'correct' scenario for this case. A previously presented 'vertex algorithm' by the author for this tough problem was derived under the misunderstood mapping of the parameter space to the matrix element space that currently exists in the literature (in the absence of the explicit expressions derived in this paper). Based on the correct mapping presented in this paper, a thorough and correct vertex solution is offered for this tough problem. Several examples are given which clearly demonstrate effectiveness of the new, corrected algorithm.**

I. Introduction

The problem of analyzing the stability of matrix families arises in many applications of systems and control theory [1]. The matrix family that has attracted considerable amount of research over the last two decades is the one arising in the area of linear state space systems with structured real parameters varying within given intervals. Let us start by considering a linear state space system with uncertain interval parameters q_i $(i = 1,2,,..r)$ given by

$$\dot{x}(t) = A(q)x(t) \qquad q \in Q \tag{1}$$

where $x(t) \in R^n$ is the state vector and $q \in R^r$ is a vector of r parameters varying in the prescribed compact set Q. Specifically , let the parameters q_i be given apriori bounds as

$$q_{iL} \quad \leq q_i \leq \;\; q_{iU} \quad i = 1,2,...r \tag{2}$$

In this paper, we restrict our attention to the linear dependent variations q_i in the entries of $A(q)$, and write the matrix $A(q)$ as

$$A(q) \;\; = \;\; A_0 + \sum_{i=1}^{r} q_i A_i \tag{3}$$

where A_0 is the 'nominal' matrix and A_i are constant, specified matrices, reflecting the 'structure' of the perturbation (i.e. reflect the presence of the uncertain parameters q_i in the different elements of A). Thus, the 'nominal' matrix A_0 is the matrix $A(q)$ when the perturbation structure matrices A_i are all zero. In this situation, in the current literature such as [2], it is stated that the set of possible $A(q)$ matrices $[A(q) : q\varepsilon Q]$, labeled as a 'polytope of matrices' in $R^{n\times n}$, can be represented as convex combination of matrices as follows:

Denoting q^i as the i^{th} extreme point of the set Q and the extreme matrix with $A(q^i)$ as A^{vi} , the above matrix family is deemed to be written as a convex combination of these vertex matrices given by

$$\mathscr{A} \;\; = \;\; \left\{ A = \sum_{i=1}^{h} \alpha_i A^{vi}, \alpha_i \geq 0, \sum \alpha_i = 1 \right\}, \tag{4}$$

It is crucial to notice that in the above expression, the convex combination coefficients α_i are not explicitly expressed as a function of the uncertain parameters q_i. Instead, what is guaranteed is that all the matrices in the family are completely captured by the above convex combination. Till now in the literature, in the above convex combination, it was taken that all α_i are ≥ 0 and that they add up to one for any given number of α_is. This representation was accepted since it does capture all the matrices of the matrix family but the absence of functional dependence of these convex combination coefficients in terms of the uncertain parameters q_i was never questioned. Since this functional dependence could hold the key to the answers to many discrepancies occurring in the examples, it was taken up as a topic of further research by this author. As a result of this research, for the first time in the literature, in [3], explicit expressions for these convex combination coefficients are derived in terms of the interval parameters. However in that paper no proofs and elaborate clarifications were offered. In this paper, we provide these expressions with proofs and more clarifications. The main insight provided by these explicit expressions is that the convex combination nature exists only for combinations with number of vertex matrices

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being equal to exponent of two. This is a significant observation which has profound influence on the final solution offered. In other words,the most important feature of this 'structured' convex combination is that these 'vertex' matrices A^{vi} are all formed and predetermined by the structure of the uncertainty and are thus interrelated and that the property of these coefficients adding up to one (which is the property that makes these nonnegative coefficients labeled as *convex* combination coefficients) happens only when the number of these coefficients is an exponent of two. In other words, the convex combination happens only for number of these coefficients being two (one parameter varying at a time), four (two parameters varying at a time), eight (three parameters varying at a time) and and so on.

Also what is not discussed in the literature concerning these convex combination representation is the nature of the vertex matrices and the linear (in)dependence among the columns of these matrices. Note that in a structured convex combination arising out of the interval parameter nature, the interrelationship between the 'vertex' matrices is clearly brought out with a 'linear (in)dependence' test as follows: Consider a convex combination of h vertex matrices, with $h = 2^r$, with $r = 1,2,3...$ Since the vertex matrices are interrelated it turns out that we always find linear dependency of these column vectors of these vertex matrices. When there are h vertex matrices of order n, then the h by n matrix consisting of the first columns of the above summation matrix set is always rank deficient indicating linear dependency among the column vectors. Similar situation occurs for the second, third and so on for all the n column vectors of this matrix set. Thus if we denote the h by n matrix formed with column i vectors of this h matrix set as S^i, for each i we have $Rank[S^i] < Min(h,n)$. This is the characteristic feature of a 'structured convex combination' arising out of 'interval parameter' matrices.

