2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009



# Two-Time-Scale Averaging of Systems Involving Operators and Its Application to Adaptive Control of Hysteretic Systems

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*Abstract*— **Motivated by the adaptive control problem for systems with hysteresis, a two-time-scale averaging framework is presented in this paper for systems involving operators, by extending the work of Teel and co-workers. The developed averaging theory is applied to the analysis of a model reference adaptive inverse control scheme for a system consisting of linear dynamics preceded by a Prandtl-Ishlinkskii (PI) hysteresis operator. The fast component of the closed-loop system involves the coupling of an ordinary differential equation and a hysteresis operator derived from the PI operator and its inverse, while the slow component is the parameter update rule. The stability of the boundary-layer system and that of the average system are established under suitable conditions, which implies practical regulation of the parameter error and tracking error under the adaptive scheme.**

## I. INTRODUCTION

This paper deals with the analysis of two-time-scale systems where the fast dynamics involves the coupling of ordinary differential equations (ODEs) and nonlinear operators (e.g., hysteresis operators). The class of systems studied is motivated by the adaptive control problem for systems with hysteresis. Consider a dynamic hysteretic system consisting of a rate-independent hysteresis operator [1] preceding a linear plant. A fundamental approach to the control of such systems is inverse compensation, where an approximate inverse of the hysteresis operator is used to cancel the effect of hysteresis, and a feedback controller is then designed to ensure desired closed-loop behavior [2]. Adaptation can be introduced when the parameters of the hysteresis operator and/or the linear dynamics are unknown [2].

In our prior work [3], [4], slow adaptation has been proposed as a potential mechanism to overcome overparametrization when uncertainties exist in both linear dynamics and the hysteresis operator. The closed-loop system under slow adaptation demonstrates multi-time-scale dynamics. The fast dynamics involves the states of the plant and the filters, described by ODEs, and the state of a hysteresis operator, coupled in a feedback manner. The slow dynamics involves the parameters being updated. While averaging seems to be a natural tool to analyze the system under consideration, to our best knowledge, existing work does not address the aforementioned class of systems. Classical two-time-scale averaging assumes that the fast dynamics is dominated by a linear term and uses the explicit expression of the integral manifold to obtain the average system [5]. In our earlier work [4], linearization was adopted to remove the hysteresis operator from the picture. While the latter allowed the use of results from [5], it is limited to local analysis. Teel *et al.* developed a general framework for analyzing two-time-scale systems [6]; however, the latter still cannot accommodate hysteresis operators in the dynamics. Averaging of hysteretic systems has been considered by Pokrovskii *et al.* [7], but the fast dynamics therein involves the hysteresis operator only.

In this paper we first present an averaging framework that is capable of handling fast dynamics involving both ODEs and hysteresis operators, by extending the work in [6]. The properties of the closed-loop system under model reference adaptive inverse control are then analyzed, in order to verify the major assumptions in the averaging theory. For this purpose, the Prandtl-Ishlinkskii (PI) operator [1] is adopted for the hysteresis model. We establish that, under suitable conditions, adaptive inverse control with slow adaptation can achieve ultimate boundedness for both the parameter error and the tracking error, where the bound can be made arbitrarily small by making the adaptation gain sufficiently small.

## II. AVERAGING OF TWO-TIME-SCALE SYSTEMS INVOLVING OPERATORS

Consider the system

$$\begin{array}{rcl} \dot{x} & = & \varepsilon f(x,\xi,w,\eta) \\ \dot{\xi} & = & A(x)\xi + g(x,w,\eta) \\ \alpha(t) & = & \phi(x(t),\xi(t),\eta(t)) \\ w(t) & = & \mathscr{W}[\alpha(\cdot);w(0);x(t)](t) \end{array} \quad (1)$$

where $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^{m_1}$, $w \in \mathbb{R}^{m_2}$, and $\varepsilon$ is a small positive parameter. The functions $f$, $g$, and $\phi$ are locally Lipschitz and $\eta(t)$ is a bounded, measurable function of $t$ for $t \geq 0$. The function $w$ is the output of an operator $\mathscr{W}$ and is determined by the initial condition $w(0)$, the current state $x(t)$ and the past history of $\alpha$, that is, $\alpha(\tau)$ for $\tau \in [0,t]$. Because of the smallness of $\varepsilon$, $x(t)$ is slowly varying relative to $(\xi(t),w(t))$. We will refer to $x$ as the slow state and $z = (\xi,w) \in \mathbb{R}^{m_3}$ as the fast state, where $m_3 = m_1 + m_2$.[1] The system is assumed to have at least one solution, locally in time, for each initial condition $(x(0),z(0))$ and each $\eta(t)$. Furthermore, we assume that $f(x,0,w,\eta) = 0$ and $g(0,w,\eta) = 0$ so that $(x(t) \equiv 0, \xi(t) \equiv 0)$ is a solution of (1), with the corresponding $w(t)$ determined by $\mathscr{W}$ driven by $\alpha(t) = \phi(0,0,\eta(t))$ and $x(t) = 0$.

This work was supported by NSF (CMMI 0824830).

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[1] We write $z = (\xi,w)$ in place of $z = [\xi^T,w^T]^T$. Similar notation will be used throughout the paper.

