2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Certainty Equivalence Adaptive Control of Plants with Unmatched Uncertainty using State Feedback[1]

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***Abstract*—A systematic design procedure using state-feedback Certainty Equivalence Adaptive Control (CEAC) technique is developed for linear plants and a class of nonlinear plants with unmatched uncertainty. It is shown that a reduced order observer and adaptive laws with normalization in conjunction with the CEAC law result in a stable overall system in the case of linear plants of any relative degree, and a class of nonlinear plants of relative degree two. In the case of higher relative degrees a CEAC approach based on multiple observers is proposed. The proposed schemes guarantee overall system stability and asymptotic tracking.**

I. Introduction

The problem of slow actuator dynamics arises in many control applications. One particularly important case is that of fault-tolerant flight control when the objective is to compensate for flight-critical actuator failures or wing damage using collective or differential engines. Engine dynamics is generally much slower than that of the actuators moving flight control surfaces and, in general, cannot be neglected during the control design. If there is uncertainty in the aircraft dynamics due to faults or failures, this type of systems falls into the category of plants with unmatched uncertainty.

The control design for plants with unmatched uncertainty has been addressed theoretically using the Adaptive Backstepping technique [4]. This approach is of direct adaptive control type [9] where controller parameters are adjusted directly based on the response of the system.

Going back to the problem of fault-tolerant flight control, in some cases there is a need to implement indirect adaptive control schemes where plant parameters are estimated on-line and used at every instant in the control law. This approach is also referred to as the Certainty Equivalence Adaptive Control (CEAC) since the control law for the case of known parameters is used, and the true parameters are replaced with their estimates at every instant. CEAC approach is useful in the case of fault-tolerant control since it includes a parameter estimation subsystem and can, therefore, be used to explicitly estimate failure or damage related parameters, and to provide information to the pilot regarding what exactly has happened with the aircraft.

Another motivation for this paper comes from the attempts to implement adaptive control using multiple models to control the plants with unmatched uncertainty. In order to apply such a control strategy, an indirect adaptive control approach is required.

Indirect adaptive control designs are available in the case of nonlinear multivariable relative degree one plants [1], [2], [3]. In the case of plants with unmatched uncertainty and higher relative degrees, available designs are of direct adaptive control type [5], [6], [7]. Literature search has revealed that a thorough indirect adaptive control approach is not yet available for nonlinear plants with unmatched uncertainty. Existing output feedback techniques (see e.g. [8] and references therein) do not appear easily adaptable to this case even in the linear case. Indirect adaptive controllers for linear plants with unmatched uncertainty result in complicated designs based on the so called augmented error and large-order signal filtering [9].

The main contribution of this paper, that addresses the above mentioned problems, is the development of a stable Certainty Equivalence Adaptive Control (CEAC) design for plants with unmatched uncertainty.

The CEAC strategy proposed in this paper is intuitively straightforward and practically easy to implement. The proposed design procedure consists of the following steps: (i) Design an ideal control law assuming that the plant parameters are known; (ii) Design a suitable observer and adaptive laws to estimate the unknown parameters on-line; and (iii) Replace the true parameters in the ideal control law with their estimates at every instant. This approach has been shown to result in a stable closed-loop system for linear plants [9]. However, extensions of this approach to the nonlinear plants with unmatched uncertainty are lacking in the existing literature.

In this paper, a systematic control design procedure for plants with unmatched uncertainty using the CEAC approach is proposed. Specific focus is on the plants where the uncertainty is concentrated in the output equation, while the "actuator" dynamics is assumed known. The analysis of the CEAC designs for this class of plants has revealed the following:

• A reduced-order observer and "pure" CEAC suffice in the case of linear plants of any relative degree and a class of nonlinear plants of relative degree two. In this case an adaptive algorithm with normalization can be used to bound

[1]This work was partially supported by NASA under contracts #NNL07AA02C and NNX08CA51P to SSCI

the adaptive law thus avoiding its repeated differentiation; and

• A full-order observer and an "augmented" CEAC are needed in the case of nonlinear plants of higher relative degrees. In this case a proper coordinate transformation is needed in order to assure the overall system stability.

The paper is organized as follows: The control problem for linear plants of an arbitrary relative degree is addressed in Section II. In section IV, the control design procedure for linear plants is presented. Extensions of results to a class of nonlinear plants is discussed in Section 4. In the same section, a procedure for a class of nonlinear plants with general continuously differentiable nonlinearities is given, while concluding remarks are given in Section V.

II. Problem Statement

Let the plant dynamics be described by the following model:

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta f(x_1) \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= u, \end{aligned} \tag{1}$$

where x_1 is the plant output, θ is an unknown constant parameter, x and u are measurable, f is a sufficiently smooth known function with well defined $n-1$ partial derivatives with respect to x_1, and bounds on the plant parameter are known, i.e. $\theta \in \mathcal{S}_\theta = \{\theta : \underline{\theta} \leq \theta \leq \overline{\theta}\}$.

