2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

A Characterization of the Hurwitz Stability of Metzler Matrices

Kumpati S. Narendra[1] and Robert Shorten[2]

Abstract— **It is well known that a Hurwitz Metzler matrix is also diagonally stable. We obtain a necessary and sufficient condition for a matrix A to be diagonally stable from the Kalman-Yacubovich-Popov lemma. This condition is equivalent to requiring that a pair of LTI systems, of lower dimension, have a common Lyapunov function. This fact is made use of to derive very simple conditions for the Hurwitz stability of a Metzler matrix. These conditions are stated in terms of the signs of the diagonal entries of a sequence of lower dimensional matrices that are easily constructed.**

I. Introduction

A matrix $A \in \mathbb{R}^{n\times n}$ is called a Metzler matrix if its off diagonal elements are non-negative. Metzler matrices are important as they arise in a number of application areas [1], [2], [3], [4], [5]. For example, continuous time linear dynamic systems that are constrained to evolve in the positive orthant, *positive dynamic systems*, are characterised by Metzler matrices. More precisely, if A is a Hurwitz Metzler matrix, then all solutions of the differential equation

$$\Sigma_A : \dot{x} = Ax \tag{1}$$

that start from initial conditions in the positive orthant, will satisfy $x_i(t) \geq 0$ for all $t > 0$, where i denotes the i'th component of the vector $x(t)$. In this paper we present a simple characterization of the Hurwitz stability of Metzler matrices.

It is well known that a stable dynamic system Σ_A is also diagonally stable if the matrix A is Metzler [2]. That is, any Metzler matrix that has eigenvalues in the open left half of the complex plane also satisfies $A^TD + DA < 0$ for some positive diagonal matrix D. In this case the quadratic function $V(x) = x^TDx$ is a called a diagonal Lyapunov function for Σ_A. Therefore, for Metzler matrices, the conditions for Hurwitz stability are equivalent to those for diagonal stability. In this paper, necessary and sufficient conditions are first derived for any matrix A to be diagonally stable. These conditions, which are obtained as a consequence of the Kalman-Yacubovich-Popov lemma, state that a matrix A is diagonally stable, if an only if two lower dimensional matrices matrices (A_1, A_2), derived from the entries of A, satisfy,

$$A_1^T\tilde{D} + \tilde{D}A_1 < 0,$$
$$A_2^T\tilde{D} + \tilde{D}A_2 < 0,$$

for some diagonal $\tilde{D} > 0$. While this condition is in general difficult to test, it becomes very simple in the case of Metzler matrices. This is because it can be shown that any diagonal Lyapunov function for Σ_{A_2} is also a diagonal Lyapunov function for Σ_{A_1}. Thus A is diagonally stable, if and only if a matrix of lower dimension is diagonally stable. This result can be applied repeatedly until a scalar is obtained. Since a necessary condition for a matrix A to be diagonally stable is that all its diagonal elements are negative, at every stage the signs of the diagonal elements of the reduced order matrices are checked. If at any stage the condition is not satisfied, the matrix A is not diagonally stable. The procedure for deriving a lower dimensional matrix is simple, and merely involves the sum of two matrices.

The paper is organized as follows. In section III, a necessary and sufficient condition for a matrix A to be diagonally stable is obtained using the Kalman-Yakubovich-Popov lemma. This assures diagonal stability in terms of a common diagonal Lyapunov function for two lower order matrices. In section IV it is shown that for a Metzler matrix, the diagonal stability of one of the derived matrices implies the diagonal stability of the second matrix (with the same diagonal Lyapunov function). This observation leads to a very simple test for the Hurwitz stability of Metzler matrices. In section V the procedure is applied a number of examples.

II. Notation

Throughout this paper, the following notation is adopted: $\mathbb{R}$ denotes the field of real numbers; $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space; $\mathbb{R}^{n\times n}$ denotes the space of $n \times n$ matrices with real entries.

The matrix A is said to be Hurwitz if all its eigenvalues lie in the open left half of the complex plane. A real symmetric matrix P is said to be positive definite if all its eigenvalues are positive. We use $P > 0$ to denote that P is positive definite. Vectors (matrices) that are entry-wise positive are denoted $x \succ 0$ $(A \succ 0)$, and vectors (matrices) that are entry-wise nonnegative $x \succeq 0$ $(A \succeq 0)$.