Assuming these 'vertex' matrices are Hurwitz stable, the issue of research is to analyze if every matrix within this family is also Hurwitz stable or not. Very informative accounts of various aspects of this research are summarized in [1], [4], [5]. Note that the stability of the 'vertex' matrices is not sufficient to guarantee stability of the entire matrix family. The author recently presented a necessary and sufficient vertex algorithm in terms of new set of matrices formed at the 'vertices' of the parameter space as a solution to the above linear interval parameter case in the journal publication [6], without realizing that α_is add up to one only when the number of α_is is equal to an exponent of two. Also in the subsequent papers,[7], [8], the mapping between the parameter space and matrix element space was not clearly understood as the current literature assumed convex combination nature to exist for any given number of coefficients. This resulted in some 'discrepancy' in the satisfaction of the necessary and sufficient conditions for some examples. In this paper, we present the 'final' version of this vertex solution with complete clarifications. With this backdrop, the paper is organized as follows. In the next section, we carry out the most important step of deriving the explicit expressions for the convex combination coefficients in terms of the interval parameters, which in turn clarifies the misconceptions that exist in the current literature with respect to this mapping. Then in section III, we revisit the strategy of converting the stability problem to that of checking nonsingularity via the 'Kronecker Lyapunov' matrix space, along with the preliminaries needed to state all the upcoming theorems. In section IV, we present the final version of the necessary and sufficient vertex solution for 'interval parameter matrix families', with the new insight provided by the correct mapping. Then in section V, few examples are presented illustrating the final result. Finally in section VI, some concluding remarks are presented along with future directions of research.

II. Problem Formulations: Interval Parameter Matrix Familes:

A. Single Parameter (i.e. Two Vertex) Case:

Let us consider an interval parameter matrix family with only a single interval parameter as follows:

$$\dot{x}(t) = A(q)x(t) \qquad q \in Q \tag{5}$$

where

$$A(q) \quad = \quad A_0 + q_1 A_1 \tag{6}$$

with

$$q_{1L} \quad \leq q_1 \leq \quad q_{1U} \tag{7}$$

Then we get two vertex matrices, and the convex combination is given by

$$\mathscr{A} \quad = \quad \left\{A = \alpha_1 A^{v1} + \alpha_2 A^{v2}, \alpha_i \geq 0, \sum \alpha_i = 1\right\}, \tag{8}$$

In this case, it may be noted that $\alpha_2 = 1 - \alpha_1$ (or $\alpha_1 = 1 - \alpha_2$) and thus there is only one freely varying coefficient.

1) Explicit Expression of αs in terms of interval parameter q:: It can be shown that, for the single parameter case, the relationship between the convex combination coefficients and the interval parameters is given as follows:

$$\alpha_1 = (q_{1U} - q_1)/(q_{1U} - q_{1L}) \tag{9}$$

$$\alpha_2 = (q_1 - q_{1L})/(q_{1U} - q_{1L}) \tag{10}$$

Here, the vertex matrices A^{v1},and A^{v2} are given by

$$A^{v1} = A_0 + q_{1L} A_1 \tag{11}$$

$$A^{v2} = A_0 + q_{1U} A_1 \tag{12}$$

Notice that the coefficient α_1 which has q_{1U} in the numerator is multiplying A^{v1} which has q_{1L} in it and vice versa. In other words, which coefficients are multiplying which vertex matrices is important.

B. Multiple Parameter Case: r interval parameters:

Next, we consider the case of multiple interval parameters, say r parameters. Thus we have

$$\dot{x}(t) = A(q)x(t) \qquad q \in Q \tag{13}$$

where

$$A(q) \quad = \quad A_0 + \sum_{i=1}^{r} q_i A_i \tag{14}$$

with

$$q_{iL} \quad \leq q_i \leq \; q_{iU} \quad i = 1, 2, \ldots r \tag{15}$$

Then, for this case, we obtain a convex combination of matrices given by

$$\mathscr{A} \quad = \quad \left\{ A = \sum_{i=1}^{h} \alpha_i(q_i) A^{vi}, \sum \alpha_i = 1 \right\}, \tag{16}$$

In the current literature, it is stated that $\alpha_i \geq 0$ and that this matrix family is a 'polytope of matrices' in $R^{n \times n}$(see [2]).

However, *if we derive the explicit expressions for these convex combination coefficients in terms of the interval parameters, it turns out that these α_i add up to one whenever the number of α_is is equal to 2^r where $r = 1, 2, 3\ldots$ and that too with some restrictions on the allowable summations.* Thus a more accurate way of representing this matrix family is as follows :

Denoting q^i as the i^{th} extreme point of the set Q and the extreme matrix with $A(q^i)$ as A^{vi} , we can write the above matrix family as

$$\mathscr{A} \quad = \quad \left\{ A = \sum_{i=1}^{h} \alpha_i A^{vi}, \sum_{i=1}^{c} \alpha_i = 1, , c = 2^k, k = 1, \ldots r \right\} \tag{17}$$

where $0 \leq \alpha_i \leq 1$ and with only specific designated α_is adding up to one. This can be seen if we express the convex combination coefficients as a function of the interval parameters. In fact, explicit expressions for these convex combination coefficients in terms of the interval parameters are available with the author and space considerations preclude presenting those long expressions for the general case but are now shown here for a two parameter case. Let us denote the four 'vertex matrices' as follows, where the nomenclature for the indices of the vertex matrices follows the pictorial representation of the mapping given in Figure 1.