The above plant is referred to as the plant with unmatched uncertainty since the uncertainty does not appear in the same equation as the control input. In addition, of interest is to define a relative degree of the plant n^*. Loosely speaking, the relative degree is a number of integrators between the input and the output. In this case $n^* = n$, where n is the plant order.

The reference model is described by

$$\begin{aligned} \dot{x}_{m1} &= x_{m2} \\ \dot{x}_{m2} &= x_{m3} \\ &\vdots \\ \dot{x}_{mn} &= -k^T x_m + k_1 r, \end{aligned}$$

where $x_m = [x_{m1}\ x_{m2}\ ...\ x_{mn}]^T$, $k = [k_1\ k_2\ ...\ k_n]^T$, its elements k_i are chosen such that the matrix:

$$A_m = \begin{bmatrix} 0 \mid I \\ \hline -k^T \end{bmatrix}$$

is Hurwitz, and r denotes a bounded piece-wise continuous reference input.

The control objective is to design a control input $u(t)$ such that all the signals in the system are bounded and, in addition, $\lim_{t\to\infty}[x_1(t) - x_{m1}(t)] = 0$.

As shown in the existing literature [5], [6], [7], this is a difficult problem due to the fact that the uncertainty appears in the first equation, while the control input is separated from the uncertainty by $n-1$ integrators.

Due to the complexity of the problem, the analysis and subsequent control design will be divided into two parts. In the following section the case of linear plants will be considered, followed by the analysis of the nonlinear plants of the form (1).

III. CEAC Design for Linear Plants

In this case $f(x_1) = x_1$. The first step in CEAC design is the design of an ideal controller, i.e. a controller that assumes that θ is known. This is discussed next.

Ideal Controller: To design such a controller, the following coordinate transformation is introduced first:

$$\begin{aligned} z_1^* &= x_1 \\ z_2^* &= x_2 + \theta z_1^* \\ &\vdots \\ z_n^* &= x_n + \theta z_{n-1}^*. \end{aligned}$$

Taking the derivative of z^* yields: $\dot{z}_i^* = z_{i+1}^*,\ i = 1, 2, ..., n-1$, and $\dot{z}_n^* = u + \theta z_n^*$. Hence the control law that achieves the tracking control objective is of the form:

$$u^* = -\theta z_n^* - k^T z^* + k_1 r,$$

since it results in $\dot{z}^* = A_m z^* + b_m r$, where $b_m = [0\ 0\ ...\ 0\ 1]^T$, and $\lim_{t\to\infty}[z^*(t) - x_m(t)] = 0$.

The CEAC law: In the case when θ is unknown, the following CEAC law is suggested:

$$u = -\hat{\theta} z_n - k^T z + k_1 r, \tag{2}$$

where variables z_i are defined by the recursion:

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= x_2 + \hat{\theta} z_1 \\ &\vdots \\ z_n &= x_n + \hat{\theta} z_{n-1}, \end{aligned} \tag{3}$$

and where $\hat{\theta}$ denotes an estimate of θ. It is noted that z_n can be expressed as $z_n = \sum_{i=1}^{n} x_i \hat{\theta}^{n-i}$.

The main question now is as to how to generate the estimate of θ to assure that the control objective is met.

To address this issue, the following observer and adaptive law are proposed:

$$\dot{\hat{x}}_1 = x_2 + \hat{\theta}\hat{x}_1 - \lambda e \tag{4}$$

$$\dot{\phi} = \dot{\hat{\theta}} = \text{Proj}_{\hat{\theta}\in\mathcal{S}_\theta}\{\frac{-\gamma x_1 e}{1 + \omega^T\omega}\}, \tag{5}$$

where $\lambda > 0$ denotes the observer gain, $e = \hat{x}_1 - x_1$, $\phi = \hat{\theta} - \theta$, $\gamma > 0$ denotes an adaptive gain, $\text{Proj}_{(\cdot)}\{(\cdot)\}$ denotes a projection operator, and where $\omega = [x_1, x_2,\ ...\ x_n]^T$. Properties of the Projection Operator can be found e.g. in [10]. It is seen that an adaptive law with normalization is used, resulting in a bounded $\dot{\hat{\theta}}(t)$, which will be shown to be crucial in avoiding differentiation of the adaptive laws to arrive at a stable closed-loop system.

It will be shown in the subsequent analysis that if the observer gain is sufficiently large, the adaptive law with normalization guarantees overall system stability within the CEAC framework.

The following proposition is useful for proving the main result of this section.

Proposition 1: Let $\omega = [x_1,\ x_2,\ ...\ x_n]^T$. Then

$$\|\omega^T \dot{\omega}(\hat{\theta}, \theta)\| \leq \alpha_1 \|\omega\|^2 + \alpha_2 \|\omega\|, \tag{6}$$

holds $\forall \omega \in \mathbb{R}^n$ and $\forall(\hat{\theta}, \theta) \in \mathcal{S}_\theta$, where $\alpha_i > 0$ are known constants.