1. Center for Systems Science, Yale University, USA
2. The Hamilton Institute, NUI Maynooth, Ireland

978-1-4244-4524-0/09/$25.00 ©2009 AACC

Given m linear time-invariant (LTI) dynamic systems, $\dot{x} = A_i x, \quad i \in \Omega = \{1, ..., m\}$, the positive definite matrix P is said to be a common Lyapunov solution for A_i if $A_i^T P + P A_i = -Q_i < 0,\ i \in \Omega$. In this case $V(x) = x^T P x$ defines a common quadratic Lyapunov function (CQLF), for the m LTI systems Σ_{A_i}. In the case that P is a diagonal matrix, $V(x)$ is a common diagonal quadratic Lyapunov function (CDLF) for Σ_{A_i}.

III. Diagonal stability and the KYP lemma

The following well known result [6] plays an important role in developing our result.

Lemma 3.1: Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix that is partitioned as

$$A = \begin{bmatrix} A_{n-1} & b_{n-1} \\ c_{n-1}^T & d_{n-1} \end{bmatrix} \tag{2}$$

where $A_{n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$, $b_{n-1}, c_{n-1} \in \mathbb{R}^{n-1}$, $d_{n-1} \in \mathbb{R}$; the subscript $(n-1)$ denotes the dimensions of the matrix A_{n-1} and the vectors b_{n-1}, c_{n-1} respectively. Let $d_{n-1} \neq 0$. Then, A^{-1} can be expressed in partitioned form as

$$A^{-1} = \begin{bmatrix} B_{n-1} & l_{n-1} \\ m_{n-1}^T & \eta_{n-1} \end{bmatrix} \tag{3}$$

where $B_{n-1} \in \mathbb{R}^{(n-1)\times(n-1)}$, $l_{n-1}, m_{n-1} \in \mathbb{R}^{n-1}$ and $\eta_{n-1} \in \mathbb{R}$. Since $AA^{-1} = I_n$, where I_n denotes the unit matrix in $\mathbb{R}^{n\times n}$, it follows that

$$\begin{aligned} A_{n-1}B_{n-1} + b_{n-1}m_{n-1}^T &= I_{n-1} && (4)\\ A_{n-1}l_{n-1} + \eta_{n-1}b_{n-1} &= 0 && (5)\\ c_{n-1}^T B_{n-1} + d_{n-1}m_{n-1}^T &= 0 && (6)\\ c_{n-1}^T l_{n-1} + d_{n-1}\eta_{n-1} &= 1 && (7) \end{aligned}$$

From equation (6), we have

$$m_{n-1}^T = -\frac{c_{n-1}^T B_{n-1}}{d_{n-1}} \tag{8}$$

and it follows from equation (4) that

$$\Big(A_{n-1} - \frac{b_{n-1}c_{n-1}^T}{d_{n-1}}\Big)B_{n-1} = I_{n-1}$$

or equivalently that

$$B_{n-1} = \Big(A_{n-1} - \frac{b_{n-1}c_{n-1}^T}{d_{n-1}}\Big)^{-1}. \tag{9}$$

The form of B_{n-1} in Equation (9) will play an important role in the discussion of the following result [7].

Theorem 3.1: Let A_{n-1} and B_{n-1} be defined as in the previous lemma with $d_{n-1} < 0$. Then, A is diagonally stable if and only if $A_{n-1}^T D_{n-1} + D_{n-1}A_{n-1} < 0$ and $\big(B_{n-1}^{-1}\big)^T D_{n-1} + D_{n-1}B_{n-1}^{-1} < 0$ for some positive definite diagonal matrix $D_{n-1} \in \mathbb{R}^{n-1\times n-1}$; i.e. if both A_{n-1} and B_{n-1}^{-1} have a common diagonal Lyapunov function (CDLF).

