2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

Backstepping Controller Synthesis for Piecewise Polynomial Systems: A Sum of Squares Approach

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Abstract— **This paper defines a new class of hybrid systems called piecewise polynomial (PWP) systems in strict form and develops a backstepping controller synthesis methodology for these systems. The main contribution of the paper is to formulate controller design for a large class of PWP systems as a convex feasibility problem. The controller synthesis problem for PWP systems in *strict feedback form* is divided into two cases. The first case consists of the construction of a sum of squares (SOS) Lyapunov function for PWP systems with discontinuous vector fields. The second case addresses the construction of a piecewise polynomial Lyapunov function for PWP systems with continuous vector fields. After constructing a (piecewise) polynomial Lyapunov function, controller synthesis for a PWP system can be formulated as an SOS program, which is a convex optimization problem and can be solved efficiently using available software. One major advantage of the proposed method is the fact that it can handle systems with discontinuous vector fields and sliding modes. The new synthesis method is applied to a numerical example.**

I. INTRODUCTION

PWP or spline approximation of curves and surfaces has been widely used in many different scientific contexts and engineering applications [1], [2]. However, the lack of efficient methods to check the sign of polynomials has prevented PWP systems to be commonly used in the field of control systems. To the best of our knowledge, one of the first attempts to design controllers for PWP systems was made in [3]. Paul proposed in [3] to partition the state space of an affine-in-the-input nonlinear system into cells and to approximate the dynamics of the system in each cell by a model that is polynomial in the state. A controller is designed for each cell using feedback linearization. A global controller is then formed by joining the individual cell controllers. The proposed method was employed in [3] to design controllers for a few nonlinear systems. However, there is no guarantee that the closed loop system is stable because a switched system consisting of stable subsystems can be unstable in general [4].

For continuous time PWP systems, a stability analysis method was proposed in [5] and [6] using piecewise polynomial Lyapunov functions. The advantage of the proposed method is that the analysis problem is formulated as a sum of squares (SOS) programming, which is a convex optimization problem. There exist numerical tools such as SOSTOOLS [7] and Yalmip [8] to solve SOS programming problems efficiently by converting them to semidefinite programs. However, systems with infinitely fast switching or sliding modes are excluded from the discussion in [5] and [6].

The main contribution of this paper is to propose a backstepping technique to construct control Lyapunov functions for a class of PWP systems. The proposed method formulates the control synthesis problem for PWP systems in *strict feedback form* as an SOS feasibility problem. The synthesis of PWP controllers is formulated for two cases. The first case addresses the construction of (SOS) Lyapunov functions for PWP systems with discontinuous vector fields. The second case deals with the construction of piecewise polynomial Lyapunov functions for PWP systems with continuous vector fields. After constructing a (piecewise) polynomial Lyapunov function, controller synthesis for a PWP system can be formulated as an SOS program. One major advantage of the proposed method is the fact that it can handle systems with discontinuous vector fields and sliding modes.

The paper is organized as follows. Mathematical preliminaries are addressed in section II. Controller design for PWP systems in strict feedback form is then described in section III. Finally, a numerical example is presented in section IV and conclusions are drawn in section V.

II. MATHEMATICAL PRELIMINARIES

A. SOS polynomials

An SOS polynomial is defined in the following.

Definition 1: [9] A multivariate polynomial $p(x_1, \dots, x_n) \triangleq p(x)$ is a sum of squares, if there exist polynomials $p_1(x), \dots, p_m(x)$ such that $p(x) = \sum_{i=1}^{m} p_i^2(x)$.

SOS polynomials $p(x)$ are globally nonnegative. Although verifying nonnegativity of a polynomial is an NP-hard problem [10], the SOS condition can be formulated as a convex problem in polynomial coefficients [11]. However, note that not all nonnegative polynomials are SOS. For a tutorial about recent system analysis techniques based on sum of squares decomposition see [6].

B. PWP systems

The dynamics of a PWP system can be written as follows.

$$\dot{x}(t) = f_i(x(t)), \text{ if } x(t) \in \mathcal{P}_i \tag{1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector and $f_i(x) \in \mathbb{R}^n$ are polynomial functions of x. The cells, $\mathcal{P}_i$, $i \in \mathcal{I} = \{1, \dots, M\}$, partition a subset of the state space $\mathcal{X} \subset \mathbb{R}^n$ such that $\cup_{i=1}^{M} \overline{\mathcal{P}}_i = \mathcal{X}$, $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$, $i \neq j$, where $\overline{\mathcal{P}}_i$ denotes the closure of $\mathcal{P}_i$. Each cell is described by

$$\mathcal{P}_i = \{x | E_i(x) \succ 0\} \tag{2}$$

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where $E_i(x) \in \mathbb{R}^{p_i}$ is a vector polynomial function of x and "$\succ$" represents an elementwise inequality.