Proof: Based on the definitions of z and ω, it is first noted that:

$$z = M(\hat{\theta})\omega,$$

where

$$M = \begin{bmatrix} 1 & 0 & 0 & ... & 0 & 0 \\ \hat{\theta} & 1 & 0 & ... & 0 & 0 \\ \hat{\theta}^2 & \hat{\theta} & 1 & ... & 0 & 0 \\ & \vdots & & & & \vdots \\ \hat{\theta}^{n-1} & \hat{\theta}^{n-2} & \hat{\theta}^{n-3} & \dots & \hat{\theta} & 1 \end{bmatrix}.$$

Further, based on the above relationship, z_n is expressed as:

$$z_n = v^T(\hat{\theta})\omega,$$

where $v(\hat{\theta}) = [\hat{\theta}^{n-1}\ \hat{\theta}^{n-2}\ ...\ \hat{\theta}\ 1]^T$. Hence

$$u = -[\hat{\theta}v^T(\hat{\theta}) + k^T M(\hat{\theta})]\omega + k_1 r$$

It now follows that:

$$\dot{\omega} = C(\hat{\theta}, \theta)\omega + dr,$$

where

$$C = \begin{bmatrix} \theta & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & \vdots & & & \vdots \\ C_{n,1} & C_{n,2} & C_{n,3} & \dots & C_{n,n} \end{bmatrix}$$

where $C_{n,1} = -\hat{\theta}^n - \sum_{i=1}^n k_i \hat{\theta}^{i-1}$, $C_{n,2} = -\hat{\theta}^{n-1} - \sum_{i=2}^n k_i \hat{\theta}^{i-2}$, $C_{n,3} = \hat{\theta}^{n-2} - \sum_{i=3}^n k_i \hat{\theta}^{i-3}$, $C_{n,n} = -\hat{\theta} - k_n$, and $d = [0\ \dots\ 0\ k_1]^T$. It is seen that elements of $C(\hat{\theta}, \theta)$ are bounded since adaptive laws with projection are used. Hence:

$$\begin{aligned} \|\omega^T \dot{\omega}\| &= \|\omega^T C \omega + \omega^T dr\| \\ &\leq \alpha_1 \|\omega\|^2 + \alpha_2 \|\omega\| \end{aligned}$$

where $\alpha_2 = k_1 \bar{r}$, and $|r| \leq \bar{r}$. ■

It can now be concluded that

$$\frac{\|\omega^T \dot{\omega}(\hat{\theta}, \theta)\|}{1 + \omega^T \omega} \leq \alpha_0, \tag{7}$$

holds $\forall \omega \in \mathbb{R}^n$ and $\forall(\hat{\theta}, \theta) \in \mathcal{S}_\theta$, where $\alpha_0 = \alpha_1 + \alpha_2/2$, with α_i, $i = 1, 2$, being known constants.

Now the following Theorem is considered:

Theorem 1: *Let the plant* (1)*, where* $f(x_1) = x_1$*, be controlled by the control law* (2) *where the parameter estimate is generated using the observer* (4) *and adaptive law* (5)*. Then, if the observer gain satisfies*

$$\lambda > \bar{\theta} + \alpha_0,$$

where α_0 *is given by* (7)*, all the signals in the system are bounded, and* $lim_{t \to \infty}[x_1(t) - x_{m1}(t)] = 0$.

Proof: Upon subtracting the first equation of the plant (1) from the observer dynamics (4), the error equation is obtained as:

$$\dot{e} = -(\lambda - \hat{\theta})e + \phi x_1, \tag{8}$$

where $e = \hat{x}_1 - x_1$ and $\phi = \hat{\theta} - \theta$.

Let a coordinate transformation ζ be defined as:

$$\zeta = \frac{e}{\sqrt{1 + \omega^T \omega}}.$$

It follows that:

$$\begin{aligned} \zeta\dot{\zeta} &= \zeta(\frac{\partial \zeta}{\partial e}\dot{e} + \frac{\partial \zeta}{\partial \omega}\dot{\omega}) \\ &= \frac{-e^2 \omega^T \dot{\omega}}{(1 + \omega^T \omega)^2} + \frac{-(\lambda - \hat{\theta})e^2 + \phi \hat{x}_1 e}{(1 + \omega^T \omega)}. \end{aligned}$$

Now the following tentative Lyapunov function is chosen:

$$V(\zeta, \phi) = \frac{1}{2}(\zeta^2 + \frac{\phi^2}{\gamma}),$$

where, as mentioned earlier, $\gamma > 0$ is the adaptive gain which can be chosen at the designer's discretion. The derivative of V along the solutions of the system yields

$$\begin{aligned} \dot{V}(\zeta, \phi) &= \zeta\dot{\zeta} + \phi\dot{\phi}/\gamma \\ &= \frac{-e^2 \omega^T \dot{\omega}}{(1 + \omega^T \omega)^2} + \frac{-(\lambda - \hat{\theta})e^2 + \phi \hat{x}_1 e}{(1 + \omega^T \omega)} + \frac{\phi\dot{\phi}}{\gamma} \\ &\leq \frac{-e^2}{1 + \omega^T \omega}(\lambda - \hat{\theta} + \frac{\omega^T \dot{\omega}}{1 + \omega^T \omega}). \end{aligned}$$

The last inequality follows from applying the adaptive law with projection (5).