In analyzing two-time-scale systems such as (1), it is usual to define boundary layer and average systems that capture the behavior of the fast and slow states, respectively. We follow closely the definitions of [6], which presents a general framework for analyzing two-time-scale systems. We note, however, that the class of systems in [6] does not allow for an operator of the form in (1). The boundary-layer system is obtained by setting $\varepsilon = 0$ in (1), which freezes the slow state $x$ at its initial value and yields

$$\begin{array}{rcl} \dot{x} & = & 0 \\ \dot{\xi} & = & A(x)\xi + g(x,w,\eta) \\ \alpha(t) & = & \phi(x,\xi(t),\eta(t)) \\ w(t) & = & \mathscr{W}[\alpha(\cdot);w(0);x](t) \end{array} \qquad (2)$$

We denote the solution of (2) by $x_{\text{bl}}(t)$, $\xi_{\text{bl}}(t)$ and $w_{\text{bl}}(t)$ and set $z_{\text{bl}} = (\xi_{\text{bl}}, w_{\text{bl}})$. Let $B_r$ denote the closed ball $\{x : \|x\| \le r\}$ for some $r > 0$.

*Assumption 1:* There exist continuous functions $h_1(t,x)$ and $h_2(t,x)$, differentiable in $t$ and bounded on compact sets of $x$ for all $t \ge 0$, that satisfy

$$\begin{array}{rcl} \frac{\partial h_1}{\partial t} & = & A(x)h_1 + g(x,h_2(t,x),\eta(t)) \\ \alpha_h(t,x) & = & \phi(x,h_1(t,x),\eta(t)) \\ h_2(t,x) & = & \mathscr{W}[\alpha_h(\cdot,x);h_2(0,x);x](t) \end{array} \qquad (3)$$

for all $t \ge 0$, $x \in B_b$, for some $b > 0$. Set $h = (h_1, h_2)$.

Because $g(0,w,\eta) = 0$, Assumption 1 implies $h_1(t,0) \equiv 0$. Given positive constants $r$ and $\rho$, define the sets $\Sigma_\rho(t,x)$ and $\Omega_{r\times\rho}(t)$ by

$$\Sigma_\rho(t,x) = \{z \;:\; \|z - h(t,x)\| \le \rho\}$$

$$\Omega_{r\times\rho}(t) = \{(x,z) \;:\; \|x\| \le r \text{ and } \|z - h(t,x)\| \le \rho\}$$

The set $\Omega_{r\times\rho}(t)$ is compact for all $t \ge 0$.

*Assumption 2:* There exists a class $\mathscr{K}\mathscr{L}$ function $\beta_{\text{f}}$ such that for each $x \in B_b$ and for all initial conditions $z_{\text{bl}}(0) \in \Sigma_c(0,x)$, for some $c > 0$, the solutions of the boundary-layer system (2) satisfy

$$\|z_{\text{bl}}(t) - h(t,x)\| \le \beta_{\text{f}}(\|z_{\text{bl}}(0) - h(0,x)\|, t), \qquad \forall\; t \ge 0 \quad (4)$$

Assumption 1 states that the boundary-layer system (2) has $z = h(t,x)$ as an integral manifold. Assumption 2 confirms asymptotic stability of the manifold.

*Assumption 3:* The limit

$$f_{\text{av}}(x) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} f(x,h_1(t,x),h_2(t,x),\eta(t))\; dt \qquad (5)$$

exists and the convergence to the limit is uniform with respect to $(x,t_0)$ for all $x \in B_b$ and $t_0 \ge 0$.

The average system is defined by

$$\dot{x} = \varepsilon f_{\text{av}}(x) \qquad (6)$$

and we denote its solution by $x_{\text{av}}(t)$. We assume that $f_{\text{av}}$ is locally Lipschitz. We note that $f_{\text{av}}(0) = 0$ because $f(x,0,w,\eta) = 0$ and $h_1(t,0) = 0$.

*Assumption 4:* There exists a class $\mathscr{K}\mathscr{L}$ function $\beta_{\text{s}}$ such that for all initial conditions $x_{\text{av}}(0) \in B_a$, the solutions of the average system (6) satisfy

$$\|x_{\text{av}}(t)\| \le \beta_{\text{s}}(\|x_{\text{av}}(0)\|, \varepsilon t), \qquad \forall\; t \ge 0 \qquad (7)$$

*Assumption 5:* There exists a positive constant $\sigma$ such that the constants $a$ and $b$ in Assumptions 1 to 4 satisfy $b \ge \beta_{\text{s}}(a,0) + \sigma$.

Let $d \ge \beta_{\text{f}}(c,0) + \sigma$. Assumptions 2, 4, 5 and the definition of $d$ ensure that for each $x_{\text{av}}(0) \in B_a$, the solution $x_{\text{av}}(t)$ of the average system (6) will be in the interior of $B_b$, and for each $x \in B_b$, and $z_{\text{bl}}(0) \in \Sigma_c(0,x)$, the solution of the boundary-layer system (2) will be in the interior of $\Sigma_d(t,x)$.