Proof : *(a) Necessity :* Let A be a diagonally stable matrix. Then so is A^{-1}. Furthermore, if $V(x) = x^T D x$ is a diagonal Lyapunov function for Σ_A, it is is also a diagonal Lyapunov function for $\Sigma_{A^{-1}}$ where we denote

$$D = \begin{bmatrix} D_{n-1} & 0 \\ 0 & \alpha \end{bmatrix},$$

where D_{n-1} is a positive diagonal square matrix of dimension $[n-1]$ and $\alpha > 0$. It therefore follows from Sylvester's criterion that $A_{n-1}^T D_{n-1} + D_{n-1}A_{n-1} < 0$, and that $B_{n-1}^T D_{n-1} + D_{n-1}B_{n-1} < 0$. But if B_{n-1} is diagonally stable then so is B_{n-1}^{-1} with the same Lyapunov function. Hence, a necessary condition for the matrix A to be diagonally stable is that $\Sigma_{A_{n-1}}$ and $\Sigma_{B_{n-1}^{-1}}$ have a common diagonal Lyapunov function.

(b) Sufficiency : Let there exist a diagonal matrix $D_{n-1} > 0$ that simultaneously satisfies

$$A_{n-1}^T D_{n-1} + D_{n-1}A_{n-1} < 0; \tag{10}$$

$$\Big(A_{n-1} - \frac{b_{n-1}c_{n-1}^T}{d_{n-1}}\Big)^T D_{n-1} + D_{n-1}\Big(A_{n-1} - \frac{b_{n-1}c_{n-1}^T}{d_{n-1}}\Big) < 0. \tag{11}$$

To establish *sufficiency* we wish to show that there exists a positive scalar $\alpha > 0$ such that

$$\begin{bmatrix} A_{n-1} & b_{n-1} \\ c_{n-1}^T & d_{n-1} \end{bmatrix}^T \begin{bmatrix} D_{n-1} & 0 \\ 0 & \alpha \end{bmatrix} + \begin{bmatrix} D_{n-1} & 0 \\ 0 & \alpha \end{bmatrix}\begin{bmatrix} A_{n-1} & b_{n-1} \\ c_{n-1}^T & d_{n-1} \end{bmatrix} < 0,$$

It follows from the Kalman-Yacubovich-Popov lemma, that the above matrix inequality is satisfied if and only if the matrix

$$-\Big(A_{n-1}^T D_{n-1} + D_{n-1}A_{n-1}\Big) + \frac{D_{n-1}b_{n-1}c_{n-1}^T}{j\omega + d_{n-1}} + \frac{c_{n-1}b_{n-1}^T D_{n-1}}{-j\omega + d_{n-1}} > 0, \tag{12}$$

(is positive definite for all frequencies). Equations (10) and (11) imply that the matrix defined in Equation (12) has positive eigenvalues for $\omega = 0$ and for $\omega = \pm\infty$. Let there exist $\omega = \omega_c$ such that Equation (12) ceases to be positive definite. This means that at least one eigenvalue of

$$-\Big(A_{n-1}^T D_{n-1} + D_{n-1}A_{n-1}\Big) + \frac{D_{n-1}b_{n-1}c_{n-1}^T}{j\omega_c + d_{n-1}} + \frac{c_{n-1}b_{n-1}^T D_{n-1}}{-j\omega_c + d_{n-1}} \tag{13}$$

is non-positive. However, since the eigenvalues of a Hermitian matrix are always real, it follows by continuity that the matrix (13) will have a negative eigenvalue at ω_c if and only if

$$\det\Big[-\Big(A_{n-1}^T D_{n-1} + D_{n-1}A_{n-1}\Big) + \frac{D_{n-1}b_{n-1}c_{n-1}^T}{j\omega + d_{n-1}} + \frac{c_{n-1}b_{n-1}^T D_{n-1}}{-j\omega + d_{n-1}}\Big] = 0 \tag{14}$$

for some real $\omega = \omega^*$. Assuming that such an $\omega = \omega^*$ exists, it follows that

$$\det\Big[-\Big(A_{n-1}^T D_{n-1} + D_{n-1}A_{n-1}\Big) + \frac{D_{n-1}b_{n-1}c_{n-1}^T}{j\omega^* + d_{n-1}} + \frac{c_{n-1}b_{n-1}^T D_{n-1}}{-j\omega^* + d_{n-1}}\Big] = 0.$$

