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A Modified Dubins Method for Optimal Path Planning of a Miniature Air Vehicle Converging to a Straight Line Path

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Abstract— **This paper presents a Dubins model based strategy to determine the optimal path of a Miniature Air Vehicle (MAV), constrained by a bounded turning rate, that would enable it to fly along a given straight line, starting from an arbitrary initial position and orientation. The method is then extended to meet the same objective in the presence of wind which has a magnitude comparable to the speed of the MAV. We use a modification of the Dubins' path method to obtain the complete optimal solution to this problem in all its generality.**

I. INTRODUCTION

Optimal path planning is an important area of interest in the field of Unmanned Air Vehicle (UAV) and robotics. In the classical work by Dubins [1], the smooth shortest path for fixed initial and final positions and orientations has been obtained geometrically. Reeds and Shepp [2] solved a similar problem using advanced calculus in which a vehicle can move forward as well as backward. Boissonnat et al. [3] proved the same result as Dubins using the powerful Pontryagin's minimum principle. Shkel and Lumelsky [4] classified Dubins path for different sets of initial and final configurations. Thomaschewski [5] solved a Dubins problem where terminal direction is not prescribed. McGee et al. [6],[7] used Dubins path to explore the problem of finding an optimal path for an UAV in wind condition. Nelson et al. [8] propose a method of straight line path-following and circular path-following by an MAV in the presence of wind, based on a vector field approach. However, the path followed by the MAV is not optimal. In [9] optimal path for MAV in the presence of wind is derived for the case when the MAV is situated sufficiently away from the straight line path. Osborne et al. [10] solved a problem of waypoint following by an UAV in the presence of wind. Wong et al. [11] determine C-C-C class paths for an UAV performing target touring, using a hybrid algorithm to solve a parameter optimization problem for path planning. Ceccarelli et al. [12] addressed the problem of controlling an MAV for the purpose of obtaining a video footage of a set of known ground targets in the presence of known constant wind. McNeely et al. [13] present results on the existence and uniqueness of minimum time solution for a Dubins vehicle flying in a general time-varying wind vector field. Stolle et al. [14] solved a problem where an UAV is equipped with a nose mounted camera for an observation of ground targets in the presence of wind.

The contribution of the present paper is to obtain a solution for the optimal path followed by an MAV in both wind and windless environment for the case where the terminal straight line path is specified but not the final position. The solution is based upon a modification of the Dubins method to account for free final position that lies on the desired straight line.

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II. ABSENCE OF WIND CONDITION

In this section we present a time optimal trajectory for an MAV from its known initial position (x_0, y_0) and orientation (χ_0) to its final position that lies on the given straight line in the absence of wind (see Fig. 1 (a)). Without any loss of generality, one can assume the final straight line and orientation to be the Y axis of the coordinate system. Hence, we have the final orientation, $\chi_f = 0$, and final x coordinate, $x_f = 0$. The Y coordinate of the final position (y_f) is free.

In Dubins' paper [1] it is shown that the shortest path consists of three consecutive path segments of the Dubins set, D, which includes six paths D={LSL, RSR, RSL, LSR, RLR, LRL}, where left and right turn with minimal allowed radius of turn (r) are denoted by L and R, respectively, and the straight line path segment is denoted by S.

Initial orientation can be any angle $(-\pi \leq \chi_0 \leq \pi)$ (see Fig. 1 (b)) and the MAV can be situated at any position, but without loss in generality, we will discuss those cases when the MAV is situated on the left side of the terminal straight line path. We will divide all the possible cases into sixteen categories (see Table I) based on the required change in orientation angle and distance from the initial to the final X-coordinate.

For the sake of clarity, we discuss in detail the cases when initial orientation is less than $\pi/2$ and the MAV is situated on the left side of the straight line. For other cases results will be given without detailed analysis due to lack of space.

The variable y_{ij} is used to denote the Y-coordinate of the final point on the straight line path, that yields the optimal air path. The notations l_{ij} is used to describe the total length corresponding to minimum air path, where i represents the quadrants $(i = 1, ..., 4)$ and j represents the particular class of path in the Dubins set as given in Table II.