$$A^{v1} = A_0 + q_{1L}A_1 + q_{2L}A_2 \tag{18}$$

$$A^{v2} = A_0 + q_{1L}A_1 + q_{2U}A_2 \tag{19}$$

$$A^{v3} = A_0 + q_{1U}A_1 + q_{2L}A_2 \tag{20}$$

$$A^{v4} = A_0 + q_{1U}A_1 + q_{2U}A_2 \tag{21}$$

Then if we denote the convex combination coefficients α_i as the coefficients multiplying the vertex matrix A^{vi}, then the explicit expressions for those coefficients are as follows:

$$\alpha_1 = \frac{(q_{1U} - q_1)(q_{2U} - q_2)}{(q_{1U} - q_{1L})(q_{2U} - q_{2L})} \tag{22}$$

$$\alpha_2 = \frac{(q_{1U} - q_1)(q_2 - q_{2L})}{(q_{1U} - q_{1L})(q_{2U} - q_{2L})} \tag{23}$$

$$\alpha_3 = \frac{(q_1 - q_{1L})(q_{2U} - q_2)}{(q_{1U} - q_{1L})(q_{2U} - q_{2L})} \tag{24}$$

$$\alpha_4 = \frac{(q_1 - q_{1L})(q_2 - q_{2L})}{(q_{1U} - q_{1L})(q_{2U} - q_{2L})} \tag{25}$$

It interesting and important to notice the specific association of each convex combination coefficient with the appropriate vertex matrix and also the nature of that specific association. In other words, notice that α_1 which has q_{1U} and q_{2U} in the numerator is multiplying the vertex matrix formed by q_{1L} and q_{2L} whereas α_4 which has q_{1L} and q_{2L} in the numerator is multiplying the vertex matrix formed by q_{1U} and q_{2U}. This type of complementary or opposite nature follows for the other two coefficients and vertex matrices as well. In fact, it is this realization that was lacking in the previous descriptions and caused confusion in the proof of the solution. Another critical point to emphasize is that, in the above description of the convex combination coefficients, we have

$$\alpha_1 + \alpha_2 = 1 \tag{26}$$

$$\alpha_2 + \alpha_4 = 1 \tag{27}$$

$$\alpha_4 + \alpha_3 = 1 \tag{28}$$

$$\alpha_3 + \alpha_1 = 1 \tag{29}$$

which are obtained by making only one parameter vary at a time (i.e. edges in the parameter space, i.e by making one of the parameters take on its extreme value) and

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1 \tag{30}$$

which is obtained by making both parameters vary simultaneously. Note that, in this representation,

$$\alpha_1 + \alpha_4 \neq 1 \tag{31}$$

$$\alpha_2 + \alpha_3 \neq 1 \tag{32}$$

$$\alpha_1 + \alpha_2 + \alpha_3 \neq 1 \tag{33}$$

$$\alpha_1 + \alpha_3 + \alpha_4 \neq 1 \tag{34}$$

$$\alpha_2 + \alpha_3 + \alpha_4 \neq 1 \tag{35}$$

$$\alpha_1 + \alpha_2 + \alpha_4 \neq 1 \tag{36}$$

1) An alternative, differential parameter representation of the matrix family: In this representation, which we call 'differential parameter' representation, given the interval ranges as above, we can take the 'center' value (average value) of each of these interval parameters and evaluate the matrix at these 'center' values of the parameters and denote that matrix as the 'center' matrix. Then the resulting 'differential parameter' interval range can be described as

$$-\overline{\delta q_i} \quad \leq \delta q_i = 0 \leq +\overline{\delta q_i} \quad i = 1, 2, \ldots r \tag{37}$$

Then the vertex matrices can be denoted, for a two parameter case, as

$$A^{v1} = A_c - \overline{\delta q_1} A_1 - \overline{\delta q_2} A_2 \tag{38}$$
$$A^{v2} = A_c - \overline{\delta q_1} A_1 + \overline{\delta q_2} A_2 \tag{39}$$
$$A^{v3} = A_c + \overline{\delta q_1} A_1 - \overline{\delta q_2} A_2 \tag{40}$$
$$A^{v4} = A_c + \overline{\delta q_1} A_1 + \overline{\delta q_2} A_2 \tag{41}$$

The advantage with representation is that the parameter space rectangle can have symmetric bounds within the interval ranges with 'center' parameter serving as the zero value for the 'differential parameter' δq_i. Thus the 'rectangle' of the 'differential parameter' space is symmetric. For this situation, the explicit expressions for the convex combination coefficients are given by

$$\alpha_1 = \frac{(\overline{\delta q_1} + \delta q_1)(\overline{\delta q_2} + \delta q_2)}{2(\overline{\delta q_1} \cdot \overline{\delta q_2})} \tag{42}$$
$$\alpha_2 = \frac{(\overline{\delta q_1} + \delta q_1)(\overline{\delta q_2} - \delta q_2)}{2(\overline{\delta q_1} \cdot \overline{\delta q_2})} \tag{43}$$
$$\alpha_3 = \frac{(\overline{\delta q_1} - \delta q_1)(\overline{\delta q_2} + \delta q_2)}{2(\overline{\delta q_1} \cdot \overline{\delta q_2})} \tag{44}$$
$$\alpha_4 = \frac{(\overline{\delta q_1} - \delta q_1)(\overline{\delta q_2} - \delta q_2)}{2(\overline{\delta q_1} \cdot \overline{\delta q_2})} \tag{45}$$

Note that both representations above are equivalent and for a given set of interval ranges produce the same vertex matrices and same convex combination coefficient values.