Then, $\det\big[M\big]\times$

$$\det\Big[I + \frac{M^{-1}D_{n-1}b_{n-1}c_{n-1}^T}{j\omega^* + d_{n-1}} + \frac{M^{-1}c_{n-1}b_{n-1}^T D_{n-1}}{-j\omega^* + d_{n-1}}\Big] = 0, \tag{15}$$

where the matrix $M = -\Big(A_{n-1}^T D_{n-1} + D_{n-1}A_{n-1}\Big)$ is invertible since it is positive definite (by assumption). Consequently,

$$\det\big[M\big]\det\Big[I + \Big[M^{-1}D_{n-1}b_{n-1}, M^{-1}c_{n-1}\Big]\times \begin{bmatrix} \frac{1}{j\omega^*+d_{n-1}} & 0 \\ 0 & \frac{1}{-j\omega^*+d_{n-1}} \end{bmatrix} \begin{bmatrix} c_{n-1}^T \\ b_{n-1}^T D_{n-1} \end{bmatrix}\Big] = 0,$$

This latter equation implies $\det\big[M\big]\times$

$$\det\Big[I_{2\times 2} + \begin{bmatrix} \frac{1}{j\omega^*+d_{n-1}} & 0 \\ 0 & \frac{1}{-j\omega^*+d_{n-1}} \end{bmatrix} \times \begin{bmatrix} c_{n-1}^T \\ b_{n-1}^T D_{n-1} \end{bmatrix} \begin{bmatrix} M^{-1}D_{n-1}b_{n-1}, M^{-1}c_{n-1} \end{bmatrix}\Big] = 0,$$

where the fact is used that $\det[I_{n\times n} + UV] = \det[I_{p\times p} + VU]$, with $U \in \mathbb{R}^{n\times p}$ and $V \in \mathbb{R}^{p\times n}$. Finally, this latter equation can be written as $\det\big[M\big]\big(\omega^{*2} + d_{n-1}^2\big)^{-1}\times$

$$\det\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = 0.$$

where

$$\begin{aligned} z_{11} &= j\omega^* + d_{n-1} + c_{n-1}^T M^{-1} D_{n-1} b_{n-1} \\ z_{12} &= c_{n-1}^T M^{-1} c_{n-1} \\ z_{21} &= b_{n-1}^T D_{n-1} M^{-1} D_{n-1} b_{n-1} \\ z_{22} &= -j\omega^* + d_{n-1} + b_{n-1}^T D_{n-1} M^{-1} c_{n-1} \end{aligned}$$

Since both $\det[M] \neq 0$ (M is positive definite) and $\omega^2 + d_{n-1}^2 > 0$ (for all ω), the above equation can only be satisfied only if the determinant of the matrix

$$\begin{bmatrix} d_{n-1} + c_{n-1}^T M^{-1} D_{n-1} b_{n-1} & c_{n-1}^T M^{-1} c_{n-1} \\ b_{n-1}^T D_{n-1} M^{-1} D_{n-1} b_{n-1} & d_{n-1} + b_{n-1}^T D_{n-1} M^{-1} c_{n-1} \end{bmatrix}$$

is negative. But this determinant of the matrix defined by (14) evaluated at $\omega = 0$, (scaled by a strictly positive number), which by assumption is positive. Hence, ω^* does not exist and by contradiction sufficiency is proven. □

Comment : Theorem 3.1 bears a strong resemblance a theorem on diagonal stability given by Redheffer in [8] and in fact this theorem can be derived as a direct consequence of Theorem 3.1. Redheffer demonstrated that if A_{n-1} and B_{n-1} share a CDLF, and if $d_{n-1} < 0$, then A is diagonally stable. In Theorem 3.1, necessary and sufficient conditions for Diagonal Stability are given in terms of A_{n-1} and B_{n-1}^{-1} (rather than B_{n-1}). Recall from Lemma 3.1 that B_{n-1}^{-1} is a matrix that differs from A_{n-1} by a matrix of rank 1. Further, this matrix is constructed directly from b_{n-1}, c_{n-1} and d_{n-1}. Consequently, Theorem 3.1, provides insight as to the nature of diagonal stability of A directly in terms its elements.

IV. On the Hurwitz stability of Metzler matrices

Theorem 3.1 (and the result of Redheffer [8]), both replace the problem of determining a diagonal Lyapunov function for a matrix A to the equivalent problem of determining a common diagonal Lyapunov function (CDLF) for two lower dimensional matrices. In general, the latter problem is not simpler than the former. However, in special cases, due to the structure of the two matrices A_{n-1} and B_{n-1}^{-1} described in Section 3, the determination of a CDLF is substantially simplified. One such class is the class of Metzler matrices.