Using the result of the Proposition 1 and expression (7), it follows that:

$$\dot{V}(\zeta, \phi) \leq \frac{-e^2}{1 + \omega^T \omega}(\lambda - \bar{\theta} - \alpha_0) \leq 0,\ \forall(\zeta, \phi) \neq (0, 0).$$

It is seen that $\dot{V} \leq 0$ is obtained since $\lambda > \alpha_0 + \bar{\theta}$. It can now be concluded that $\zeta \in \mathcal{L}^\infty \cap \mathcal{L}^2$, $\phi \in \mathcal{L}^\infty$. It also follows from (5) that $\dot{\phi} = \dot{\hat{\theta}} \in \mathcal{L}^\infty \cap \mathcal{L}^2$.

To prove the overall stability, the following coordinate transformation is chosen:

$$\begin{aligned} \hat{z}_1 &= \hat{x}_1 \\ \hat{z}_2 &= x_2 + \hat{\theta}\hat{z}_1 \\ &\vdots \\ \hat{z}_n &= x_n + \hat{\theta}\hat{z}_{n-1}. \end{aligned}$$

It is seen that the relationship between z and $\hat{z}$ is of the form: $\hat{z} = z + v(\hat{\theta})e$.

The above transformation is next differentiated to obtain:

$$\dot{\hat{z}}_1 = \hat{z}_2 - \lambda e$$
$$\dot{\hat{z}}_2 = \hat{z}_3 + \dot{\hat{\theta}}\hat{z}_1 - \lambda\hat{\theta}e$$
$$\vdots$$
$$\dot{\hat{z}}_n = u + \hat{\theta}\hat{z}_n + \dot{\hat{\theta}}\hat{z}_{n-1} + \hat{\theta}\dot{\hat{\theta}}z_{n-2} + \ldots + \hat{\theta}^{n-2}\dot{\hat{\theta}}z_1 - \lambda\hat{\theta}^{n-1}e.$$

It is next noted that the control law (2) can be rewritten as:

$$u = -\hat{\theta}(\hat{z}_n - \hat{\theta}^{n-1}e) - k^T(\hat{z} - v(\hat{\theta})e) + k_1 r.$$

Substituting this control law into the above equation yields

$$\dot{z} = [A_m + N(t)]z + L(\hat{\theta})e + br, \tag{9}$$

where

$$N(t) = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ \hat{\theta}(t) & 1 & 0 & & \\ \vdots & \hat{\theta}(t) & 1 & 0 & \\ \hat{\theta}^{n-2}(t) & \cdots & \hat{\theta}(t) & 1 & 0 \end{bmatrix} \cdot \dot{\hat{\theta}}(t), \; b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ k_1 \end{bmatrix},$$

and $L(\hat{\theta}) = [-\lambda, -\lambda\hat{\theta}, -\lambda\hat{\theta}^2, \cdots, -\lambda\hat{\theta}^{n-1} + \hat{\theta}^n + k^T v(\hat{\theta})]^T$. Since A_m is Hurwitz and $\|N(t)\| \in \mathcal{L}^\infty \cap \mathcal{L}^2$ (which follows from the fact that $\hat{\theta} \in \mathcal{L}^\infty$ and $\dot{\hat{\theta}} \in \mathcal{L}^\infty \cap \mathcal{L}^2$), it can now be concluded that, for $e = 0$ and $r = 0$, the above system is exponentially stable, and, therefore, BIBO stable. Hence to demonstrate signal boundedness, one needs to show that e is bounded.

Since $\hat{\theta}$ is bounded and $\dot{\hat{\theta}} \in \mathcal{L}^\infty \cap \mathcal{L}^2$, it follows that (9) is a linear time varying system with bounded parameters in which the signals can grow at most exponentially. Hence analysis based on the growth rates of signals (see [9], pp. 476-479) can be used. Based on the fact that $\zeta = e/\sqrt{1 + \omega^T\omega} \in \mathcal{L}^\infty \cap \mathcal{L}^2$, it follows that $e = \beta(t)\sqrt{1 + \omega^T\omega}$, where $\beta \in \mathcal{L}^2$ [9].