In the absence of the operator $\mathscr{W}$, the local Lipschitz property of $f$ and $h$ ensures continuous dependence of the solution of (1) on initial conditions and right-hand-side functions. The following assumption requires the same property in the presence of $\mathscr{W}$. In particular, the assumption requires the fast state $z(t)$ to depend continuously on the initial state $z(0)$ and the slow state $x(t)$, which is viewed as input to the fast equations.

*Assumption 6:* For each $T > 0$, there exist positive constants $K_1$ and $K_2$ such that for all $(x(0),z(0)) \in \Omega_{a\times c}(0)$ and $(x_{\text{bl}}(0),z_{\text{bl}}(0)) \in \Omega_{a\times c}(0)$, the solutions of (1) and (2) satisfy the inequality

$$\|z(t) - z_{\text{bl}}(t)\| \le K_1\|z(0) - z_{\text{bl}}(0)\| + K_2 \sup_{\tau\in[0,t]} \|x(\tau) - x_{\text{bl}}(0)\| \qquad (8)$$

for all $t \in [0,T]$.

*Proposition 2.1:* Let Assumptions 1 to 6 hold and take $x_{\text{av}}(0) = x_{\text{bl}}(0) = x(0)$ and $z_{\text{bl}}(0) = z(0)$. Then there exists $T^* > 0$ so that for each $T \ge T^*$ and $\delta > 0$, there exists $\varepsilon^* > 0$ such that for each $\varepsilon \in (0,\varepsilon^*]$ and each initial condition $(x(0),z(0)) \in \Omega_{a\times c}(0)$,

$$\|z(t) - z_{\text{bl}}(t)\| \le \delta, \qquad \forall\; t \in [0,T] \qquad (9)$$

$$(x(t),z(t)) \in \Omega_{b\times c}(t), \qquad \forall\; t \in [T^*,T] \qquad (10)$$

$$\|x(t) - x_{\text{av}}(t)\| \le \delta, \qquad \forall\; t \in [0,T/\varepsilon] \qquad (11)$$

and

$$x(t) \in B_a, \qquad \forall\; t \in [T^*/\varepsilon, T/\varepsilon] \qquad (12)$$

*Proposition 2.2:* Under Assumptions 1 to 6, for each $\delta > 0$ there exists $\varepsilon^* > 0$ such that, for all $\varepsilon \in (0,\varepsilon^*]$ and all $(x(0),z(0)) \in \Omega_{a\times c}(0)$, the solutions of (1) exist and satisfy the following two inequalities for all $t \ge 0$,

$$\|x(t)\| \le \beta_{\text{s}}(\|x(0)\|, t) + \delta \qquad (13)$$

$$\|z(t) - h(t,x(t))\| \le \beta_{\text{f}}(\|z(0) - h(0,x(0))\|, t) + \delta \qquad (14)$$

Inequality (13) shows that $x(t)$ is ultimately bounded by $\delta$, which can be made arbitrarily small by choosing $\varepsilon$ small enough. Inequality (14) shows that the deviation of $z$ from the slow manifold $h(t,x)$ is also ultimately bounded by $\delta$. Noting that $h_1(t,0) = 0$, we see that $\xi(t)$ is also ultimately bounded by a constant that can be made arbitrarily small by choosing $\varepsilon$ small enough.

With the help of Assumption 6, the proof of Propositions 2.1 and 2.2 follows closely the argument of [6] and it will be omitted here due to space limitation.

## III. Preisach-like Hysteresis Operators and Their Inversion

In this paper we consider Preisach-like operators for the hysteresis model. Preisach-like operators are phenomenological models that have proven effective in capturing complex, hysteretic behaviors in physical systems, such as smart materials [8]–[10], ferromagnetism, and superconductivity [11]. A Preisach-like operator consists of weighted superposition of many (and even a continuum of) basic hysteretic elements, called *hysterons*. For example, the hysteron for a classical Preisach operator is a relay with a pair of thresholds $(\beta,\alpha)$. On the other hand, the hysteron for Prandtl-Ishlinskii (PI) operator, another Preisach-like operator, is a play (or backlash) operator $P_r$, the width of which is characterized by parameter $r$. Since the PI operator will be used in later analysis, it is described in more detail here. For a monotone, continuous input $v$ on $[0,T]$, the corresponding output $u_r$ of $P_r$ can be written as, $\forall t\in[0,T]$,

$$u_r(t)=P_r[v;u_r(0)](t)=\max\{\min\{v(t)+r,u_r(0)\},v(t)-r\} \tag{15}$$

For a general, continuous input $v$ on $[0,T]$, one can first divide $[0,T]$ into subintervals such that $v$ is monotone on each interval and then apply (15) recursively to evaluate the output $u$ of $P_r$. Note that the output $u_r$ also represents the state of the operator $P_r$, when the latter is considered as a dynamical system.