We now note the following result concerning a pair of Metzler matrices one of which is entry-wise *bigger* than the other [9].

Lemma 4.1: Let $A, A + bc^T$ be a pair of Metzler matrices, where b, c are non-negative vectors. Let $\big(A + bc^T\big)^T D + D\big(A + bc^T\big) < 0$ for some diagonal matrix $D > 0$. Then $A^T D + DA < 0$.

Proof : The proof of the lemma follows immediately from the fact that $\big(A + bc^T\big)^T D + D\big(A + bc^T\big) \succ A^T D + DA$. Since both matrices are Metzler, and since $\big(A + bc^T\big)^T D + D\big(A + bc^T\big) < 0$, the assertion of the lemma follows immediately from basic properties of Metzler matrices. □

A consequence of Theorem 3.1 is the following.

Theorem 4.1: Let the matrix $A \in \mathbb{R}^{n\times n}$ be a Metzler matrix. Define the sequence of matrices $\{A[n], A[n-1], ..., A[1]\}$ as follows. $A[n] = A$. For $k = 2, ..., n$ partition

$$A[k] = \begin{bmatrix} A_{k-1} & b_{k-1} \\ c_{k-1} & d_{k-1} \end{bmatrix} \tag{16}$$

where $A_{k-1} \in \mathbb{R}^{(k-1)\times(k-1)}, b_{k-1}, c_{k-1} \in \mathbb{R}^{k-1}$. The matrix $A[k-1]$ is defined to be $A[k-1] = A_{k-1} - \frac{b_{k-1}c_{k-1}^T}{d_{k-1}}$. Then, a necessary and sufficient condition for the matrix A to be Hurwitz, is that the diagonal entries of the matrices $A[1], .., A[n]$ are all strictly negative.

Proof : The assertions of the Theorem follow from Theorem 3.1, Lemma 4.1, and from the fact that a Metzler matrix is Hurwitz if and only if it is diagonally stable [9]. If A is diagonally stable, by the principal theorem, the dynamic systems

$$\dot{x} = A_{n-1}x, \text{ and } \dot{x} = \Big(A_{n-1} - \frac{b_{n-1}c_{n-1}^T}{d_{n-1}}\Big)x$$

must have a common diagonal Lyapunov function. But, since $A[n-1]$ is Metzler, and since b_{n-1}, c_{n-1} are both entrywise nonnegative, by Lemma 4.1 these dynamic systems will have a common diagonal Lyapunov function if and only if $A[n-1]$ is Hurwitz stable. Thus, A is Hurwitz if and only if $A[n-1]$ is diagonally stable (or equivalently Hurwitz). But by the previous argument $A[n-1]$ is Hurwitz stable if and only if $A[n-2]$ is diagonally stable. Hence, it follows that the diagonal entries of $A[n-2]$ must also be negative. By repeating this argument for all $A[i]$, $i = n, ..., 2$ one concludes that the diagonal entries of all of the aforementioned matrices must be negative. Further, if the scalar $A[1] < 0$, then this in turn implies that $A[2]$ is Hurwitz, and hence that all $A[i]$ are Hurwitz . $\square$

V. Examples

An attractive feature of the proof of the previous theorem is that it provides a very simple procedure to check whether a Metzler matrix is Hurwitz. Given a Metzler matrix, we recursively construct its Schur complement. At every stage we only verify whether or not the diagonal elements of the matrix of lower dimension are negative. The process is continued until only a single element remains. A necessary and sufficient condition for A to be diagonally stable is that all diagonal entries of the lower dimensional matrices are negative. The following examples illustrate the procedure discussed in the paper.

Example 5.1: Let A be a Metzler matrix:

$$A = \begin{bmatrix} -2 & 1 \\ 3 & -3 \end{bmatrix}.$$

This matrix is diagonally stable, since its diagonal elements are negative, and its determinant is positive [10]. Using the procedure outlined in the paper, the lower dimensional matrix is the scalar -2 + $\frac{3}{3}$= -1 which assures that A is diagonally stable. Note that the latter condition is identical to the condition on the (negative) determinant.