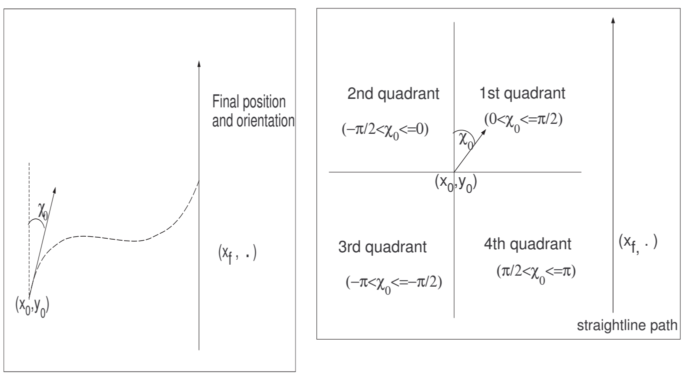
A. I-LP (First Quadrant, Long Path)

1) *Type of path*: In [4] it has already been proved that for two given points in a plane, each with the prescribed direction of motion, the optimal time path is CSC and not CCC for the long path case. Here 'C' stands for a circular turn either to the right (R) or to the left (L). As the straight line path is situated on the right side of the MAV, it will reach the desired path earlier if it takes a right turn first. Now the question is whether

TABLE I

POSSIBLE CASES

	Long Path $(x_f - x_0) > 4r$	Medium Path $2r < (x_f - x_0) \leq 4r$	Short Path $r < (x_f - x_0) \leq 2r$	Very Short Path $0 \leq (x_f - x_0) \leq r$
Q I $(0 < \chi_0 \leq \pi/2)$	I-LP	I-MP	I-SP	I-VSP
Q II $(-\pi/2 < \chi_0 \leq 0)$	II-LP	II-MP	II-SP	II-VSP
Q III $(-\pi < \chi_0 \leq -\pi/2)$	III-LP	III-MP	III-SP	III-VSP
Q IV $(\pi/2 < \chi_0 \leq \pi)$	IV-LP	IV-MP	IV-SP	IV-VSP



(a) (b)

Fig. 1. (a) The geometry of initial and final configurations (b) Definition of quadrants

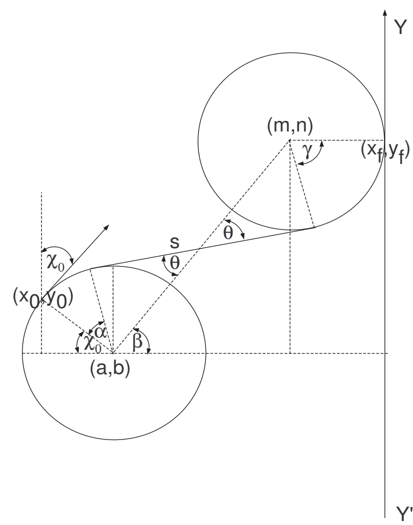


Fig. 2. Dubins' RSL path for initial orientation $(\chi_0) < \pi/2$ and final orientation $(\chi_f) = 0$.

TABLE II

TABLE FOR 'j'

j	Dubins path type
0	L
1	RSL
2	RL
3	LR
4	LSL
5	SL

the optimal path is of the RSL type or RSR type. If the MAV takes a left turn just before a distance r from the desired line, the path will be shorter than that of RSR type. This is because, in the RSR case, the MAV crosses the desired line and enters the right side and then takes a right turn to come back to the desired line with the desired final orientation. So, the optimal path will be of RSL type if χ_0 is less than $\pi/2$, otherwise it will be of SL type.

2) *Length of the path*: For the configuration given in Figure 2, we arbitrarily fix the final point on the straight line. To get the minimum path among all feasible RSL path we have to minimize the total path length. Total path length (l) is given by,

$$l = s + (r\alpha + r\gamma) \tag{1}$$

where, s is the length of the straight line path and $(r\alpha + r\gamma)$ is the total length of the two arcs.

To calculate the straight line path (s) we do the following: The center of the first circle is

$$(a, b) = (x_0 + r\cos\chi_0, y_0 - r\sin\chi_0) \tag{2}$$

Center of the second circle is

$$(m, n) = (x_f - r\cos\chi_f, y_f + r\sin\chi_f) \tag{3}$$

Length of the straight line path will be

$$s = \sqrt{(a-m)^2 + (b-n)^2 - 4r^2} \tag{4}$$

For this problem $x_f = 0$ and $\chi_f = 0$. So s will be

$$\sqrt{(x_0 + r\cos\chi_0 + r)^2 + (y_0 - y_f - r\sin\chi_0)^2 - 4r^2}$$

To calculate the arc length we consider Figure 2, from which we will get, total arc length=$(\alpha + \gamma)r$, where