C. Lyapunov stability

Consider the following piecewise smooth (PWS) system

$$\dot{x} = f_i(x),\ x \in \mathcal{P}_i \tag{3}$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector and the functions $f_i(x) : \overline{\mathcal{P}}_i \to \mathbb{R}^n$ for $i = 1, \ldots, M$ are continuous in x and locally bounded. The Filippov definition [12] of trajectories is considered for the solution of (3).

The following theorem describes sufficient conditions for stability of system (3) in the sense of Lyapunov based on a continuous Lyapunov function that is not necessarily differentiable everywhere.

Theorem 1 ([13]): For the PWS system (3), if there exists a continuous function $V(x)$ such that

$$\begin{aligned} &V(0) = 0 \\ &V(x) > 0 \text{ for all } x \neq 0 \text{ in } \mathcal{X} \\ &t_1 \leq t_2 \Rightarrow V(x(t_1)) \geq V(x(t_2)) \end{aligned}$$

then $x = 0$ is a stable equilibrium point. Moreover if there exists a continuous function $W(x)$ such that

$$\begin{aligned} &W(0) = 0 \\ &W(x) > 0 \text{ for all } x \neq 0 \text{ in } \mathcal{X} \\ &t_1 \leq t_2 \Rightarrow V(x(t_1)) \geq V(x(t_2)) + \int_{t_1}^{t_2} W(x(\tau))d\tau \end{aligned}$$

and

$$\|x\| \to \infty \Rightarrow V(x) \to \infty \tag{4}$$

then all trajectories in $\mathcal{X}$ asymptotically converge to $x = 0$. □

Two propositions are provided in the following to describe the sufficient conditions for the system (3) to be stable in two cases of discontinuous and continuous vector fields. The importance of these propositions lies in the fact that to check the stability of the system, it suffices to verify a condition on the candidate Lyapunov function and the vector field of the subsystem in each region separately. There is therefore no need to examine the candidate Lyapunov function in one region with the vector field of another region, which would make the problem much more complicated.

Proposition 1: (Smooth Lyapunov functions) The PWS system (3) is asymptotically stable if there exists a positive definite $\mathcal{C}^1$ function $V(x)$ and a positive definite continuous function $W(x)$ so that $V(0) = 0$, $W(0) = 0$ and for all $x \in \overline{\mathcal{P}}_i$, $i = 1, \ldots, M$

$$\nabla V(x)^{\mathrm{T}} f_i(x) \leq -W(x) \tag{5}$$

Proposition 2: (PWS Lyapunov functions) The PWS system (3) is asymptotically stable if its vector field is continuous in x, i.e. for any $i, j \in \{1, \ldots, M\}$ such that $\overline{\mathcal{P}}_i \bigcap \overline{\mathcal{P}}_j \neq \emptyset$,

$$f_i(x) = f_j(x),\ \forall x \in \overline{\mathcal{P}}_i \bigcap \overline{\mathcal{P}}_j \tag{6}$$

and there exists positive definite functions $V(x)$ and $W(x)$ so that $V(0) = 0$, $W(0) = 0$ and

- $V(x)$ is a continuous function where
$$V(x) = V_i(x), x \in \overline{\mathcal{P}}_i \tag{7}$$
where $V_i : \overline{\mathcal{P}}_i \to \mathbb{R}$ is a $\mathcal{C}^1$ function,
- $W(x)$ is a continuous function,
- for all $x \in \overline{\mathcal{P}}_i$, $i = 1, \ldots, M$
$$\nabla V(x)_i^{\mathrm{T}} f_i(x) \leq -W(x) \tag{8}$$

The Propositions 1 and 2 are not proved here due to lack of space, but they can be obtained using the results in [14].

III. Recursive Backstepping Controller Design

In this section, a recursive PWP controller synthesis method is proposed for *strict feedback* PWP systems, which consist of different polynomial vector fields in different regions of operation. The dynamics of this new class of systems can be written in the form

$$\begin{cases} \dot{x}_1 = f_{1i_1}(x_1) + g_{1i_1}(x_1)x_2, \quad \text{for } x_1 \in \mathcal{P}_{1i_1} \\ \dot{x}_2 = f_{2i_2}(x_1, x_2) + g_{2i_2}(x_1, x_2)x_3, \quad \text{for } \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathcal{P}_{2i_2} \\ \vdots \\ \dot{x}_k = f_{ki_k}(x) + g_{ki_k}(x)u, \text{for } x \in \mathcal{P}_{ki_k} \end{cases} \tag{9}$$

where x is the state vector of the system (9) and is divided into k subvectors:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} \in \mathbb{R}^n,\ x_j \in \mathbb{R}^{n_j}, \tag{10}$$

We use a notation for the regions in which the first index represents the variable whose differential equation we are interested in and the second index represents the number of the region where the differential equation is valid. Following this notation, for each $j \in \{1, 2, \ldots, k\}$, the regions $\mathcal{P}_{ji_j}$ for $i_j = 1, \ldots, M_j$ are disjoint sets defined as

$$\mathcal{P}_{ji_j} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_j \end{bmatrix} \middle| E_{ji_j}(x_1, \ldots, x_j) \succ 0 \right\} \tag{11}$$

where $E_{ji_j}(x_1, \ldots, x_j) \in \mathbb{R}^{p_j}$ is a vector polynomial function and $\succ$ denotes an elementwise inequality. For a given j, the regions $\mathcal{P}_{ji_j}$ for $i_j = 1, \ldots, M_j$ partition the projection of the state space $\mathcal{X} \subset \mathbb{R}^n$ onto the $(x_1, \ldots, x_j)$ space.