Thus the most important feature of the above convex combination representation is that the convex combination coefficients add up to one only when the number of those α coefficients is equal to an exponent of two. It is important to note that one vertex cannot be represented as a convex combination of other vertex matrices. However, it is interesting to realize that one vertex matrix can be expressed as a *linear* combination of the other vertex matrices.For example, it can be easily seen that

$$A^{v1} = A^{v2} + A^{v3} - A^{v4} \tag{46}$$
$$A^{v2} = A^{v1} - A^{v3} + A^{v4} \tag{47}$$
$$A^{v3} = A^{v1} + A^{v4} - A^{v2} \tag{48}$$
$$A^{v4} = A^{v2} + A^{v3} - A^{v1} \tag{49}$$

Recall the pictorial representation of the mapping from the parameter space rectangle to the matrix element space rectangle for the case of an 'interval matrix' with two varying elements, depicted in Figure 1. Unfortunately, all these years, with the absence of these explicit expressions derived for the convex combination coefficients of this paper, the fact that the convexity nature exists only when the number of coefficients take on a value of exponent of two was never recognized! The above 'correct' interpretation explains the reason why the 'vertex' algorithm presented by the author under the assumption of a general convex combination of matrices [6], [9] requires slight modification,and in the next few sections, this modification of the vertex algorithm for checking the robust stability of this matrix family is presented. Since the 'vertex' matrices A^{vi} are all formed and predetermined by the structure of the uncertainty and are thus interrelated imparting a special structure to them, from now on, we continue to label this convex combination as '*Structured*' convex combination.

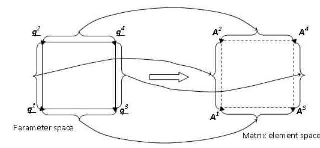

Fig. 1. Mapping from parameter space to matrix element space

III. Stability Problem as a Nonsingularity Problem via the 'Kronecker Lyapunov' Matrix:

It is known from references [10], [11], [12], [13] that the stability assessment problems posed in the introduction for both formulations can be converted to a nonsingularity problem involving Kronecker based matrix operations.The above cited literature presents these conditions in terms of three matrices (each of which employs different Kronecker based operations) namely: (i) Kronecker Sum matrix (denoted as K matrix) (ii) Lyapunov matrix (later in this paper labeled as 'Kronecker Lyapunov' matrix to distinguish it from the standard and familiar Lyapunov matrix equation solution) denoted by L and iii) 'Bialternate Sum' matrix, denoted by B matrix. In this research, we consider only the second of these matrices i.e. the 'Kronecker Lyapunov' matrix denoted by L i.e.

$$L \quad = \quad A^{\dagger} = A \times I_n + I_n \times A \tag{50}$$

where '$\times$' denotes an operation similar to the Kronecker Sum (see Jury [12]) for details). In order to conserve notation, henceforth, we will label the matrix given in (50) as the 'Kronecker Lyapunov' matrix and denote the matrix operation as 'dagger' operation. Note that L is a square matrix of dimension $m = \frac{1}{2}n(n+1)$,whose eigenvalues are the pairwthe pairwise summation of the eigenvalues of A. In Tesi and Vicino [14] and in Jury [12] , simple computer amenable methodologies are given to form L matrix from the given matrix A.

Example 1: For n=2,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

with μ_1 and μ_2 as eigenvalues, the Kronecker Lyapunov matrix L is given by

$$L = \begin{bmatrix} 2a_{11} & 2a_{12} & 0 \\ a_{21} & a_{11} + a_{22} & a_{12} \\ 0 & 2a_{21} & 2a_{22} \end{bmatrix}$$

with eigenvalues $\mu_1 + \mu_2$, $2\mu_1$ and $2\mu_2$.

Note that there is an alternative form for the above L matrix where the element a_{ij} could be interchanged with the element a_{ji}.

Mathematical Preliminaries Related to 'Kronecker Lyapunov' Operation:

- **Property 1:**
  For two square matrices A_1 and A_2,
  $$(k_1A_1+k_2A_2)^{\dagger} \quad = \quad k_1{A_1}^{\dagger}+k_2{A_2}^{\dagger} \tag{51}$$
  where k_1 and k_2 are scalars.

We **define** the vertex matrices L^{vi} as follows:
$$L^{vi}=(A^{vi})^{\dagger}. \tag{52}$$

Machinery and Concepts Needed to State the Main Theorems:

In this section we present the necessary concepts needed to set the stage to state the 'Theorems' of this paper. The following holds for any $h \geq 2$.

'Virtual Center' matrix:: Let $L_{vc,h}$ denote the 'virtual center' matrix formed with all the h vertex matrices L^{v1} ,L^{v2}..L^{vh} taken together at a time given by
$$L_{vc,h} \quad = \quad (L^{v1}+L^{v2}+L^{v3}+...L^{vh}) \tag{53}$$
The corresponding 'virtual center'(or 'summation') matrix $A_{vc,h}$ can also be easily defined for the original matrix space.

A Necessary Condition For Stability: All the matrices belonging to the given matrix family are Hurwitz stable **only if**
the 'virtual center' matrix $A_{vc,h}$ is Hurwitz stable.

So in the rest of the statements of the main theorems, we assume that this necessary condition is satisfied.