$$\alpha = \gamma - \chi_0 \quad , \qquad \gamma = \pi/2 + \theta - \beta$$

and

$$\theta = \sin^{-1}\left\{(2r)/\sqrt{(a-m)^2 + (b-n)^2}\right\},$$

$$\beta = \sin^{-1}\left\{(n-b)/\sqrt{(a-m)^2 + (b-n)^2}\right\}$$

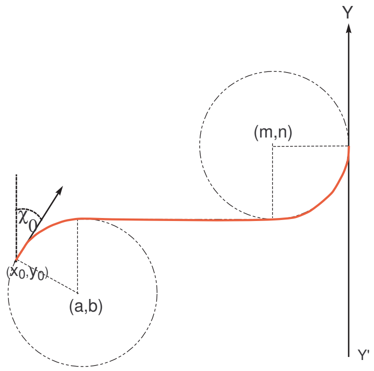


Fig. 3. First quadrant: Long Path (I-LP) case

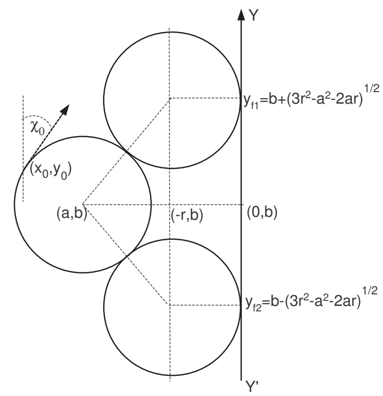


Fig. 4. First quadrant: Analysis for medium path (I-MP) case

So, total arc length=$(\pi + 2\theta - 2\beta - \chi_0)r$. The length of the path traveled by the MAV will be

$$l = s + (\pi + 2\theta - 2\beta - \chi_0)r \tag{5}$$

To get minimum path length, we differentiate l with respect to y_f,

$$\frac{dl}{dy_f} = -\frac{(b-n)}{s} + \frac{2r[-(a-m)s + 2r(b-n)]}{s[(a-m)^2 + (b-n)^2]} = 0 \tag{6}$$

Solving this equation we will get $(b - n) = \pm 2r$, but for this configuration $(b - n) < 0$, so we get

$$n = b + 2r \tag{7}$$

since $\chi_f = 0$, hence the Y-coordinate of the final position of MAV on the straight line that yields the optimal path is

$$y_{fmin} = y_0 - r \sin \chi_0 + 2r \tag{8}$$

So, the minimum length of the optimal path (l_{min}) (Fig. 3) in this case is obtained as follows:

a) If path is of RSL type: Total length is, $l_{11} = (\pi - \chi_0)r - x_0 - r(1 + \cos \chi_0)$, which is a sum of the first arc length $(\pi/2 - \chi_0)r$, the straight line length $-x_0 - r(1 + \cos \chi_0)$ and the second arc length $(\pi/2)r$.

b) If path is of SL type: Total length is, $l_{15} = (\pi/2)r - x_0 - r$.

3) *The point at which the optimal path will meet the straight line path (y_{fmin})*:

a) If path is of RSL type: $y_{11} = y_0 - r \sin \chi_0 + 2r$

b) If path is of SL type: $y_{15} = y_0 + r$

B. I-MP (First Quadrant, Medium Path)

For this configuration, there are many types of optimal Dubins paths when we vary y_f (see Fig. 4) along the straight line.

(i) When $b - \sqrt{3r^2 - a^2 - 2ar} < y_f < b + \sqrt{3r^2 - a^2 - 2ar}$ there is no RSL type path, and the optimal path is either LSL or RSR type.

(ii) When $y_f = b \pm \sqrt{3r^2 - a^2 - 2ar}$ the optimal path is of RL type.

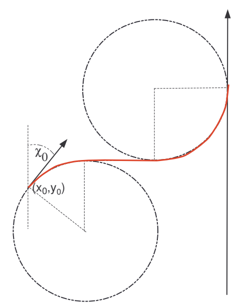


Fig. 5. First quadrant: Medium Path (I-MP) case

(iii) For other values of y_f, the optimal path is of RSL or LSL type.

However, among all these possible paths, minimum path length is of RSL (SL, if $\chi_0 = \pi/2$) type (see Fig. 5) and expressions for the total length of the path and expressions for y_{fmin} remain the same as in I-LP.

C. I-SP (First Quadrant, Short Path)

With reference to Fig. 6 we get,

1) Optimal path will be RSL (SL, if $\chi_0 = \pi/2$) if at that point the orientation is more than χ_{011}. The expressions for l_{min} and y_{fmin} will remain same as in the I-LP case.