It is assumed that for $1 \leq j_1 < j_2$, the projection of each region $\mathcal{P}_{j_2 i_{j_2}}$ for $i_{j_2} = 1, \ldots, M_{j_2}$ on the $(x_1, \ldots, x_{j_1})$ space is a subset of only *one* of the regions $\mathcal{P}_{j_1 i_{j_1}}$ for $i_{j_1} = 1, \ldots, M_{j_1}$. In other words, for each j_1, j_2 and $i_{j_2} \in \{1, \ldots, M_{j_2}\}$, where $1 < j_2 \leq k$ and $j_1 < j_2$, there exists a unique number $i(j_1, j_2, i_{j_2})$ in $\{1, \ldots, M_{j_1}\}$ such that

$$\begin{bmatrix} x_1 \\ \vdots \\ x_{j_2} \end{bmatrix} \in \mathcal{P}_{j_2 i_{j_2}} \Rightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_{j_1} \end{bmatrix} \in \mathcal{P}_{j_1 i(j_1, j_2, i_{j_2})} \tag{12}$$

In addition, it is assumed that

$$f_{ji_j^\star}(0, \ldots, 0) = 0,\ \forall i_j^\star \in \mathcal{I}_j(0, \ldots, 0) \tag{13}$$

where

$$\mathcal{I}_j(x_1, \ldots, x_j) := \left\{ i_j \middle| \begin{bmatrix} x_1 \\ \vdots \\ x_j \end{bmatrix} \in \mathcal{P}_{ji_j} \right\} \tag{14}$$

In what follows the stabilization problem for PWP systems in strict feedback form is solved for two cases of PWP systems: discontinuous and continuous vector fields.

A. PWP systems with discontinuous vector fields

To design a PWP controller for (9), we start from the following subsystem

$$\dot{x}_1 = f_{1i_1}(x_1) + g_{1i_1}(x_1)x_2, \text{ for } x_1 \in \mathcal{P}_{1i_1}, \tag{15}$$

with $i_1 = 1, \dots, M_1$. It is assumed that there exist a polynomial Lyapunov function $V_1(x_1)$, a polynomial controller $x_2 = \gamma_1(x_1)$ and a polynomial vector $\Gamma_{1i_1}(x_1) \in \mathbb{R}^{p_1}$ such that for $i_1 = 1, \dots, M_1$

$$\begin{cases} \gamma_1(0) = 0 \\ V_1(0) = 0 \\ V_1(x_1) - \lambda(x_1) \text{ is SOS} \\ -\nabla V_1(x_1)^{\mathrm{T}}(f_{1i_1}(x_1) + g_{1i_1}(x_1)\gamma_1(x_1)) \\ -\Gamma_{1i_1}(x_1)^{\mathrm{T}} E_{1i_1}(x_1) - \alpha V_1(x_1) \text{ is SOS} \\ \Gamma_{1i_1}(x_1) \text{ is SOS} \end{cases} \tag{16}$$

where $\alpha > 0$ is fixed and $\lambda(x_1)$ is a positive definite polynomial. Note that we call a vector $\Gamma_{1i_1}(x_1) \in \mathbb{R}^{p_1}$ SOS if all the entries of the vector are SOS polynomials.

A polynomial controller can then be designed for the following subsystem

$$\begin{cases} \dot{x}_1 = f_{1i_1}(x_1) + g_{1i_1}(x_1)x_2, \text{ for } x_1 \in \mathcal{P}_{1i_1} \\ \dot{x}_2 = f_{2i_2}(x_1, x_2) + g_{2i_2}(x_1, x_2)x_3, \text{ for } \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathcal{P}_{2i_2} \end{cases} \tag{17}$$

Note that if $f_{2i_2}(x_1, x_2) = 0$ and $g_{2i_2}(x_1, x_2) = 1$, this would be an integrator backstepping problem.

- *Lyapunov function construction*: We consider the following candidate Lyapunov function

  $$V_2(x_1, x_2) = V_1(x_1) + \frac{1}{2}(x_2 - \gamma_1(x_1))^{\mathrm{T}}(x_2 - \gamma_1(x_1)) \tag{18}$$

  The synthesis problem can then be formulated as the following SOS program.
- *Controller synthesis*:

  $$\begin{aligned} &\text{Find } \; x_3 = \gamma_2(x_1, x_2),\; \Gamma_{2i_2}(x_1, x_2) \\ &\text{such that} \\ &-\nabla_{x_1} V_2(x_1, x_2)^{\mathrm{T}}(f_{1i(1,2,i_2)}(x_1) + g_{1i(1,2,i_2)}(x_1)x_2) \\ &-\nabla_{x_2} V_2(x_1, x_2)^{\mathrm{T}}(f_{2i_2}(x_1, x_2) + g_{2i_2}(x_1, x_2)x_3) \\ &-\Gamma_{2i_2}(x_1, x_2)^{\mathrm{T}} E_{2i_2}(x_1, x_2) - \alpha V_2(x_1, x_2) \text{ is SOS}, \\ &\Gamma_{2i_2}(x_1, x_2) \text{ is SOS} \\ &\gamma_2(0, 0) = 0 \end{aligned} \tag{19}$$

  where $\alpha > 0$, $i_2 = 1, \dots, M_2$ and $\gamma_2(x_1, x_2)$ is a polynomial function of x_1 and x_2.