In order to state the theorems in later sections, we need a set of '**specific'** matrices labelled 'Kronecker Nonsingularity Matrices' as follows:

'Kronecker Nonsingularity' Matrices: These special matrices are 'specific' matrices given by
$$L(ns;h;j)=-[(L_(vc;h))^{-1}L^{vj}] \tag{54}$$
$$(j=1,2,..h) \tag{55}$$
For example, in a 4 vertex case, $L(ns;4;1)$ denotes the matrix $-[(L^{v1}+L^{v2}+L^{v3}+L^{v4})^{-1}L^{v1}]$ and $L(ns;4;2)$ denotes the matrix $-[(L^{v1}+L^{v2}+L^{v3}+L^{v4})^{-1}L^{v2}]$. Note that there are h KN matrices. For the two vertex case, the two KN matrices are given by $-[(L^{v1}+L^{v2})^{-1}L^{v1}]$ and $-[(L^{v1}+L^{v2})^{-1}L^{v2}]$. Also note that in the 4 vertex case, there are 8 KN matrices with two vertex matrices taken at a time (essentially those associated with 'exposed edges').

- 'Real Axis Stability': We say a matrix is Real Axis Stable if its real eigenvalues are all negative.
- **Another Necessary Condition for Robust Stability of the matrix family** [6]: The matrix family is stable only if all the KN matrices are real axis stable.
- **A Sufficient Condition for Robust Stability of the matrix family**: The matrix family is stable if all the KN matrices are Hurwitz stable.

Proofs are omitted in this version of the paper but they follow the lines of proofs given in [6].

IV. Necessary and Sufficient Vertex Solutions for Checking the Robust Stability of Interval Parameter Matrix Families:

A. *Theorem for Single parameter (Two Vertex) Case:*

Theorem 4.1: All the matrices belonging to the convex combination matrix family (with vertex matrices A^{v1} and A^{v2} and the 'virtual center' matrix $A^{v1}+A^{v2}$ being Hurwitz stable) are Hurwitz stable
if and only if
the two 'Kronecker Nonsingularity Matrices' (KN matrices), namely
$$-[(L^{v1}+L^{v2})^{-1}L^{v1}]\ and \tag{56}$$
$$-[(L^{v1}+L^{v2})^{-1}L^{v2}] \tag{57}$$
are Real Axis Stable.
Proof: As in [6].
Next, we switch our attention to the multiple parameter case. To state the main theorem for this case, we introduce the concept of 'virtual stability'. Since the correct convex combination representation holds for only the cases when the number of coefficients is equal to an exponent of two, and each of the convex combination coefficients is nonnegative, it turns out that the complete Hurwitz stability of KN matrices with four taken at a time is not a necessity any more. Instead,what is necessary is 'Virtual Stability'. Let us elaborate on this 'virtual stability' concept. In the two parameter case, the number of convex combination coefficients is four but as each of these coefficients is nonnegative, and the three coefficients taken at a time is not a convex combination any more (but only a positive real combination), there is slack in the KN matrix eigenvalue distribution and this slack is manifested as positive real parts in the complex eigenvalues of the KN matrices. In the case this happens, the dilemma would be to decide how much slack or how much positive real part can be tolerated in these complex eigenvalues of these KN matrices with the matrix family could still be hurwitz stable. We say that the KN matrix is 'Virtually stable' if the sum of the positive real parts of a complex conjugate pair added is less than a 'coupling bound', κ_s. In this paper, finally we present a correct threshold for this bound κ_s which interestingly gets determined by the nature and explicit expressions we derived for the convex combination coefficients before, which in turn explicitly bring in the size of the rectangle in the parameter space being considered! This insight was missing in the previous solutions presented by this author. In this paper, for brevity, this threshold for the bound κ_s is presented only for the two parameter case and not for the general r parameter case. This threshold value for the bound for the two parameter case is $\kappa_s=1/[3(q_{1U}-q_{1L})(q_{2U}-q_{2L})]$. Note that the product $[(q_{1U}-q_{1L})(q_{2U}-q_{2L})]$ amounts to the area of the parameter space rectangle. For brevity, we write $\kappa_s = 1/(3{\times}Area)$. Again for brevity a formal proof of this bound is not presented here. It suffices to note that the number $1/3$ appears in the bound expression because when four α_i are present, when one of the coefficients is very close to zero,

only three coefficients add up to close to one but because three coefficients adding up does not represent a convex combination situation, each of the α_is can accommodate a maximum of $1/3$ and this slack appears as a positive real part in KN matrix eigenvalue distribution. In previous papers [8], this bound was derived to be $1/h$ which unfortunately now turns out to be incorrect because it was derived under the lack of insight about the nature of convex combination being considered. An eigenvalue distribution in which the summation of all the positive real parts of the complex eigenvalues of KN matrices is less than this bound κ_s is lebeled as 'practical stability'. Let us now label the concepts of 'practical stability' and 'hurwitz stability' together as 'Virtual Stability'. That is, 'Virtual stability' includes both 'practical stability' as well as 'hurwitz stability'.

B. Theorem for the Two Parameter Case:

Now we state the main result of this paper specializing it to the two parameter case to make the theorem more coherent and understandable.