Consider a PI operator consisting of $m+1$ play operators, characterized by threshold parameters $0=r_0<r_1<\cdots<r_m<\infty$. Let $W=(W_0,W_1,\cdots,W_m)^T$ denote the vector of the states of the play operators. For a continuous input $v$ on $[0,T]$, the output $u$ of the PI operator $\Gamma$ can be defined as, $\forall t\in[0,T]$,

$$u(t)=\Gamma[v,W(0)](t)=\sum_{i=0}^{m}\theta_{Hi}P_{r_i}[v;W_i(0)](t) \tag{16}$$

where $\theta_H=(\theta_{H0},\theta_{H1},\cdots,\theta_{Hm})^T\geq 0$ denotes the vector of weights for the play operators. Later on, we will also consider the associated vector hysteresis operator, $\mathscr{P}=\left(P_{r_0},\cdots,P_{r_m}\right)^T$, i.e., $\mathscr{P}$ captures the evolution of $W$ under the input $v$:

$$W(t)=\mathscr{P}[v;W(0)](t) \tag{17}$$

The inversion of Preisach-like operators has been extensively studied [8], [11]–[13]. For a wide class of Preisach-like operators that are Lipschitz continuous and piecewise monotone [14], a fixed-point iteration-type algorithm exists such that the error between the desired output $u_d$ and the actual output $u$ satisfies $|u_d-u|\leq\varepsilon$, for a given tolerance $\varepsilon>0$.

For a PI operator $\Gamma$ with a finite number of hysterons with $\theta_{H0}>0$, the inverse operator $\Gamma^{-1}$ can be represented by another PI operator $\Gamma'$ and thus $|u_d-u|=0$ can be achieved. The thresholds $\{r_i'\}$, weights $\{\theta_{Hi}'\}$, and initial conditions $\{W_i'(0)\}$ for $\Gamma'$ can be obtained from those of $\Gamma$ through a continuous transformation [13].

Now suppose that the exact parameters $\theta_H$ for $\Gamma$ are unknown. Instead, one has to perform inversion based on the parameter estimate $\hat{\theta}_H$. Equivalently, an operator $\hat{\Gamma}$ with parameter $\hat{\theta}_H$ is inverted, with its output $\hat{u}=\hat{\Gamma}[v,W(0)]=\hat{\theta}_H^TW$ satisfying $|\hat{u}-u_d|\leq\varepsilon$. Since $\Gamma$ and $\hat{\Gamma}$ have common input $v(\cdot)$, they share $W(\cdot)$ and hence $\hat{u}-u=\tilde{\theta}_H^TW$, where $\tilde{\theta}_H\overset{\triangle}{=}\hat{\theta}_H-\theta_H$. This implies that, in the presence of parameter error,

$$u_d(t)-u(t)=\tilde{\theta}_H(t)^TW(t)+d \tag{18}$$

where $d$ represents a small, bounded error.

## IV. Model Reference Adaptive Inverse Control Scheme

### A. The Error State Equation

Let $G(s)=k_p\frac{Z_p(s)}{P_p(s)}$, where $k_p$ is the high-frequency gain, $P_p(s)$ and $Z_p(s)$ are monic polynomials of degree $n$ and $m$, respectively. The plant has state space representation $(A_p,B_p,C_p,0)$, so that

$$\begin{aligned}\dot{x}_p &= A_px_p(t)+B_pu(t)\\ y(t) &= C_px_p(t)\end{aligned} \tag{19}$$

The goal of the controller design is to make the output of the plant, $y$, track the output of the model, $y_m$. The output of the model is given by $y_m(t)=G_m(s)[r](t)$, where $r$ is a bounded, piecewise continuous reference input and $G_m(s)[r](t)$ denotes the time-domain output of the transfer function $G_m(s)$ operating on $r$.

*Assumption 7:* The following assumptions are made about the plant and the model:

– $Z_p(s)$ is a stable polynomial;
– The degrees $n$ and $m$ are known;
– $k_p>0$;
– $G_m(s)=\frac{1}{P_m(s)}$ where $P_m(s)$ is a stable polynomial of degree $n^*=n-m$.

Fig. 1 shows the classical model reference control paradigm [2]. With exact hysteresis cancellation and knowledge of the plant parameters, perfect model following can be achieved with

$$u(t)=\theta_1^Tw_1(t)+\theta_2^Tw_2(t)+\theta_{20}y(t)+\theta_3r(t)$$

where

$$w_1(t)\overset{\triangle}{=}\frac{a(s)}{\lambda(s)}[u](t),\ w_2(t)\overset{\triangle}{=}\frac{a(s)}{\lambda(s)}[y](t),$$

$a(s)=\left[\begin{array}{cccc}1, & s, & \cdots, & s^{n-2}\end{array}\right]^T$ and $\lambda(s)$ is a stable polynomial of degree $n-1$. The parameters $\theta_1\in\mathbb{R}^{n-1}$, $\theta_2\in\mathbb{R}^{n-1}$, $\theta_{20}\in\mathbb{R}$ and $\theta_3\in\mathbb{R}$ are determined by the matching equation:

$$\begin{aligned}\theta_1^Ta(s)P_p(s)+[\theta_2^Ta(s)+\theta_{20}\lambda(s)]\,k_pZ_p(s)=\\ \lambda(s)\,[P_p(s)-\theta_3k_pZ_p(s)P_m(s)]\end{aligned}$$

Now define a controllable canonical pair $(\Lambda,B_\lambda)$ such that

$$(sI-\Lambda)^{-1}B_\lambda=\frac{a(s)}{\lambda(s)} \tag{20}$$

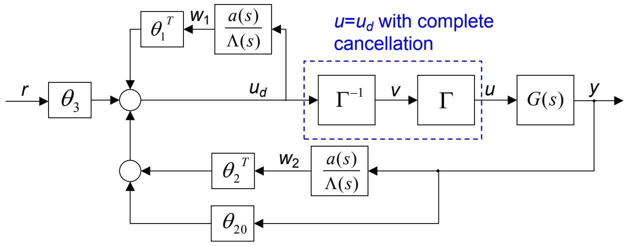


Fig. 1. Model reference controller when $G(s)$ is known and an exact inverse $\Gamma^{-1}$ exists.