It is now assumed that ω grows in an unbounded fashion. The equation (8) is rewritten as:

$$\dot{e} = -(\lambda - \hat{\theta})e + \bar{\phi}^T\omega,$$

where $\bar{\phi} = [0, 0, \phi]^T$. Since $\lambda > \bar{\theta}$, the above first-order system is exponentially stable for $\bar{\phi} = 0$. This, along with the fact that $\bar{\phi}$ is bounded, implies that $e = \mathcal{O}(sup_{\tau \le t}\|\omega(\tau)\|)$. This, along with (8), also implies that $\dot{e} = \mathcal{O}(sup_{\tau \le t}\|\omega(\tau)\|)$. From (9) it follows that $\|z\| = \mathcal{O}(sup_{\tau \le t}|e(\tau)|)$. Since $\hat{x}_1 = z_1$, $x_2 = z_2 - \hat{\theta}\hat{x}_1$, and $x_1 = \hat{x}_1 - e$, it follows that $\hat{x}_1 = \mathcal{O}(sup_{\tau \le t}|e(\tau)|)$, $x_2 = \mathcal{O}(sup_{\tau \le t}|e(\tau)|)$, and $x_1 = \mathcal{O}(sup_{\tau \le t}|e(\tau)|)$. Thus, $\|\omega\| = \mathcal{O}(sup_{\tau \le t}|e(\tau)|)$. It can now be concluded that e and $\|\omega\|$ grow at the same rate, i.e. $sup_{\tau \le t}|e(\tau)| \sim sup_{\tau \le t}\|\omega(\tau)\|$.

On the other hand, since $e = \beta(t)\sqrt{1 + \omega^T\omega}$, where $\beta \in \mathcal{L}^2$, and $\dot{e} = \mathcal{O}(sup_{\tau \le t}\|\omega(\tau)\|)$, it follows that $e = \mathbf{o}(sup_{\tau \le t}\|\omega(\tau)\|)$ [9], i.e. e and $\|\omega\|$ grow at different rates, which is a contradiction. Hence all the signals in the closed-loop system are bounded. It can now be readily shown using the standard arguments that $lim_{t\to\infty}e(t) = 0$. ∎

Comments:

- The key element in the proposed design is the use of the normalization in the adaptive law (5). Since the normalization bounds a product of $\dot{\hat{\theta}}$ and its regressor, this property is used to avoid repeated differentiation of the adaptive law during the control design.
- While the properties of adaptive laws with normalizations in the case of static observers have been well established, their use in the context of dynamic observers remains less understood. In the paper the condition on the observer gain is given to guarantee overall system stability with such adaptive law.
- The resulting closed-loop system is linear in $\dot{\hat{\theta}}$; this, along with the property $\dot{\hat{\theta}} \in \mathcal{L}^\infty \cap \mathcal{L}^2$ that arises from the use of adaptive laws with normalization, is the key in proving signal boundedness and, eventually, asymptotic tracking.

IV. Nonlinear Plants

In this section, the case of plants (1) is considered.

A. Ideal Control Design

To design an ideal controller, in this case the following coordinate transfromation is used:

$$\begin{aligned} z_1^* &= x_1 \\ z_2^* &= x_2 + \theta f(x_1) \\ z_3^* &= x_3 + \theta \dot{f}(x_1) \\ &\vdots \\ z_n^* &= x_n + \theta \frac{d^{n-2}}{dt^{n-2}} f(x_1), \end{aligned}$$

resulting in:

$$\begin{aligned} \dot{z}_1^* &= z_2^* \\ \dot{z}_2^* &= z_3^* \\ &\vdots \\ \dot{z}_n^* &= u + \theta \frac{d^{n-1}}{dt^{n-1}} f(x_1). \end{aligned}$$

For instance, for $n^* = 3$, $\dot{z}_3^* = u + \theta(f'' z_2^{*2} + f' z_3^*)$, etc.

The ideal controller is now of the form:

$$u^* = -\theta \frac{d^{n-1}}{dt^{n-1}} f(x_1) - k^T z^* + k_1 r,$$

resulting in $\lim_{t\to\infty}[z^*(t) - x_m(t)] = 0$.

B. Continuously differentiable f with bounded $|f'|$

In this section it will be shown that an analysis similar to that for the case of linear plants can be used in the case of nonlinear plants where the nonlinearity has a bounded first partial derivative. It will be also shown that this analysis holds only for $n^* = 2$, while for higher relative degrees a different design is needed.

1) Case $n^* = 2$: In this case the plant is described by

$$\dot{x}_1 = x_2 + \theta f(x_1), \quad \dot{x}_2 = u, \tag{10}$$

where $|f'(x_1)|$ is bounded for all x_1. The CEAC law is:

$$u = -\hat{\theta} f'(x_2 + \hat{\theta} f) - k_1\hat{x}_1 - k_2(x_2 + \hat{\theta} f) + k_1 r. \tag{11}$$

Following the design procedure for linear plants, the following observer is used:

$$\dot{\hat{x}}_1 = x_2 + \hat{\theta}\hat{f} - \lambda e. \tag{12}$$

The resulting error equation is then: $\dot{e} = -\lambda e + \phi\hat{f} + \hat{\theta}(\hat{f} - f)$, where $\hat{f} = f(\hat{x}_1)$.