Theorem 4.2: All the matrices belonging to the convex combination matrix family (with four vertex matrices A^{vi} and the 'virtual center' matrix $A^{v1}+A^{v2}+A^{v3}+A^{v4}$ being Hurwitz stable) are Hurwitz stable

if and only if

the 8 exposed 'Kronecker Nonsingularity Matrices' (KN matrices) of two vertex matrices taken at a time, namely $-[(L^{v1}+L^{v2})^{-1}L^{v1}]$ and $-[(L^{v1}+L^{v2})^{-1}L^{v2}]$;$-[(L^{v2}+L^{v4})^{-1}L^{v2}]$ and $-[(L^{v2}+L^{v4})^{-1}L^{v4}]$;$-[(L^{v4}+L^{v3})^{-1}L^{v4}]$ and $-[(L^{v4}+L^{v3})^{-1}L^{v3}]$;$-[(L^{v3}+L^{v1})^{-1}L^{v3}]$ and $-[(L^{v3}+L^{v1})^{-1}L^{v1}]$ are 'Real Axis stable' and the 4 'Kronecker Nonsingularity Matrices' (KN matrices) of all four vertex matrices taken at a time, namely $-[(L^{v1}+L^{v2}+L^{v3}+L^{v4})^{-1}L^{v1}]$; $-[(L^{v1}+L^{v2}+L^{v3}+L^{v4})^{-1}L^{v2}]$; $-[(L^{v1}+L^{v2}+L^{v3}+L^{v4})^{-1}L^{v3}]$; $-[(L^{v1}+L^{v2}+L^{v3}+L^{v4})^{-1}L^{v4}]$ are all 'Virtually stable'.

Proof: It is essentially in the lines of the proof available in [6]. However, here we discuss some salient points of that proof. In [6], it is shown that, because of the special nature of the dagger space matrices, in the linear domain of dagger space (i.e. addition of matrices), nonsingularity and stability are equivalent. In other words, the real parts of complex eigenvalues and the real eigenvalues are coupled and for nonsingularity (i.e. stability) the real parts are required to behave the same way as the real eigenvalues. Then the necessity of stability of the product domain KN matrices is established based on the generalized eigenvalue problem of the 'virtual ray' matrix $L_{vc}+\rho L^i$, where the positive scalar variable ρ varies within the open interval $(0,\infty)$.

V. Illustrative Example:

Example 1: Let us consider an 'interval matrix' with two entries (a_{11} and a_{44}) varying in an interval as follows: $-1.5026 \le a_{11} \le -0.5026$ and $-4.0026 \le a_{44} \le -1.5026$. This gives rise to four vertex matrices given by

$$A^1 = \begin{bmatrix} -1.5026 & -12.06 & -0.06 & 0 \\ -0.25 & -0.0329 & 1.0 & 0.5 \\ 0.25 & -4.0 & -1.0329 & 0 \\ 0 & 0.5 & 0 & -4.0026 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -1.5026 & -12.06 & -0.06 & 0 \\ -0.25 & -0.0329 & 1.0 & 0.5 \\ 0.25 & -4.0 & -1.0329 & 0 \\ 0 & 0.5 & 0 & -1.5026 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -0.5026 & -12.06 & -0.06 & 0 \\ -0.25 & -0.0329 & 1.0 & 0.5 \\ 0.25 & -4.0 & -1.0329 & 0 \\ 0 & 0.5 & 0 & -4.0026 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -0.5026 & -12.06 & -0.06 & 0 \\ -0.25 & -0.0329 & 1.0 & 0.5 \\ 0.25 & -4.0 & -1.0329 & 0 \\ 0 & 0.5 & 0 & -1.5026 \end{bmatrix}$$

Note that the absolute parameter ranges are 1 and 2.5 making the Area of the parameter space rectangle equal to 2.5 and thus *the κ_s bound for virtual stability testing is 0.13333.* A brute force simulation shows the region of stability/instability in the parameter space as shown in Fig 2. The region formed by the convex combination of the above vertex matrices is labeled as c1d1c26d26. Also we know that this interval family is formed due to the variation of the elements $A(1,1)=a_{11}$ and $A(4,4)=a_{44}$ of the original $A(q)$ matrix. In the figure the a_{11} variation is marked along the Y-axis and the a_{44} variation is marked along the X-axis. So we can denote a member matrix in the region as an ordered pair (a_{11},a_{44}). So by the region c1d1c26d26we mean the rectangle in the parameter space formed by the points $c1(-1.5026,-4.0026)$, $c26(-1.5026,-1.5026)$, $d26(-0.5026,-1.5026)$ and $d1(-0.5026,-4.0026)$. Let us denote The Kronecker Nonsingularity matrices, as KNL_i i=1,2,3,4 . The eigenvalue distribution of the KN matrices corresponding to two exposed edge vertex matrices taken at a time indicate that they are all real axis stable. For brevity that distribution is not reported here. Now let us consider the case of the KN matrices formed with all four vertices taken at a time. The eigenvalue distribution of the KN matrices are given on Table 1 where $zkn1$ denote the eigenvalues of KNL_1 matrix and similarly for the other matrices.

$zkn1$	$zkn2$	$zkn3$	$zkn4$
-0.6812+0.6012i	-0.1918+0.7495i	0.1812-0.6012i	-0.3082-0.7495i
-0.6812-0.6012i	-0.1918-0.7495i	0.1812+0.6012i	-0.3082+0.7495i
-0.3296+0.0335i	-0.3759	-0.1704-0.0335i	-0.1241
-0.3296-0.0335i	-0.1835+0.0682i	-0.1704+0.0335i	-0.3165-0.0682i
-0.3768	-0.1835-0.0682i	-0.1232	-0.3165+0.0682i
-0.3752	-0.2073	-0.1248	-0.2927
-0.3655	-0.1337	-0.1345	-0.3663
-0.2500	-0.2500	-0.2500	-0.2500
-0.2500	-0.2500	-0.2500	-0.2500
-0.2500	-0.2500	-0.2500	-0.2500

TABLE I
EIGENVALUE DISTRIBUTION OF KN MATRICES FOR REGION C1D1C26D26

This instability in the interior of the parameter space is verified by the theorem in the paper as one of the KN matrices is unstable with a complex conjugate pair with positive real parts adding upto 0.3625 (0.18125×2) which is greater than $\kappa_s = 0.1334$.