Using (19) and (20), the state space form of the system with perfect matching is

$$\begin{aligned} \dot{x}_m &= A_m x_m + B_m r \\ y_m &= C_m x_m, \end{aligned} \tag{21}$$

where $x_m \triangleq [x_{pm}, w_{m1}, w_{m2}]$,

$$A_m \triangleq \begin{bmatrix} A_p + B_p\theta_{20}C_p & B_p\theta_1^T & B_p\theta_2^T \\ B_\lambda\theta_{20}C_p & \Lambda + B_\lambda\theta_1^T & B_\lambda\theta_2^T \\ B_\lambda C_p & 0 & \Lambda \end{bmatrix},$$

$$B_m \triangleq \begin{bmatrix} B_p\theta_3 \\ B_\lambda\theta_3 \\ 0 \end{bmatrix}, \; C_m \triangleq \begin{bmatrix} C_p & 0 & 0 \end{bmatrix}$$

The true parameters are unknown, so

$$u_d(t) = \hat{\theta}_1^T w_{11} + \hat{\theta}_2^T w_2 + \hat{\theta}_{20} y(t) + \hat{\theta}_3 r(t) \tag{22}$$

is the implemented control, where $\hat{\theta}_k$ is an estimate of $\theta_k$ and

$$w_{11}(t) \triangleq \frac{a(s)}{\lambda(s)}[u_d](t)$$

When the parameters of $\Gamma$ are unknown, the estimated parameters must be used in the hysteresis inversion. The following assumption is made for ease of discussion:

*Assumption 8:* The inversion error $u_d - u$ satisfies (18) with $d = 0$.

With the control signal $u_d$ in (22) and using (18), the system has the state space representation

$$\dot{e} = \begin{bmatrix} A_m + \frac{B_m\tilde{\theta}_G^T Q}{\theta_3} & \frac{B_m(\theta_1+\tilde{\theta}_1)^T}{\theta_3} \\ 0 & \Lambda \end{bmatrix} e + \begin{bmatrix} \frac{B_m(\tilde{\theta}_G^T w_m - \tilde{\theta}_H^T W)}{\theta_3} \\ B_\lambda(\tilde{\theta}_H^T W) \end{bmatrix} \tag{23}$$

where

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} \triangleq \begin{bmatrix} x_p - x_{pm} \\ w_1 - w_{m1} \\ w_2 - w_{m2} \\ w_{11} - w_1 \end{bmatrix}, \; \tilde{\theta}_k = \hat{\theta}_k - \theta_k$$

$$\tilde{\theta}_G \triangleq \begin{bmatrix} \tilde{\theta}_3 \\ \tilde{\theta}_{20} \\ \tilde{\theta}_1 \\ \tilde{\theta}_2 \end{bmatrix}, \; w_m \triangleq \begin{bmatrix} r \\ C_p x_{pm} \\ w_{m1} \\ w_{m2} \end{bmatrix}, \; Q \triangleq \begin{bmatrix} 0 & 0 & 0 \\ C_p & 0 & 0 \\ 0 & \mathbb{I} & 0 \\ 0 & 0 & \mathbb{I} \end{bmatrix},$$

$\mathbb{I}$ is the identity matrix of suitable dimension, and the dimensions of remaining entries in $Q$ are defined appropriately. Since $w_m$ is determined by $r$ and the reference model $G_m$ only, it can be treated as an exogenous input.

### B. State Evolution of the Preisach-like Operator

In (23), $W(t)$ is the state of the Preisach-like operator $\Gamma$ determined by $W(0)$ and the input $v$ to the operator, as in (17). The input $v$ is in turn obtained through inversion of $u_d$ based on $\hat{\Gamma}$:

$$v(t) = \hat{\Gamma}^{-1}[u_d, W(0)](t) \tag{24}$$

Expressing $u_d$ (22) in terms of the error state and the exogenous input, we can write

$$u_d = (\tilde{\theta}_G + \theta_G)^T (M_1 e + M_2 w_m) \tag{25}$$

for appropriately defined constant matrices $M_1$ and $M_2$, where $\theta_G = (\theta_3, \theta_{20}, \theta_1, \theta_2)$. Combining (17), (24), and (25), we have

$$W(t) = \mathscr{W}[u_d; W(0); \tilde{\theta}_H(t)](t) \tag{26}$$

where $\mathscr{W} \triangleq \mathscr{P} \circ \hat{\Gamma}^{-1}$, "$\circ$" denotes the composition of operators, and the explicit dependence of $\hat{\Gamma}^{-1}$ on the current estimate $\hat{\theta}_H$ of hysteresis parameters is indicated in (26).