Note that if this SOS program is feasible then the procedure can be repeated for the next steps by adding the dynamics of x_i for $i = 3, \dots, k$. Assume that all SOS programs in the backstepping procedure are feasible and at the last step the following candidate Lyapunov function is used:

$$\begin{aligned} V_k(x) &= V_{k-1}(x_1, \dots, x_{k-1}) \\ &+ \frac{1}{2}(x_k - \gamma_{k-1}(x_1, \dots, x_{k-1}))^{\mathrm{T}}(x_k - \gamma_{k-1}(x_1, \dots, x_{k-1})) \end{aligned} \tag{20}$$

where $\gamma_{k-1}(x_1, \dots, x_{k-1})$ is a polynomial function. Since the final controller $u = \gamma_k(x)$ will not be used to construct another SOS Lyapunov function, it does not have to be continuously differentiable. Therefore, one can search for a PWP control

$$u = \gamma_{ki_k}(x), \text{ for } x \in \mathcal{P}_{ki_k} \tag{21}$$

for $i_k = 1, \dots, M_k$. This step can be formulated as the following SOS program:

$$\begin{aligned} &\text{Find } \; u = \gamma_{ki_k}(x),\; \Gamma_{ki_k}(x) \\ &\text{such that} \\ &-\nabla_{x_1} V_k^{\mathrm{T}}(f_{1i(1,k,i_k)}(x_1) + g_{1i(1,k,i_k)}(x_1)x_2) \\ &-\nabla_{x_2} V_k^{\mathrm{T}}(f_{2i(2,k,i_k)}(x_1, x_2) + g_{2i(2,k,i_k)}(x_1, x_2)x_3) \\ &-\ldots - \nabla_{x_k} V_k^{\mathrm{T}}(f_{ki_k}(x) + g_{ki_k}(x)u) \\ &-\Gamma_{ki_k}(x)^{\mathrm{T}} E_{ki_k}(x) - \alpha V_k \text{ is SOS}, \\ &\Gamma_{ki_k}(x) \text{ is SOS} \end{aligned} \tag{22}$$

for $i_k = 1, \dots, M_k$. The following theorem shows that if the SOS program (22) is feasible then the PWP controller (21) stabilizes the PWP system (9).

Theorem 2: Let there exist polynomial functions $V_1(x_1)$ and $\gamma_1(x_1)$ satisfying (16). Let also $V_j(x_1, \dots, x_j)$ for $j = 2, \dots, k$ be defined as

$$\begin{aligned} V_j(x_1, \dots, x_j) &= V_{j-1}(x_1, \dots, x_{j-1}) \\ &+ \frac{1}{2}(x_j - \gamma_{j-1}(x_1, \dots, x_{j-1}))^{\mathrm{T}}(x_j - \gamma_{j-1}(x_1, \dots, x_{j-1})) \end{aligned} \tag{23}$$

where

$$\gamma_j(\overbrace{0, \dots, 0}^{j \text{ arguments}}) = 0, \; j = 1, \dots, k-1 \tag{24}$$

and $\gamma_2(x_1, x_2)$ to $\gamma_{k-1}(x_1, \dots, x_{k-1})$ satisfy the corresponding SOS conditions. Also assume that the PWP control (21) satisfies the conditions of the SOS program (22). Then the PWP control (21) makes the trajectories of the PWP system (9) in $\mathcal{X}$ asymptotically converge to the origin.

Proof: It follows from (16) that $V_1(x_1) \geq \lambda(x_1)$ and since $\lambda(x_1)$ is positive definite,

$$V_1(x_1) > 0, \text{ if } x_1 \neq 0 \tag{25}$$

From (23) we have

$$\begin{aligned} V_k(x) &= V_1(x_1) + \sum_{j=2}^{k} \frac{1}{2}(x_j - \gamma_{j-1}(x_1, \dots, x_{j-1}))^{\mathrm{T}} \\ &(x_j - \gamma_{j-1}(x_1, \dots, x_{j-1})) \end{aligned} \tag{26}$$

Therefore, $V_k(x) \geq 0$. Now assume for some $x_1, x_2, \ldots, x_k$ we have $V_k(x) = 0$. It follows from (26) that

$$V_1(x_1) = 0 \tag{27}$$

and

$$x_j = \gamma_{j-1}(x_1, \ldots, x_{j-1}),\ j = 2, \ldots, k \tag{28}$$

From (24) and positive definiteness of $V_1(x_1)$ it follows that $x_1 = 0, x_2 = 0, \ldots, x_k = 0$. Therefore $V_k(x)$ is a positive definite function.