To test the proposed necessary and sufficient vertex solution algorithm with the virtual stability concept, we now deliberately view the same above rectangle in the parameter space as divided into several sub rectangles of different size , as shown in Fig 2, and observe the eigenvalue distribution of the corresponding KN matrices. Few of those are presented here which corroborate the successful satisfaction of the new theorem.

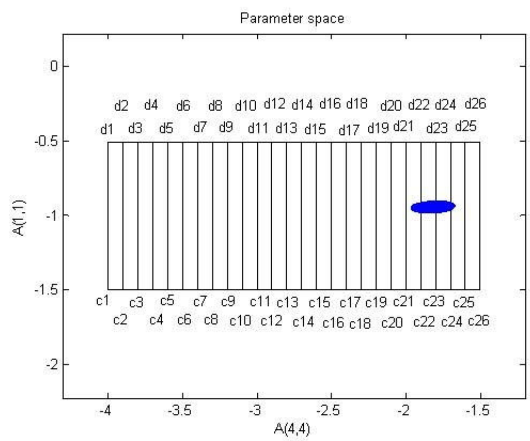

Fig. 2. Stability/Instability in Parameter Space

As shown in the figure, the region c1c26d26d1 was divided into 25 narrow strips along the horizontal axis with each strip width being equal to 0.1. Note that the vertical length of each strip is 1. Let us now investigate the application of the proposed vertex solution of this paper to the various sub rectangle regions in the parameter space. For brevity, let us consider some specific sub rectangle regions, denoted by c1c2d2d1, c6c7d7d6, c1c19d19d1, c1c20d20d1, c1 c21d21d1, c1c22d22d1, c1d23d23d1, c1c24d24d1, c1c25d25d1. It turns out that as expected, for regions c1c22d22d1, c1d23d23d1, c1c24d24d1, the KN matrices with two vertex matrices taken at a time are real axis unstable thereby not even requiring to go for the KN matrix distribution of all four vertices taken at a time. So in what follows, we present the eigenvalue distribution of the KN matrices with all four vertices taken at a time, where it is verified (but not reported here) that for all these cases, the KN matrices with two vertices taken at a time turn out to be real axis stable. Specifically Table 2 gives the KN matrix eigenvalue distribution for region c1c2d2d1; Table 3 gives for region c6c7d7d6; Table 4 gives for region c1c19d19d1,Table 5 for region c1c20d20d1; Table 6 gives for region c1c21d21d1; and finally Table 7 for region c1c25d25d1. The results in Tables 2,3,4 are as expected. The theorem verifies the fact that these regions are stable family regions. Consider the results in Table 5. There is a KN matrix with positive real part eigenvalues. So we need to apply the virtual stability criterion. The sum of the positive real parts is 0.0166 which is less than the coupling bound κ_s given by 0.1754 for this region, rendering this KN matrix virtually stable. Thus the theorem correctly predicts that this region is a stable region. Next let us look at the results in Table 6. Again there is a KN matrix with positive real parts whose sum is equal to 0.0592 but is less than the bound 0.167 for this region, again rendering it virtually stable. Thus the theorem verifies that this region is indeed a stable region. Finally, let us look at Table 7. There is a KN matrix with positive real part whose real part sum is equal to 0.2864 but is greater than the bound 0.139 for this region, rendering this KN matrix virtually unstable. Thus the new vertex solution theorem verifies that this region contains some unstable matrices which is indeed the case. Similar computational experiments were carried out on the above matrix family with different parameter space regions and all these computational experiments validate the proposed theorem given in terms of virtual stability of this paper.

$zkn1$	$zkn2$	$zkn3$	$zkn4$
-0.3291+0.4296i	-0.3229+0.4300i	-0.1709-0.4296i	-0.1771-0.4300i
-0.3291-0.4296i	-0.3229-0.4300i	-0.1709+0.4296i	-0.1771+0.4300i
-0.3759	-0.3759	-0.1241	-0.1241
-0.2807	-0.2748	-0.2193	-0.2252
-0.2524+0.0010i	-0.2476-0.0011i	-0.2476-0.0010i	-0.2524+0.0011i
-0.2524-0.0010i	-0.2476+0.0011i	-0.2476+0.0010i	-0.2524-0.0011i
-0.2532	-0.2468	-0.2468	-0.2532
-0.2500	-0.2500	-0.2500	-0.2500
-0.2500	-0.2500	-0.2500	-0.2500
-0.2500	-0.2500	-0.2500	-0.2500

TABLE II

EIGENVALUE DISTRIBUTION OF KN MATRICES FOR REGION C1C2D2D1

$zkn1$	$zkn2$	$zkn3$	$zkn4$
-0.3503+0.4977i	-0.3412-0.4989i	-0.1497-0.4977i	-0.1588+0.4989i
-0.3503-0.4977i	-0.3412+0.4989i	-0.1497+0.4977i	-0.1588-0.4989i
-0.3759	-0.3759	-0.1241	-0.1241
-0.2847	-0.2781	-0.2153	-0.2219
-0.2526+0.0013i	-0.2474-0.0013i	-0.2474-0.0013i	-0.2526+0.0013i
-0.2526-0.0013i	-0.2474+0.0013i	-0.2474+0.0013i	-0.2526-0.0013i
-0.2537	-0.2463	-0.2463	-0.2537
-0.2500	-0.2500	-0.2500	-0.2500
-0.2500	-0.2500	-0.2500	-0.2500
-0.2500	-0.2500	-0.2500	-0.2500