### C. Adaptation Rule

A gradient algorithm was *formally* derived in [3]:

$$\dot{\hat{\theta}} = -\gamma_G e_0 G_m \left( \begin{bmatrix} r \\ y \\ w_{11} \\ w_2 \\ \gamma' \left( \left( \frac{a(s)}{\lambda(s)}[W^T] \right)^T \hat{\theta}_1 - W \right) \end{bmatrix} \right) \tag{27}$$

where $\gamma' = \frac{\gamma_H}{\gamma_G}$. One can choose $\gamma' << 1$ to separate the slow adaptation of the plant and hysteresis parameters into two time scales. All of the signals required in (27) are available online.

Eqs. (23), (25), (26), and (27) form a complete description of the closed-loop system under adaptation, which fits the class of systems (1), with $\tilde{\theta} = (\tilde{\theta}_G, \tilde{\theta}_H)$, $e$, $u_d$, $W$, $w_m$, $\gamma_G$ corresponding to $x$, $\xi$, $\alpha$, $w$, $\eta$, $\varepsilon$ in (1), respectively.

## V. Stability of the Boundary Layer System

Under slow adaptation, i.e., when $\gamma_G << 1$ in (27), the closed-loop system will demonstrate a two-time-scale behavior, with the dynamics of $(e, W)$ at a faster scale than that of $\tilde{\theta}$. When the slow variable $\tilde{\theta}$ is frozen, i.e., $\gamma_G = 0$ in (27), the boundary layer system is obtained from (23), (25), (26), which is summarized below for convenience:

$$\begin{aligned} \dot{e}(t) &= A_0(\tilde{\theta}_G)e(t) + B_0(\tilde{\theta}_G)w_m(t) + B_1\tilde{\theta}_H^T W(t) && (28) \\ u_d(t) &= (\tilde{\theta}_G + \theta_G)^T (M_1 e(t) + M_2 w_m(t)) && (29) \\ W(t) &= \mathscr{W}[u_d(\cdot); W(0); \tilde{\theta}_H](t) && (30) \end{aligned}$$

where

$$A_0(\tilde{\theta}_G) = \begin{bmatrix} A_m + \frac{B_m\tilde{\theta}_G^T Q}{\theta_3} & \frac{B_m(\theta_1+\tilde{\theta}_1)^T}{\theta_3} \\ 0 & \Lambda \end{bmatrix}$$

$$B_0(\tilde{\theta}_G) = \begin{bmatrix} \frac{B_m\tilde{\theta}_G^T}{\theta_3} \\ 0 \end{bmatrix}, \; B_1 = \begin{bmatrix} -\frac{B_m}{\theta_3} \\ B_\lambda \end{bmatrix}$$

The following assumptions are made:

*Assumption 9:* The reference signal $r$ is $T-$periodic.

*Remark 5.1:* The results of this paper can be extended to the case where $r$ is almost periodic.

*Assumption 10:* The vector hysteresis operator $\mathscr{W}$ is associated with a PI operator, as described in Section III.

*Assumption 11:* $A_0(\tilde{\theta}_G)$ is Hurwitz for every fixed $\tilde{\theta}_G$ in the domain of interest.

*Remark 5.2:* Since $A_m$ is Hurwitz, Assumption 11 will be satisfied if $\|\tilde{\theta}_G\|$ is sufficiently small.

The goal is to show that for a given $\tilde{\theta}$, for any initial condition $(e(0), W(0))$, the solution $(e(t), W(t))$ of the system (28) - (30) converges asymptotically to a unique pair $(h_1(t,\tilde{\theta}), h_2(t,\tilde{\theta}))$, where $h_1$ and $h_2$ are periodic in $t$. Before we do this, however, we need to show that, given $(e(0), W(0))$, there exists a unique solution to (28) - (30), furthermore, the solution depends continuously on $(e(0), W(0))$. The latter can be proven by first establishing the Lipschitz continuity of the operator $\mathscr{W}$ in (30), and applying the standard contraction argument. The details are omitted due to the space limitation.

While the continuous dependence of solution to (28) - (30) on the initial condition is not equivalent to Assumption 6, one can see that Assumption 6 is reasonable to expect, considering further the continuous dependence of (28) - (30) on $\tilde{\theta}$. For convenience of later reference, we rephrase this assumption:

*Assumption 12:* Assumption 6 is satisfied when specialized to the system (23), (25), (26), and (27).

In order to analyze the stability of periodic solutions to the system (28) - (30), we view this system as perturbed from

$$\dot{e} = A_0 e + B_0 w_m \tag{31}$$

Because of Assumption 9, $w_m$ is a $T$-periodic exogenous signal (possibly with exponentially decaying transient due to the effect of initial condition in (21)). Assumption 11 implies that the solution to (31), starting from any initial condition $e(0)$, converge exponentially to a $T-$periodic function $e_T$. Since $e_T$ is an exponentially stable solution, (31) is globally $T-$*convergent near* $e_T(\cdot)$ [15]. The latter property essentially means that the solution to (31) with small perturbation $\xi(t)$

$$\dot{e} = A_0 e + B_0 w_m + \xi(t)$$

will be well behaved and stay close to $e_T$.