2) Optimal path will be RL[1] if at that point the orientation is less than or equal to χ_{011}, where χ_{011} is given by,

$$\chi_{011} = \cos^{-1}\{(-x_0 - r)/r\} \tag{9}$$

and the switching angle at which the MAV changes its turning strategy from right (R) to left (L) is denoted by γ_{1a}, where,

$$\gamma_{1a} = \cos^{-1}\{(x_0 + r + r\cos\chi_0)/2r\} \tag{10}$$

RL[1] type path consists of two arc length; the length of the first arc is $(\gamma_{1a} - \chi_0)r$ and the second arc length is $\gamma_{1a}r$. So the minimum length of the path is, $l_{12}^1 = (2\gamma_{1a} - \chi_0)r$. The expression for y_{fmin} is, $y_{12}^1 = y_0 - r\sin\chi_0 + 2r\sin\gamma_{1a}$

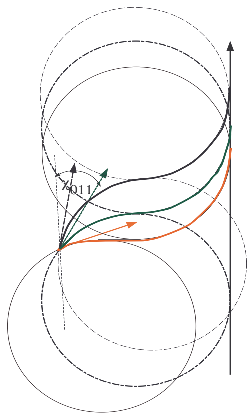


Fig. 6. First quadrant: Short path (I-SP) case

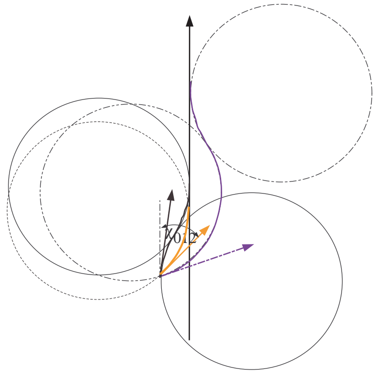


Fig. 7. First quadrant: Very Short Path (I-VSP) case

D. I-VSP (First Quadrant, Very Short Path)

With reference to Fig. 7 we get,

1) Optimal path will be RL^1 type if orientation at that point is less than χ_{012}.
2) If orientation at that point is of a value exactly same as χ_{012} then optimal path will be of L type. Total length is, $l_{10} = \chi_{012} r$, and the expression for y_{fmin} is, $y_{10} = y_0 + r \sin \chi_{012}$.
3) If orientation is more than χ_{012} optimal path will be of LR^1 type, where,

$$\chi_{012} = \cos^{-1}\{(x_0 + r)/r\} \tag{11}$$

and the switching angle at which the MAV changes its turning strategy from left (L) to right (R) is denoted by γ_{2a}, where,

$$\gamma_{2a} = \cos^{-1}\{(-x_0 + r + r\cos\chi_0)/2r\} \tag{12}$$

LR^1 type path consists of two arc length; the length of the first arc is $(\gamma_{2a} + \chi_0)r$, the length of the second arc length is $\gamma_{2a} r$. Total length is, $l^1_{13} = (2\gamma_{2a} + \chi_0)r$. The expression for y_{fmin} is, $y^1_{13} = y_0 + r\sin\chi_0 + 2r\sin\gamma_{2a}$.

E. All Quadrants and Orientations

The results for optimal path for all orientations is listed in the Table III, where,