For continuously differentiable f, it follows that $f(\hat{x}_1) - f(x_1) = f'(x^*)e$, for some $x^*(x_1, \hat{x}_1) \in \mathcal{R}$. Thus the error equation can be rewritten as:

$$\dot{e} = -(\lambda - \hat{\theta}f')e + \phi f.$$

In this case the following normalization is proposed: $\zeta = e/\sqrt{1 + \omega^T\omega}$, where $\omega = [x_1, x_2, f]^T$. The tentative Lyapunov function is chosen as:

$$V(\zeta, \phi) = \frac{1}{2}(\zeta^2 + \frac{\phi^2}{\gamma}).$$

Its derivative along the solutions of the system yields

$$\dot{V}(\zeta, \phi) \leq \frac{-e^2}{1 + \omega^T\omega}(\lambda - \hat{\theta}f' + \frac{\omega^T\dot{\omega}}{1 + \omega^T\omega}).$$

The inequality follows from applying the adaptive law

$$\dot{\phi} = \dot{\hat{\theta}} = Proj_{\hat{\theta} \in \mathcal{S}_\theta}\{\frac{-\gamma f e}{1 + \omega^T\omega}\}.$$

From (10) and (12), it follows that

$$\dot{\omega} = [x_2 + \theta f, u, f'(x_2 + \theta f)]^T.$$

Using (11), it can now be readily verified that $\left|(\omega^T\dot{\omega}(\theta, \hat{\theta}))/(1 + \omega^T\omega)\right| \leq \alpha_0, \ \forall(\theta, \hat{\theta}) \in \mathcal{S}_\theta$, where α_0 is known. Hence, by choosing $\lambda > \bar{\theta} + \alpha_0$, it can now be concluded that $\zeta \in \mathcal{L}^\infty \cap \mathcal{L}^2$, and $\phi \in \mathcal{L}^\infty$. It also follows that $\dot{\phi} = \dot{\hat{\theta}} \in \mathcal{L}^\infty \cap \mathcal{L}^2$.

By taking the following transformation $z_1 = \hat{x}_1, \quad z_2 = x_2 + \hat{\theta}\hat{f}$, it follows that:

$$\dot{z} = A_m z + L(t)e + Br,$$

where

$$A_m = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}, L(t) = \begin{bmatrix} -\lambda \\ -\lambda\hat{\theta}\hat{f}' + \varphi \end{bmatrix}, B = \begin{bmatrix} 0 \\ k_1 \end{bmatrix},$$

where $\varphi = 0$ if $\dot{\hat{\theta}} = 0$, and $\varphi = -\frac{\gamma f^2}{1 + \omega^T\omega}$ elsewhere. Since the elements of $L(t)$ are uniformly bounded, signals in the above system can grow at most exponentially [9]. Thus, the signal growth rate argument for showing the overall signal boundedness parallels that of the previous section.

Comments:

• If an additional condition $f(0) = 0$ is imposed to the nonlinearity f, then $\dot{\hat{\theta}}\hat{f} = \dot{\hat{\theta}}\hat{f}'(\hat{x}^*)\hat{x}_1$ for some $\hat{x}^*(\hat{x}_1) \in \mathcal{R}$. By using the pure CEAC control law $u = -\hat{\theta}\hat{f}'(x_2 + \hat{\theta}\hat{f}) - k_1\hat{x}_1 - k_2(x_2 + \hat{\theta}\hat{f}) + k_1 r$ instead of the augmented CEAC control law (11), overall stability and asymptotic tracking can be readily demonstrated.

• Results from this section can be readily extended to piecewise differentiable function f with $f(\hat{x}_1) - f(x_1) = g(x^*)(\hat{x}_1 - x_1)$, where $|g(x^*)|$ is bounded for all $x^* \in \mathcal{R}$.

2) Case $n^* = 3$: Previous results are based on the key fact that use of normalization in the Lyapunov function results in a term $\omega^T\dot{\omega}(\theta, \hat{\theta})/(1 + \omega^T\omega)$ that is bounded for all ω and all $(\theta, \hat{\theta}) \in \mathcal{S}_\theta$. However, in the case of $n^* = 3$ and higher, terms such as $x_3 x_2^2/(1 + \omega^T\omega)$ that cannot be shown to be bounded for all ω will appear in $\omega^T\dot{\omega}$. Therefore, in the case of $n^* \geq 3$, there is a structural obstacle preventing the use of the normalization in the adaptive law. This leads to the full-order observer approach discussed in the following section.