We will treat (28) as perturbed from (31) by $B_1\tilde{\theta}_H^T W(t)$. To facilitate the formulation, we express $\tilde{\theta}_H$ as

$$\tilde{\theta}_H = \varepsilon \tilde{\theta}_H^0$$

where $\tilde{\theta}_H^0$ is the unit vector in the direction of $\tilde{\theta}_H$, and rewrite (28) as

$$\dot{e}(t) = A_0 e(t) + B_0 w_m(t) + \varepsilon \tilde{\theta}_H^{0T} W(t) \tag{32}$$

The stability analysis for (28) - (30) will be based on a special contraction property for the operator $\mathscr{W}$. This property stems from the following characteristic of a play operator:

*Lemma 5.1:* Consider a play operator $P_r$ with threshold $r$. Let $\zeta_0^a$, $\zeta_0^b$ be two arbitrary initial conditions for $P_r$. Then, for any continuous input $v$,

$$|P_r[v;\zeta_0^a](t) - P_r[v;\zeta_0^b](t)| \le |\zeta_0^a - \zeta_0^b|, \quad \forall t \ge 0 \tag{33}$$

Furthermore, for any $t > 0$,

$$|P_r[v;\zeta_0^a](t) - P_r[v;\zeta_0^b](t)| = 0, \quad \text{if } osc_{[0,\,t]}[v] \ge 2r \tag{34}$$

where the oscillation function $osc$ is defined as

$$osc_{[t_1,t_2]}[v] = \sup_{t_1 \le \tau, \sigma \le t_2} |v(\tau) - v(\sigma)|$$

From Lemma 5.1, under a same input $v$, the distance between the state trajectories of a play operator $P_r$ starting from different initial conditions is non-increasing over time, and it drops to zero once the variation of $v$ exceeds $2r$. Lemma 5.1 can be proved through elementary analysis of $P_r$. We will not provide the proof here due to space limitation.

*Lemma 5.2:* Let $W_0^a$, $W_0^b$ be two arbitrary initial conditions for the operator $\mathscr{W}$. Then, for any continuous input $u_d$, $\forall t > 0$,

$$|\mathscr{W}[u_d;W_0^a;\tilde{\theta}_H](t) - \mathscr{W}[u_d;W_0^b;\tilde{\theta}_H](t)| \le |W_0^a - W_0^b| \tag{35}$$

where $|\cdot|$ denotes any norm in $\mathbb{R}^{m+1}$. Furthermore, denote $v_a = \hat{\Gamma}'[u_d;W_0^a]$ and $v_b = \hat{\Gamma}'[u_d;W_0^b]$, where $\hat{\Gamma}'$ represents the inverse of $\hat{\Gamma}$ that is expressed in terms of another PI operator. Let $\{r_i\}_{i=0}^m$ and $\{r_i'\}_{i=0}^m$ represent the thresholds for $\hat{\Gamma}$ (and thus $\Gamma$) and those for $\hat{\Gamma}'$, respectively. Suppose $u_d$ is such that, for $0 < t_1 < t_2$,

$$\begin{aligned} osc_{[0,t_1]}[u_d] &\ge 2r_m' && (36)\\ osc_{[t_1,t_2]}[v_a] &\ge 2r_m && (37) \end{aligned}$$

then

$$|\mathscr{W}[u_d;W_0^a;\tilde{\theta}_H](t) - \mathscr{W}[u_d;W_0^b;\tilde{\theta}_H](t)| = 0, \quad \forall t \ge t_2 \tag{38}$$

*Sketch of proof.* Recall $\mathscr{W} = \mathscr{P} \circ \hat{\Gamma}'$. Claim (35) follows directly from (33). Following (36) and Lemma 5.1, the state of $\hat{\Gamma}'$ satisfies $W_a'(t) \equiv W_b'(t)$, $\forall t \ge t_1$, implying $v_a \equiv v_b$, $\forall t \ge t_1$. Claim (38) then follows by applying (37) and Lemma 5.1 to $\mathscr{P}$ associated with $\Gamma$. $\square$

We can now state the main result of this section.

*Theorem 5.1:* Let Assumptions 7 - 11 hold. Define

$$u_d^* = \hat{\theta}_G(M_1 e_T + M_2 w_m)$$

where $e_T$ is the periodic solution of (31). For some arbitrary initial condition $W_0^c$, define $v_c = \hat{\Gamma}'[u_d^*;W_0^c]$. If

$$\begin{aligned} osc_{[0,T]}[u_d^*] &> 2r_m' && (39)\\ osc_{[T,2T]}[v_c] &> 2r_m && (40) \end{aligned}$$

then there exists $\varepsilon^* > 0$, such that, $\forall \varepsilon \in (0, \varepsilon*]$, for an arbitrary initial condition $(e(0), W(0))$, the solution $(e(t), W(t))$ of the system (32), (29), and (30) converges asymptotically to a unique periodic solution $(h_1(t,\tilde{\theta}), h_2(t,\tilde{\theta}))$. In particular, there exists a class $\mathscr{K}\mathscr{L}$ function $\beta_{\mathrm{f}}$, such that

$$\left\| \begin{array}{c} e(t) - h_1(t,\tilde{\theta}) \\ W(t) - h_2(t,\tilde{\theta}) \end{array} \right\| \le \beta_{\mathrm{f}}\left( \left\| \begin{array}{c} e(0) - h_1(0,\tilde{\theta}) \\ W(0) - h_2(0,\tilde{\theta}) \end{array} \right\|, t \right) \tag{41}$$