From (11) and (22), it follows that for $i_k = 1, \ldots, M_k$

$$\nabla_x V_k(x)^{\mathrm{T}} f_{i_k}(x) \leq -\alpha V_k(x),\ \text{for}\ x \in \mathcal{P}_{ki_k} \tag{29}$$

Now, from Proposition 1 it follows that the PWP system (9) is asymptotically stable with the Lyapunov function $V_k(x)$ and $W(x) = \alpha V_k(x)$. ∎

B. PWP systems with continuous vector fields

In this section, it is assumed that the vector field of PWP system (9) is continuous for $x \in \mathcal{X}$. It is also assumed that for the following subsystem

$$\dot{x}_1 = f_{1i_1}(x_1) + g_{1i_1}(x_1)x_2,\ \text{for}\ x_1 \in \mathcal{P}_{1i_1}, \tag{30}$$

with $i_1 = 1, \ldots, M_1$, there exist a continuous piecewise polynomial Lyapunov function $V_1(x_1)$ and a continuous PWP controller $x_2 = \gamma_1(x_1)$ with

$$\begin{cases} V_1(x_1) = V_{1i_1}(x_1) \\ \\ \gamma_1(x_1) = \gamma_{1i_1}(x_1) \end{cases},\ \text{for}\ x_1 \in \mathcal{P}_{i_1}, \tag{31}$$

such that $\gamma_{1i_1}(x_1)$ and $V_{1i_1}(x_1)$ are polynomials and that for $i_1 = 1, \ldots, M_1$ we have

$$\begin{cases} V_1(0) = 0 \\ \gamma_1(0) = 0 \\ V_{1i_1}(x_1) - \Lambda_{1i_1}(x_1)^{\mathrm{T}} E_{1i_1}(x_1) - \lambda(x_1)\ \text{is SOS} \\ -\nabla V_{1i_1}(x_1)^{\mathrm{T}}(f_{1i_1}(x_1) + g_{1i_1}(x_1)\gamma_{1i_1}(x_1)) \\ -\Gamma_{1i_1}(x_1)^{\mathrm{T}} E_{1i_1}(x_1) - \alpha V_{1i_1}\ \text{is SOS} \\ \Lambda_{1i_1}(x_1)\ \text{and}\ \Gamma_{1i_1}(x_1)\ \text{are SOS} \end{cases} \tag{32}$$

where $\alpha > 0$ and $\lambda(x_1)$ is a positive definite polynomial.

Then, a PWP controller can be designed for the following subsystem

$$\begin{cases} \dot{x}_1 = f_{1i_1}(x_1) + g_{1i_1}(x_1)x_2,\ \text{for}\ x_1 \in \mathcal{P}_{1i_1} \\ \dot{x}_2 = f_{2i_2}(x_1, x_2) + g_{2i_2}(x_1, x_2)x_3,\ \text{for}\ \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathcal{P}_{2i_2} \end{cases} \tag{33}$$

Considering the following PWP candidate Lyapunov function

$$V_2(x_1, x_2) = V_{2i_2}(x_1, x_2), \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathcal{P}_{2i_2} \tag{34}$$

where

$$V_{2i_2}(x_1, x_2) = V_{1i(1,2,i_2)}(x_1) + \frac{1}{2}(x_2 - \gamma_{1i(1,2,i_2)}(x_1))^{\mathrm{T}}(x_2 - \gamma_{1i(1,2,i_2)}(x_1)) \tag{35}$$

the synthesis problem can be formulated as the following SOS program:

$$\begin{aligned} &\text{Find}\ x_3 = \gamma_{2i_2}(x_1, x_2), \Gamma_{2i_2}(x_1, x_2), c_{i_{21}i_{22}}(x_1, x_2) \\ &\text{such that} \\ &-\nabla_{x_1} V_{2i_2}(x_1, x_2)^{\mathrm{T}}(f_{1i(1,2,i_2)}(x_1) + g_{1i(1,2,i_2)}(x_1)x_2) \\ &-\nabla_{x_2} V_{2i_2}(x_1, x_2)^{\mathrm{T}}(f_{2i_2}(x_1, x_2) + g_{2i_2}(x_1, x_2)x_3) \\ &-\Gamma_{2i_2}(x_1, x_2)^{\mathrm{T}} E_{2i_2}(x_1, x_2) - \alpha V_{2i_2}(x_1, x_2)\ \text{is SOS}, \\ &\Gamma_{2i_2}(x_1, x_2)\ \text{is SOS} \\ &\gamma_{2i_{21}}(x_1, x_2) - \gamma_{2i_{22}}(x_1, x_2) = \\ &c_{i_{21}i_{22}}(x_1, x_2) E_{2i_{21}i_{22}}(x_1, x_2) \\ &\gamma_2(0, 0) = 0 \end{aligned} \tag{36}$$

for $i_2 = 1, \ldots, M_2$ and all i_{21} and i_{22} in $\{1, \ldots, M_2\}$ such that $\mathcal{P}_{2i_{21}}$ and $\mathcal{P}_{2i_{22}}$ are neighboring cells and $E_{2i_{21}i_{22}}(x_1, x_2) = 0$ contains their boundary, i.e.,

$$\overline{\mathcal{P}}_{2i_{21}} \cap \overline{\mathcal{P}}_{2i_{22}} \subset \{ \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] | \, E_{2i_{21}i_{22}}(x_1, x_2) = 0 \} \tag{37}$$