C. General Continuously Differentiable f

For simplicity, the attention is focused on the case of relative degree $n^* = 3$. The plant of relative degree 3 is described by

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta f(x_1) \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u. \end{aligned}$$

The objective is to design a control input $u(t)$ such that $x \in \mathcal{L}^\infty$ and $\lim_{t\to\infty}[x_1(t) - x_{m1}(t)] = 0$, where $x_{m1}(t)$ is an output of the reference model

$$\begin{aligned} \dot{x}_{m1} &= x_{m2} \\ \dot{x}_{m2} &= x_{m3} \\ \dot{x}_{m3} &= -k_1 x_{m1} - k_2 x_{m2} - k_3 x_{m3} + k_1 r, \end{aligned} \tag{13}$$

where k_1, k_2, k_3 satisfy the Hurwitz condition $k_2 k_3 > k_1$.

In this case the design of the ideal control law is based on the following coordinate transformation:

$$\begin{aligned} z_1^* &= x_1 \\ z_2^* &= x_2 + \theta f \\ z_3^* &= x_3 + \theta f' z_2^*. \end{aligned}$$

It follows that

$$\begin{aligned} \dot{z}_1^* &= z_2^* \\ \dot{z}_2^* &= z_3^* \\ \dot{z}_3^* &= u + \theta f'' z_2^{*2} + \theta f' z_3^*. \end{aligned}$$

Thus the ideal control law is chosen as:

$$u = -k_1 z_1^* - k_2 z_2^* - k_3 z_3^* + k_1 r - \theta f'' z_2^{*2} - \theta f' z_3^*.$$

The main idea behind the proposed approach is to build three observers to generate three parameter estimates to avoid repeated differentiation of the adaptive law. This approach is described by the following 3 steps:

Step 1: Let $z_1 = x_1$. The first observer is chosen as

$$\dot{\hat{z}}_1 = x_2 + \hat{\theta}f - \lambda_1 e_1,$$

where $e_1 = \hat{z}_1 - z_1$. The resulting error equation and adaptive law are of the form:

$$\begin{aligned} \dot{e}_1 &= -\lambda_1 e_1 + \tilde{\theta}f, \\ \dot{\tilde{\theta}} = \dot{\hat{\theta}} &= -\gamma_1 f e_1, \end{aligned}$$

where $\tilde{\theta} = \hat{\theta} - \theta$. A tentative Lyapunov function is chosen as: $V_1(e_1, \tilde{\theta}) = \frac{1}{2}(e_1^2 + \frac{\tilde{\theta}^2}{\gamma_1})$. Its derivative yields:

$$\dot{V}_1(e_1,\tilde{\theta}) = -\lambda_1 e_1^2 \le 0,\ \forall (e_1,\tilde{\theta}) \neq (0,0).$$

Hence $e_1 \in \mathcal{L}^\infty$, $\tilde{\theta} \in \mathcal{L}^\infty$, and also $e_1 \in \mathcal{L}^2$.

Step 2: Let $z_2 = x_2 + \hat{\theta} f$. It follows that

$$\dot{z}_2 = x_3 + \dot{\hat{\theta}} f + \hat{\theta} f'(x_2 + \theta f).$$

The second observer is then designed as

$$\dot{\hat{z}}_2 = x_3 - \gamma_1 e_1 f^2 + \hat{\theta} f'(x_2 + \hat{\varphi} f) - \lambda_2 e_2,$$

where $e_2 = \hat{z}_2 - z_2$, and a new estimate of θ is denoted by $\hat{\varphi}$. The resulting error equation and adaptive law are of the form:

$$\begin{aligned} \dot{e}_2 &= -\lambda_2 e_2 + \tilde{\varphi}\hat{\theta} f' f \\ \dot{\tilde{\varphi}} = \dot{\hat{\varphi}} &= -\gamma_2 \hat{\theta} f' f e_2, \end{aligned}$$

where $\tilde{\varphi} = \hat{\varphi} - \theta$. A tentative Lyapunov function is now chosen as: $V_2(e_2,\tilde{\varphi}) = \frac{1}{2}(e_2^2 + \frac{\tilde{\varphi}^2}{\gamma_2})$. Its derivative results in

$$\dot{V}_2(e_2,\tilde{\varphi}) = -\lambda_2 e_2^2 \le 0,\ \forall (e_2,\tilde{\varphi}) \neq (0,0).$$

This implies that $e_2 \in \mathcal{L}^\infty$, $\tilde{\varphi} \in \mathcal{L}^\infty$, and also $e_2 \in \mathcal{L}^2$.