*Sketch of the proof.* For $\varepsilon > 0$, define the shift operator $S^{\varepsilon}$ via

$$S^{\varepsilon}[e(0),W(0)] = (e(2T),W(2T))$$

Using Lemma 5.2, one can show that $S^{\varepsilon}$ becomes a contraction mapping on an appropriately defined set, following similar arguments as for Theorem 2.1 in [15]. This implies that the system under consideration has an asymptotically stable, $2T-$periodic solution. Further analysis can show that this solution is also $T-$periodic. □

The following assumption is made for later analysis:

*Assumption 13:* $h_1(t,\tilde{\theta})$ and $h_2(t,\tilde{\theta})$ are continuously differentiable with respect to $\tilde{\theta}$.

## VI. Stability of the Average and Full Systems

It can be shown that the averaged system is given by

$$\dot{\tilde{\theta}}_{av} = -\gamma A_{VG}\left(K(\cdot)K^T(\cdot)\right)\tilde{\theta}_{av} - \gamma A_{VG}(R(\cdot,\tilde{\theta}_{av})) \tag{42}$$

for some $R(t,\tilde{\theta})$ of order $O(|\tilde{\theta}|^2)$. For ease of presentation, we have let $\gamma_G = \gamma_H = \gamma$. The averaging operator $A_{VG}$ is defined as, for a $T-$periodic $f$,

$$A_{VG}(f(\cdot)) = \frac{1}{T}\int_0^T f(\tau)d\tau, \quad \text{and } K(t) = \left.\frac{\partial e_0}{\partial \tilde{\theta}}\right|_0$$

Note that $K(t)$ can be explicitly evaluated as

$$K(t) = \frac{G_m(s)}{\theta_3}\begin{bmatrix} w_m \\ \left(\frac{a(s)}{\lambda(s)}[W_*^T]\right)^T \theta_1 - W_* \end{bmatrix} \tag{43}$$

where $W_*(t)$ is the periodic solution obtained by feeding the signal $\theta_G^T M_2 w_m$ to the operator $\mathscr{W}$ characterized by parameter $\theta_H$. Note that the periodic solution is unique if

$$osc_{[0,T]}[\theta_G^T M_2 w_m] > 2r_m$$

*Assumption 14:* $Q \overset{\triangle}{=} A_{VG}(K(\cdot)K(\cdot)^T)$ is positive definite.

*Theorem 6.1:* Let Assumptions 7 - 14 hold. There exists $C > 0$, such that, if $|\tilde{\theta}_{av}| < C$, the average system (42) is asymptotically stable. In particular,

$$|\tilde{\theta}_{av}(t)| \le \beta_s(|\tilde{\theta}_{av}(0)|,\gamma t) \tag{44}$$

for some class $\mathscr{K}\mathscr{L}$ function $\beta_s$. Furthermore, for any $\delta > 0$, there exists $\gamma^* > 0$ such that, $\forall \gamma \in (0,\gamma^*]$, $\forall \tilde{\theta}$ satisfying $|\tilde{\theta}| < C$, $\forall (e(0),W(0))$, the following holds:

$$|\tilde{\theta}(t)| \le \beta_s(|\tilde{\theta}(0)|,\gamma t) + \delta$$

$$\left\| \begin{array}{c} e(t) - h_1(t,\tilde{\theta}(t)) \\ W(t) - h_2(t,\tilde{\theta}(t)) \end{array} \right\| \le$$
$$\beta_f\left(\left\| \begin{array}{c} e(0) - h_1(0,\tilde{\theta}(0)) \\ W(0) - h_2(0,\tilde{\theta}(0)) \end{array} \right\|, t\right) + \delta$$

*Sketch of proof.* The local, asymptotic stability of $\tilde{\theta}_{av} = 0$ for (42) and thus (44) can be established by using a Lyapunov function $V(\tilde{\theta}_{av}) = \frac{1}{2}|\tilde{\theta}_{av}|^2$ and noting that the term $A_{VG}[R(\cdot,\tilde{\theta})]$ is of order $O(|\tilde{\theta}_{av}|^2)$. One can verify that all assumptions required for Proposition 2.2 are satisfied. The rest of the claims then follows from Proposition 2.2. □

## VII. Conclusion

We presented a framework for two-time-scale averaging for systems involving operators such as Preisach-like hysteresis operators. As a motivating example, we analyzed in detail the slow adaptation scheme for systems with hysteresis and verified the major assumptions required in the proposed averaging framework. The latter allowed us to establish the practical regulation of parameter error and tracking error, under suitable persistent excitation conditions.

Future work will be pursued in several directions, focusing on the application of the presented averaging framework in adaptive control of hysteretic systems. We will explore other adaptation rules (than the formal gradient rule) that can potentially provide larger regions of attraction. We will also investigate the behavior of the closed-loop system when the persistent excitation condition is not met. Finally, the study will be extended to the case of nonlinear plants.

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