In addition, $\gamma_{2i_2}(x_1, x_2)$ and $c_{i_{21}i_{22}}(x_1, x_2)$ are polynomial functions.

If this SOS program is feasible then the procedure can be repeated for the next steps by adding the dynamics of x_3 and so on until x_k. If all SOS programs in the backstepping procedure are feasible, a continuous PWP controller

$$u = \gamma_{ki_k}(x),\ \text{for}\ x \in \mathcal{P}_{ki_k} \tag{38}$$

can be designed using the following SOS program:

$$\begin{aligned} \text{Find}\quad & u = \gamma_{ki_k}(x),\ \Gamma_{ki_k}(x), c_{i_{k1}i_{k2}} \\ \text{s.t.}\quad & -\nabla_{x_1} V_{ki_k}^{\mathrm{T}}(f_{1i(1,k,i_k)}(x_1) + g_{1i(1,k,i_k)}(x_1)x_2) \\ & -\nabla_{x_2} V_{ki_k}^{\mathrm{T}}(f_{2i(2,k,i_k)}(x_1, x_2) \\ & +g_{2i(2,k,i_k)}(x_1, x_2)x_3) - \ldots - \\ & \nabla_{x_k} V_{ki_k}^{\mathrm{T}}(f_{ki_k}(x) + g_{ki_k}(x)u) \\ & -\Gamma_{ki_k}(x)^{\mathrm{T}} E_{ki_k}(x) - \alpha V_{ki_k}\ \text{is SOS}, \\ & \Gamma_{ki_k}(x)\ \text{is SOS}, \\ & f_{ki_{k1}}(x) + g_{ki_{k1}}(x)\gamma_{ki_{k1}} \\ & -f_{ki_{k2}}(x) + g_{ki_{k2}}(x)\gamma_{ki_{k2}} \\ & = c_{i_{k1}i_{k2}}(x) E_{ki_{k1}i_{k2}}(x) \end{aligned} \tag{39}$$

for $i_k = 1, \ldots, M_k$ and all i_{k1} and i_{k2} in $\{1, \ldots, M_k\}$ such that $\mathcal{P}_{ki_{k1}}$ and $\mathcal{P}_{ki_{k2}}$ are neighboring cells. Note that $\gamma_{ki_k}(x)$ and $c_{i_{k1}i_{k2}}(x)$ are polynomial functions.

Theorem 3: Let there exist a PWP function $V_1(x_1)$ satisfying (32). Let also $V_j(x_1, \ldots, x_j)$ for $j = 2, \ldots, k$ be defined as

$$V_j(x_1, \ldots, x_j) = V_{j-1}(x_1, \ldots, x_{j-1}) + \frac{1}{2}(x_j - \gamma_{j-1}(x_1, \ldots, x_{j-1}))^{\mathrm{T}}(x_j - \gamma_{j-1}(x_1, \ldots, x_{j-1})) \tag{40}$$

where

$$\gamma_j(\overbrace{0, \ldots, 0}^{j\ \text{arguments}}) = 0,\ j = 1, \ldots, k-1 \tag{41}$$

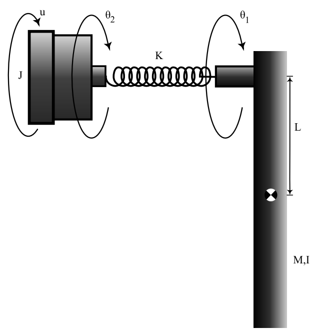

Fig. 1. Single-link flexible-joint robot

and $\gamma_2(x_1, x_2)$ to $\gamma_{k-1}(x_1, \ldots, x_{k-1})$ satisfy the corresponding SOS conditions. Also assume that the PWP control (38) satisfies the conditions of the SOS program (39). Then the PWP control (38) makes the trajectories of the PWP system (9) in $\mathcal{X}$ asymptotically converge to the origin.