TABLE III

EIGENVALUE DISTRIBUTION OF KN MATRICES FOR REGION C6C7D7D6

VI. CONCLUSIONS

This paper presents a 'vertex solution' to the problem of checking the stability of families of matrices described by interval parameters, which in turn produce convex combinations of Hurwitz stable 'vertex' matrices. In this paper,for the first time in the literature,explicit expressions for the convex combination coefficients in terms of the interval parameters are derived. These expressions help to clarify and explain the misconceptions that currently exist in the research community about the nature of the convex combination

*zkn*1	*zkn*2	*zkn*3	*zkn*4
-0.4891+0.5415i	-0.2571-0.5867i	-0.0109-0.5415i	-0.2429+0.5867i
-0.4891-0.5415i	-0.2571+0.5867i	-0.0109+0.5415i	-0.2429-0.5867i
-0.3760	-0.3759	-0.2045-0.1240	-0.1241
-0.3467	-0.2004+0.0387i	-0.1533	-0.2996-0.0387i
-0.3019+0.0205i	-0.2004-0.0387i	-0.1981-0.0205i	-0.2996+0.0387i
-0.3019-0.0205i	-0.2279	-0.1981+0.0205i	-0.2721
-0.3240	-0.1759	-0.1760	-0.3241
-0.2500	-0.2500	-0.2500	-0.2500
-0.2500	-0.2500	-0.2500	-0.2500
-0.2500	-0.2500	-0.2500	-0.2500

TABLE IV

EIGENVALUE DISTRIBUTION OF KN MATRICES FOR REGION C1C19D19D1

*zkn*1	*zkn*2	*zkn*3	*zkn*4
-0.5083+0.5499i	-0.2497+0.6034i	0.0083-0.5499i	-0.2503-0.6034i
-0.5083-0.5499i	-0.2497-0.6034i	0.0083+0.5499i	-0.2503+0.6034i
-0.3761	-0.3759	-0.1239	-0.1241
-0.3055+0.0220i	-0.1976+0.0424i	-0.1945-0.0220i	-0.3024-0.0424i
-0.3055-0.0220i	-0.1976-0.0424i	-0.1945+0.0220i	-0.3024+0.0424i
-0.3510	-0.2251	-0.1490	-0.2749
-0.3294	-0.1705	-0.1706	-0.3295
-0.2500	-0.2500	-0.2500	-0.2500
-0.2500	-0.2500	-0.2500	-0.2500
-0.2500	-0.2500	-0.2500	-0.2500

TABLE V

EIGENVALUE DISTRIBUTION OF KN MATRICES FOR REGION C1C20D20D1

*zkn*1	*zkn*2	*zkn*3	*zkn*4
-0.5296+0.5585i	-0.2418+0.6218i	0.0296-0.5585i	-0.2582-0.6218i
-0.5296-0.5585i	-0.2418-0.6218i	0.0296+0.5585i	-0.2582+0.6218i
-0.3092+0.0236i	-0.3759	-0.1908-0.0236i	-0.1241
-0.3092-0.0236i	-0.1948+0.0462i	-0.1908+0.0236i	-0.3052-0.0462i
-0.3761	-0.1948-0.0462i	-0.2063-0.1239	-0.3052+0.0462i
-0.3553	-0.2222	-0.1447	-0.2778
-0.3350	-0.1649	-0.1650	-0.3351
-0.2500	-0.2500	-0.2500	-0.2500
-0.2500	-0.2500	-0.2500	-0.2500
-0.2500	-0.2500	-0.2500	-0.2500

TABLE VI

EIGENVALUE DISTRIBUTION OF KN MATRICES FOR REGION C1C21D21D1

*zkn*1	*zkn*2	*zkn*3	*zkn*4
-0.6432+0.5931i	-0.2032+0.7178i	0.1432-0.5931i	-0.2968-0.7178i
-0.6432-0.5931i	-0.2032-0.7178i	0.1432+0.5931i	-0.2968+0.7178i
-0.3253+0.0311i	-0.3759	-0.1747-0.0311i	-0.1241
-0.3253-0.0311i	-0.1854+0.0635i	-0.1747+0.0311i	-0.3146-0.0635i
-0.3764	-0.1854-0.0635i	-0.1236	-0.3146+0.0635i
-0.3716	-0.2104	-0.1284	-0.2897
-0.3591	-0.1404	-0.1409	-0.3596
-0.2500	-0.2500	-0.2500	-0.2500
-0.2500	-0.2500	-0.2500	-0.2500
-0.2500	-0.2500	-0.2500	-0.2500

TABLE VII

EIGENVALUE DISTRIBUTION OF KN MATRICES FOR REGION C1C25D25D1

coefficients induced by the interval parameters and shed significant insight into the 'correct' scenario for this case. A previously presented 'vertex algorithm' by the author for this tough problem was derived under the misunderstood mapping of the parameter space to the matrix element space that currently exists in the literature (in the absence of the explicit expressions derived in this paper). Based on the correct mapping presented in this paper, a 'correct' vertex solution is offered explaining away the 'discrepancy' in the previous erroneous results. For better exposition, this paper concentrates on the two parameter cae but the technique presented here is easily extendable to three and other multiple parameter cases.

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