TABLE III

ALL PATH TYPE WITH ITS LENGTH

	Path Type	Length of Path	y_{fmin}	Range of χ_0
I-LP,I-MP	RSL	l_{11}	y_{11}	$\chi_0 \neq \pi/2$
	SL	l_{15}	y_{15}	$\chi_0 = \pi/2$
I-SP	RSL	l_{11}	y_{11}	$\chi_0 > \chi_{011} (\neq \frac{\pi}{2})$
	SL	l_{15}	y_{15}	$\chi_0 = \pi/2$
	RL^1	l^1_{12}	y^1_{12}	$\chi_0 \leq \chi_{011}$
I-VSP	RL^1	l^1_{12}	y^1_{12}	$\chi_0 < \chi_{012}$
	L	l_{10}	y_{10}	$\chi_0 = \chi_{012}$
	LR^1	l^1_{13}	y^1_{13}	$\chi_0 > \chi_{012}$
II-LP,II-MP	RSL	l_{21}	y_{21}	all χ_0 in QII
II-SP	RSL	l_{21}	y_{21}	$\chi_0 < \chi_{021}$
	RL^1	l^1_{22}	y^1_{22}	$\chi_0 \geq \chi_{021}$
II-VSP	RL^1	l^1_{22}	y^1_{22}	all χ_0 in QII
III-LP,III-MP	LSL	l_{34}	y_{34}	all χ_0 in QIII
III-SP	LSL	l_{34}	y_{34}	$\chi_0 > \chi_{031}$
	L	l_{30}	y_{30}	$\chi_0 = \chi_{031}$
	RL^2	l^2_{32}	y^2_{32}	$\chi_0 < \chi_{031}$
III-VSP	RL^2	l^2_{32}	y^2_{32}	$\chi_0 < \chi_{032}$
	RL^1	l^1_{32}	y^1_{32}	$\chi_0 \geq \chi_{032}$
IV-LP,IV-MP	LSL	l_{44}	y_{44}	all χ_0 in QIV
IV-SP	LSL	l_{44}	y_{44}	$\chi_0 < \chi_{041}$
	L	l_{40}	y_{40}	$\chi_0 = \chi_{041}$
	LR^1	l^1_{43}	y^1_{43}	$\chi_0 > \chi_{041}$
IV-VSP	LR^1	l^1_{43}	y^1_{43}	all χ_0 in QIV

$$
\begin{aligned}
l_{11} &= (\pi - \chi_0)r - x_0 - r(1 + \cos\chi_0) \\
y_{11} &= y_0 - r\sin\chi_0 + 2r \\
\gamma_{1a} &= \cos^{-1}\{(x_0 + r + r\cos\chi_0)/2r\} \\
l^1_{12} &= (2\gamma_{1a} - \chi_0)r \\
y^1_{12} &= y_0 - r\sin\chi_0 + 2r\sin\gamma_{1a} \\
\chi_{011} &= \cos^{-1}\{(-x_0 - r)/r\} \\
\chi_{012} &= \cos^{-1}\{(r + x_0)/r\} \\
\gamma_{2a} &= \cos^{-1}\{(-x_0 + r + r\cos\chi_0)/2r\} \\
l^1_{13} &= (2\gamma_{2a} + \chi_0)r \\
y^1_{13} &= y_0 + r\sin\chi_0 + 2r\sin\gamma_{2a} \\
l_{15} &= (\pi/2)r - x_0 - r \\
y_{15} &= y_0 + r \\
l_{10} &= r\chi_{012} \\
y_{10} &= y_0 + r\sin\chi_{012} \\
l_{1j} &= l_{2j} \text{ for } j = 1, 2 \\
y_{1j} &= y_{2j} \text{ for } j = 1, 2 \\
\chi_{021} &= -\cos^{-1}\{(-x_0 - r)/r\} \\
\gamma_{1b} &= -\pi + \cos^{-1}\{(-x_0 - r - r\cos\chi_0)/2r\} \\
l^2_{32} &= (2\gamma_{1b} + 2\pi - \chi_0)r \\
y^2_{32} &= y_0 - r\sin\chi_0 + 2r\sin\gamma_{1b} \\
l^1_{32} &= (2\gamma_{1a} - \chi_0)r \\
y^1_{32} &= y_0 - r\sin\chi_0 + 2r\sin\gamma_{1a} \\
\chi_{031} &= -\pi + \cos^{-1}\{(-x0 - r)/r\} \\
\chi_{032} &= -\pi + \cos^{-1}\{(x0 + r)/r\} \\
l_{34} &= (2\pi + \chi_0)r - x_0 - r(1 - \cos\chi_0) \\
y_{34} &= y_0 + r\sin\chi_0
\end{aligned}
$$

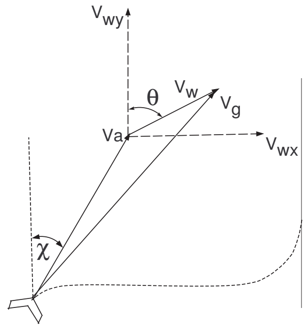


Fig. 8. Representation of airspeed (v_a), ground speed (v_g) and wind speed (v_w).

$$
\begin{aligned}
l_{30} &= (2\pi + \chi_{031})r \\
y_{30} &= y_0 + r\sin\chi_{031} \\
\chi_{041} &= \pi - \cos^{-1}\{(-x_0 - r)/r\} \\
l_{40} &= \chi_{041} r \\
y_{40} &= y_0 + r\sin\chi_{041} \\
l_{44} &= \chi_0 r - r - x_0 + r\cos\chi_0 \\
y_{44} &= y_0 + r\sin\chi_0 \\
l^1_{43} &= (2\gamma_{2a} + \chi_0)r \\
y^1_{43} &= y_0 + r\sin\chi_0 + 2r\sin\gamma_{2a}
\end{aligned}
$$