Proof: The proof follows a similar reasoning to that of the proof of Theorem 2 using Proposition 2. ■

IV. Numerical Example

Example 1: Consider the single-link flexible-joint robot in Fig. 1. The dynamic equations of the robot are given by [15]

$$\dot{x}_1 = x_2 \tag{42}$$

$$\dot{x}_2 = -\frac{MgL}{I}\sin(x_1) - \frac{K}{I}(x_1 - x_3) \tag{43}$$

$$\dot{x}_3 = x_4 \tag{44}$$

$$\dot{x}_4 = -\frac{T_f}{J} + \frac{K}{J}(x_1 - x_3) + \frac{1}{J}u \tag{45}$$

where $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, $x_3 = \theta_2$ and $x_4 = \dot{\theta}_2$, u is the motor torque and $T_f = f_2(x_4)$ denotes the motor friction which is described by [16]

$$T_f = b_m x_4 + \mathrm{sgn}(x_4)\left(F_{cm} + (F_{sm} - F_{cm})\exp(-\frac{x_4^2}{c_m^2})\right) \tag{46}$$

The numerical values of the parameters are given as follows

$$\begin{aligned} &M = 0.25\text{kg},\ L = 1\text{m},\ I = 0.0833\text{kgm}^2 \\ &K = 7.47\text{Nm/rad},\ J = 0.216\text{kgm}^2,\ g = 9.8\text{m/s}^2 \\ &c_m = 1.2\text{rad/sec},\ F_{cm} = 1.2\text{Nm},\ F_{sm} = 1.75\text{Nm} \\ &b_m = 0.17\text{Nm/(rad/sec)} \end{aligned}$$

In this example, the objective is to stabilize the nonlinear model at the origin. To build a PWP model, there are two nonlinear functions that should be approximated by PWP curves. The function $f_1(x_1) = \sin(x_1)$ is approximated by the following function for $x_1 \in [-\pi, \pi]$

$$\hat{f}_1(x_1) = \tag{47}$$

$$\begin{cases} 0.4031x_1^2 + 1.2464x_1 - 0.0211 & -\pi \le x_1 \le -\frac{2\pi}{7} \\ 0.908x_1 & -\frac{2\pi}{7} \le x_1 \le \frac{2\pi}{7} \\ -0.4031x_1^2 + 1.2464x_1 + 0.0211 & \frac{2\pi}{7} \le x_1 \le \pi \end{cases} \tag{48}$$

The nonlinear function $T_f = f_2(x_4)$ in (46) is approximated by the following PWP function for $x_4 \in [-8, 8]$

$$\hat{f}_2(x_4) = \tag{49}$$

$$\begin{cases} -0.0057x_4^3 + 0.0873x_4^2 - 0.2472x_4 + 1.8056 & x_4 > 0 \\ -0.0057x_4^3 - 0.0873x_4^2 - 0.2472x_4 - 1.8056 & x_4 < 0 \end{cases} \tag{50}$$

Next, the PWP approximation of the nonlinear model (42)-(45) can be written in the strict feedback form (9). To start the controller synthesis procedure from subsection III-B, we first consider the following system

$$\dot{x}_1 = x_2 \tag{51}$$

with

$$\mathcal{P}_{11} = \{x_1 | x_1 \in \mathbb{R}\} \tag{52}$$

The linear controller $x_2 = -2x_1$ is considered in this step to make the quadratic Lyapunov function $V_1(x_1) = \frac{1}{2}x_1^2$ decreasing with time.

In the second step, the following PWP system is considered

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{MgL}{I}\hat{f}_1(x_1) - \frac{K}{I}(x_1 - x_3) \end{aligned} \tag{53}$$

with the regions defined as

$$\mathcal{P}_{21} = \left\{ \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \middle| -\pi < x_1 < -\frac{2\pi}{7}, x_2 \in \mathbb{R} \right\} \tag{54}$$

$$\mathcal{P}_{22} = \left\{ \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \middle| -\frac{2\pi}{7} < x_1 < \frac{2\pi}{7}, x_2 \in \mathbb{R} \right\} \tag{55}$$

$$\mathcal{P}_{23} = \left\{ \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \middle| \frac{2\pi}{7} < x_1 < \pi, x_2 \in \mathbb{R} \right\} \tag{56}$$

Considering the Lyapunov function $V_2(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + 2x_1)^2$ and solving the SOS feasibility problem (36) for the PWP system (53), the following controller is computed

$$x_3 = \gamma_2(x_1, x_2) =$$

$$\begin{cases} 0.26137 + 0.85161x_1 - 0.1x_2 & \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathcal{P}_{21} \\ 0.56043x1 - 0.1x2 & \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathcal{P}_{22} \\ -0.26137 + 0.85161x1 - 0.1x2 & \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathcal{P}_{23} \end{cases} \tag{57}$$

For the next step, the following PWP system is considered

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{MgL}{I}\hat{f}_1(x_1) - \frac{K}{I}(x_1 - x_3) \\ \dot{x}_3 &= x_4 \end{aligned} \tag{58}$$

with the following regions

$$\mathcal{P}_{31} = \left\{ \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right] \middle| -\pi < x_1 < -\frac{2\pi}{7}, x_2 \in \mathbb{R}, x_3 \in \mathbb{R} \right\} \tag{59}$$

$$\mathcal{P}_{32} = \left\{ \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right] \middle| -\frac{2\pi}{7} < x_1 < \frac{2\pi}{7}, x_2 \in \mathbb{R}, x_3 \in \mathbb{R} \right\} \tag{60}$$

$$\mathcal{P}_{33} = \left\{ \left[\begin{smallmatrix} x_1 \\ x_2 \\ x_3 \end{smallmatrix}\right] \middle| \frac{2\pi}{7} < x_1 < \pi, x_2 \in \mathbb{R}, x_3 \in \mathbb{R} \right\} \tag{61}$$