Example 5.2: Let A be a Metzler matrix in $\mathbb{R}^{3\times3}$:

$$A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -6 & 1 \\ 3 & 2 & -6 \end{bmatrix}.$$

Then

$$A_2 = \begin{bmatrix} -3 & 1 \\ 1 & -6 \end{bmatrix}, \; b_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, c_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

and $d_2 = -6$;

$$A[2] = A_2 - \frac{b_2 c_2^T}{d_2} = \begin{bmatrix} -2 & 1.67 \\ 1.50 & -5.67 \end{bmatrix}.$$

To determine whether $A[2]$ is diagonally stable we repeat the operation to reduce it to a scalar $A[1] = -1.59$ which is negative. This in turn ensures that A is diagonally stable.

Example 5.3: Let A be a Metzler matrix in $\mathbb{R}^{6\times6}$

$$A = \begin{bmatrix} -14 & 2 & 3 & 4 & 5 & 6 \\ 1 & -15 & 7 & 5 & 2 & 3 \\ 1 & 1 & -14 & 1 & 1 & 1 \\ 2 & 3 & 4 & -14 & 1 & 1 \\ 1 & 2 & 4 & 5 & -9 & 7 \\ 1 & 1 & 0 & 9 & 1 & -14 \end{bmatrix}.$$

Then

$$A[5] = \begin{bmatrix} -13.57 & 2.43 & 3 & 7.86 & 5.43 \\ 1.21 & -14.79 & 7 & 6.93 & 2.21 \\ 1.07 & 1.07 & -14 & 1.64 & 1.07 \\ 2.07 & 3.07 & 4 & -13.36 & 1.07 \\ 1.5 & 2.50 & 4 & 9.5 & -8.5 \end{bmatrix};$$

$$A[4] = \begin{bmatrix} -12.61 & 4.03 & 5.55 & 13.92 \\ 1.61 & -14.13 & 8.04 & 9.40 \\ 1.26 & 1.39 & -13.50 & 2.84 \\ 2.26 & 3.39 & 4.50 & -12.16 \end{bmatrix};$$

$$A[3] = \begin{bmatrix} -10.02 & 7.90 & 10.71 \\ 3.35 & -11.52 & 11.53 \\ 1.79 & 2.18 & -12.44 \end{bmatrix};$$

$$A[2] = \begin{bmatrix} -8.49 & 9.78 \\ 5.01 & -9.50 \end{bmatrix}; A[1] = -3.33.$$

The matrices $A[5], A[4], A[3], A[2]$, and $A[1]$ are constructed as in the previous examples. It follows from the assertions of the main theorem that A is a Hurwitz Metzler matrix.

VI. Conclusions

In this paper a very simple test is developed for determining whether a given Metzler matrix is Hurwitz stable. This test is derived by noting that any Metzler matrix that is Hurwitz stable, is also diagonally stable, and by using the Kalman-Yacubovic-Popov (KYP) lemma to obtain a characterization of diagonally stable matrices. The interesting feature of the result is that it reduces the stability of a given positive dynamic systems to the stability of lower dimensional positive system.

REFERENCES

[1] L. Farina and S. Rinaldi, *Positive linear systems: Theory and applications*. Wiley, 2000.

[2] A. Berman and R. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*. SIAM, 1979.

[3] D. Siljak, *Large Scale Dynamic Systems*. North-Holland-New York, 1978.

[4] S. Meyn, *Control Techniques for Complex Networks*. University of Illinois, In preparation, 2007.

[5] C. R. Johnson, "Sufficient Condition for D-stability," *Journal of Economic Theory*, vol. 9, pp. 53–62, 1974.

[6] T. Kailath, *Linear Systems*. Prentice Hall, New Jersey, 1980.

[7] R. Shorten and K. Narendra, "On a Theorem on Diagonal Stability by Redheffer." To appear in *Linear Algebra and its Applications*, 2009.

[8] R. Redheffer, "Volterra multipliers II," *SIAM Journal on Algebraic and Discrete Methods*, vol. 4, no. 6, pp. 612–620, 1985.

[9] R. Horn and C. Johnson, *Topics in Matrix Analysis*. Cambridge University Press, 1991.

[10] N. Oleng and K. Narendra, "On the existence of diagonal solutions to the Lyapunov equation for third order systems," in *proceedings of American Control Conference*, 2003.