Step 3: Let $z_3 = x_3 - \gamma_1 e_1 f^2 + \hat{\theta} f'(x_2 + \hat{\varphi} f)$. Hence

$$\begin{aligned} \dot{z}_3 = &- \gamma_1(-\lambda_1 e_1 + (\hat{\theta} - \theta) f) f^2 - 2\gamma_1 e_1 f f'(x_2 + \theta f) \\ &- \gamma_1 f e_1 f'(x_2 + \hat{\varphi} f) + \hat{\theta} f''(x_2 + \theta f)(x_2 + \hat{\varphi} f) \\ &+ \hat{\theta} f'(x_3 - \gamma_2 \hat{\theta} f' f^2 e_2 + \hat{\varphi} f'(x_2 + \theta f)) + u. \end{aligned}$$

The third observer is then designed as

$$\begin{aligned} \dot{\hat{z}}_3 = &- \gamma_1(-\lambda_1 e_1 + (\hat{\theta} - \hat{\rho}) f) f^2 - 2\gamma_1 e_1 f f'(x_2 + \hat{\rho} f) \\ &- \gamma_1 f e_1 f'(x_2 + \hat{\varphi} f) + \hat{\theta} f''(x_2 + \hat{\rho} f)(x_2 + \hat{\varphi} f) \\ &+ \hat{\theta} f'(x_3 - \gamma_2 \hat{\theta} f' f^2 e_2 + \hat{\varphi} f'(x_2 + \hat{\rho} f)) + u - \lambda_3 e_3, \end{aligned}$$

where $e_3 = \hat{z}_3 - z_3$ and $\hat{\rho}$ is a new estimate of θ. The error equation and the adaptive law are now given as:

$$\begin{aligned} \dot{e}_3 &= \tilde{\rho}(\gamma_1 f^3 - 2\gamma_1 e_1 f^2 f' + \hat{\theta} f f''(x_2 + \hat{\varphi} f) + \hat{\theta}\hat{\varphi} f f'^2) - \lambda_3 e_3 \\ \dot{\hat{\rho}} &= -\gamma_3(\gamma_1 f^3 - 2\gamma_1 e_1 f^2 f' + \hat{\theta} f f''(x_2 + \hat{\varphi} f) + \hat{\theta}\hat{\varphi} f f'^2) e_3, \end{aligned}$$

where $\tilde{\rho} = \hat{\rho} - \theta$. A tentative Lyapunov function is chosen as: $V_3(e_3,\tilde{\rho}) = \frac{1}{2}(e_3^2 + \frac{\tilde{\rho}^2}{\gamma_3})$. Its derivative yields

$$\dot{V}_3(e_3,\tilde{\rho}) = -\lambda_3 e_3^2 \le 0,\ \forall (e_3,\tilde{\rho}) \neq (0,0).$$

It follows that $e_3 \in \mathcal{L}^\infty$, $\tilde{\rho} \in \mathcal{L}^\infty$, and $e_3 \in \mathcal{L}^2$.

By applying the following general CEAC control law

$$\begin{aligned} u = \ &\gamma_1(-\lambda_1 e_1 + (\hat{\theta} - \hat{\rho}) f) f^2 + 2\gamma_1 e_1 f f'(x_2 + \hat{\rho} f) \\ &+ \gamma_1 f e_1 f'(x_2 + \hat{\varphi} f) - \hat{\theta} f''(x_2 + \hat{\rho} f)(x_2 + \hat{\varphi} f) \\ &- \hat{\theta} f'(x_3 - \gamma_2 \hat{\theta} f' f^2 e_2 + \hat{\varphi} f'(x_2 + \hat{\rho} f)) \\ &- k_1 \hat{z}_1 - k_2 \hat{z}_2 - k_3 \hat{z}_3 + k_1 r, \end{aligned}$$

it follows that

$$\dot{\hat{z}} = A_m \hat{z} - \Lambda e + B r,$$

where $\Lambda = \text{diag}[\lambda_1\ \lambda_2\ \lambda_3]$. Since e is bounded, so is z. It can now be readily shown using standard arguments that all the signals in the system are bounded and $lim_{t\to\infty}(x_1(t) - x_{m1}(t)) = 0$.

Comment: Results from this section can be readily extended to systems in the following pure-feedback form:

$$\begin{aligned} \dot{x}_1 &= x_2 + \theta_1^T f_1(x_1) \\ \dot{x}_2 &= x_3 + \theta_2^T f_2(x_1, x_2) \\ &\vdots \\ \dot{x}_n &= u + \theta_n^T f_n(x_1, x_2, \ldots, x_n). \end{aligned}$$

V. Concluding Remarks

In this paper a Certainty Equivalence Adaptive Control (CEAC) strategy for plants with unmatched uncertainty was proposed. It was shown that, in the case of linear plants of any relative degree and a class of nonlinear plants of relative degree two, a CEAC control strategy, in conjunction with a reduced-order observer and an adaptive law with normalization, results in a stable overall system in which the tracking control objective is achieved asymptotically.

In the case of nonlinear plants with a general continuously differentiable nonlinearity, a CEAC-based control law is used along with multiple observers and adaptive laws to assure system stability. In this case the overall design is relatively complicated which is primarily due to the comlexity of the problem arising from nonlinearity, high relative degree and unmatched uncertainty. Despite the complex design, results from the paper appear to be the first ones developing a stable indirect adaptive control scheme for nonlinear plants with unmatched uncertainty in the case of state feedback.

Future work will be focused on the transient properties of the overall adaptive control systems, as well as on a study of propagation of parameter estimates through multiple adaptive observers.

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