Considering the Lyapunov function $V_3(x_1,x_2,x_3)=\frac{1}{2}x_1^2+\frac{1}{2}(x_2+2x_1)^2+\frac{1}{2}(x_3-\gamma_2(x_1,x_2))^2$ and solving the corresponding SOS feasibility problem for the PWP system (58), the following controller is computed

$$x_4=\gamma_3(x_1,x_2,x_3)=\begin{cases}2.1731-77.5789x_1-75.8241x_2-80x_3 & \text{for } \mathcal{P}_{31}\\ -80x_1-75.8241x_2-80x_3 & \text{for } \mathcal{P}_{32}\\ -2.1731-77.5789x_1-75.8241x_2-80x_3 & \text{for } \mathcal{P}_{33}\end{cases} \quad (62)$$

For the next step, the following PWP system is considered

$$\begin{aligned}\dot{x}_1&=x_2\\ \dot{x}_2&=-\frac{MgL}{I}\hat{f}_1(x_1)-\frac{K}{I}(x_1-x_3)\\ \dot{x}_3&=x_4\\ \dot{x}_4&=-\frac{\hat{f}_2(x_4)}{J}+\frac{K}{J}(x_1-x_3)+\frac{1}{J}u\end{aligned} \quad (63)$$

with the following regions

$$\begin{aligned}\mathcal{P}_{41}&=\left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\middle|-\pi<x_1<-\frac{2\pi}{7},x_2\in\mathbb{R},x_3\in\mathbb{R},x_4>0\right\}\\ \mathcal{P}_{42}&=\left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\middle|-\frac{2\pi}{7}<x_1<\frac{2\pi}{7},x_2\in\mathbb{R},x_3\in\mathbb{R},x_4>0\right\}\\ \mathcal{P}_{43}&=\left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\middle|\frac{2\pi}{7}<x_1<\pi,x_2\in\mathbb{R},x_3\in\mathbb{R},x_4>0\right\}\\ \mathcal{P}_{44}&=\left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\middle|-\pi<x_1<-\frac{2\pi}{7},x_2\in\mathbb{R},x_3\in\mathbb{R},x_4<0\right\}\\ \mathcal{P}_{45}&=\left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\middle|-\frac{2\pi}{7}<x_1<\frac{2\pi}{7},x_2\in\mathbb{R},x_3\in\mathbb{R},x_4<0\right\}\\ \mathcal{P}_{46}&=\left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\middle|\frac{2\pi}{7}<x_1<\pi,x_2\in\mathbb{R},x_3\in\mathbb{R},x_4<0\right\}\end{aligned} \quad (64)$$

Now, a PWP control of third order in x_1, first order in x_2, first order in x_3 and third order in x_4 is designed. Considering the Lyapunov function $V_4(x_1,x_2,x_3,x_4)=\frac{1}{2}x_1^2+\frac{1}{2}(x_2+2x_1)^2+\frac{1}{2}(x_3-\gamma_2(x_1,x_2))^2+\frac{1}{2}(x_4-\gamma_3(x_1,x_2,x_3))^2$ and solving the SOS feasibility problem (39) for the PWP system (63), the PWP controller is computed. The polynomials are not shown due to lack of space.

Fig. 2 shows the states of the nonlinear system in feedback connection with the PWP controller with the initial condition $x_0=[\pi\ 0\ 0.8\pi\ 0]^{\mathrm{T}}$. It can be seen in the figure that the system trajectories converge to the origin.

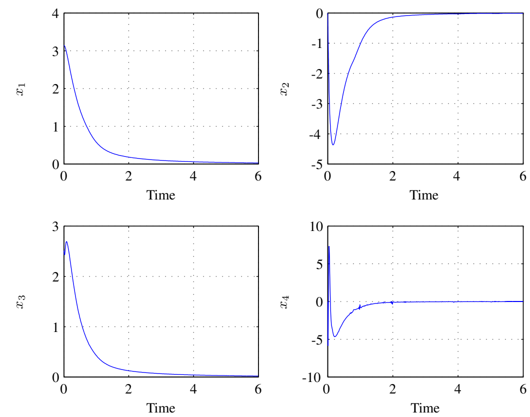

Fig. 2. State variables of the nonlinear model - PWP controller

V. Conclusions

In this paper, the strict feedback form for PWP systems was first introduced. Then backstepping controller synthesis for this large class of PWP systems was formulated as an SOS program, which is a convex program. The synthesis problem was addressed in two cases: SOS Lyapunov functions for PWP systems with discontinuous vector fields and PWP Lyapunov functions for PWP systems with continuous vector fields. One of the main advantages of the proposed method is that it addresses for the first time PWP systems with discontinuous vector fields regardless of possible attractive sliding modes.

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