III. Presence of Wind condition

Consider the simple kinematic model given in Fig. 8. The problem is defined for fixed wing MAV which cannot make instantaneous arbitrary movement. The control input here is the turning rate, $\dot{\chi}$, which is assumed to be bounded by $|\dot{\chi}| < \dot{\chi}_{max}$. The speed of the vehicle is v_a which is the air speed, the wind speed is v_w with an angle θ with respect to the vertical axis and we define $q = v_a/v_w$. We assume that wind speed and speed of the vehicle are constant. Let,

$$X = [x \quad y \quad \chi]^T \tag{13}$$

then,

$$\dot{X} = [v_a \sin\chi + v_w \sin\theta \quad v_a \cos\chi + v_w \cos\theta \quad \dot{\chi}]^T \tag{14}$$

This problem can be formulated in the way similar to McGee et al. [6]. An MAV approaching a straight line in wind condition can be considered as a virtual target straight line moving with an equal and opposite velocity to the wind acting on it in a situation where the wind is absent. The airpath is defined as the path traveled by the MAV with respect to the moving air frame and ground path is the path with respect to the ground. So, in this problem formulation, the MAV will follow its optimal air path to meet its virtual final point situated on the virtual target straight line (VTSL) and the reformulated model will be as follows:

$$\dot{X} = [v_a \sin\chi \quad v_a \cos\chi \quad \dot{\chi}]^T \tag{15}$$

where the initial position and orientation, X_i, is given. Final orientation is the same as χ_f and the final position for this reconstructed problem is

$$x_d = x_f - v_w t \sin\theta \qquad y_d = y_f - v_w t \cos\theta$$

where, t is the minimum time it takes for the MAV to meet with its virtual final point and d_{ij} is the distance traveled by the virtual final point in time t.

A. Generation of minimum time path

Under no wind condition, the long path and short path cases can be classified clearly. But in the presence of wind, it is not clear exactly how much distance will be traveled by the VTSL till the MAV achieves its goal. So, the nature of the optimal path is unknown to us. We will discuss in detail how the Dubins airpath switches from one type to another in the presence of wind.

From the discussion on the no wind case for the first quadrant, we know that the optimal air path can be one of the following set, {RSL, SL, RL^1, L, LR^1} of Dubins path. Now we will derive the optimal airpath for the reformulated problem. Then, from the state equation, we can get the ground path. These will be illustrated in the simulation results given in Section IV.

Let the VTSL be at a position S_1 when MAV achieves its goal (see Fig. 9(a)). As shown in this figure, the S_1 line is a common tangent of two circles, one at the initial position and the other at the final position. The position of S_1 is important here because if finally the virtual target straight line cannot cross this position and stays to the right side of this position, the optimal path will be of RSL or SL (if $\chi_0 = \pi/2$) type. If finally the virtual target straight line crosses this S_1 position and goes to the left side of this position (for example, position of S_3) then the optimal airpath can be one of three paths: {RL^1, L, LR^1}.

From Fig. 9(b) it is clear that if VTSL reaches S_1 position at the final time then, by taking only one left turn, the MAV can achieve its goal. But if finally VTSL lies on the right side of S_1, air path is of RL^1 type, otherwise air path is of LR^1 type.

B. Algorithm for checking path type in the wind case

We will discuss the algorithm for the case when χ_0 is in first quadrant. For all other quadrants, we can apply the same procedure to derive the optimal paths.

1) If

$$-x_0 - r\cos\chi_0 - r > \{(\pi - \chi_0) r v_w \sin\theta\}/v_a \tag{16}$$

then the airpath is of RSL type and its length is, $ql_{11}/(q + \sin\theta)$

2) If

$$-x_0 - r\cos\chi_0 - r \le \{(\pi - \chi_0) r v_w \sin\theta\}/v_a \tag{17}$$

a) If

$$-x_0 - r + r\cos\chi_0 > \{\chi_0 v_w r \sin\theta\}/v_a \tag{18}$$

then airpath is RL^1 with length $(2\gamma_{w11a} - \chi_0)r$; where, γ_{w11a} is defined in the same way as γ_{1a}

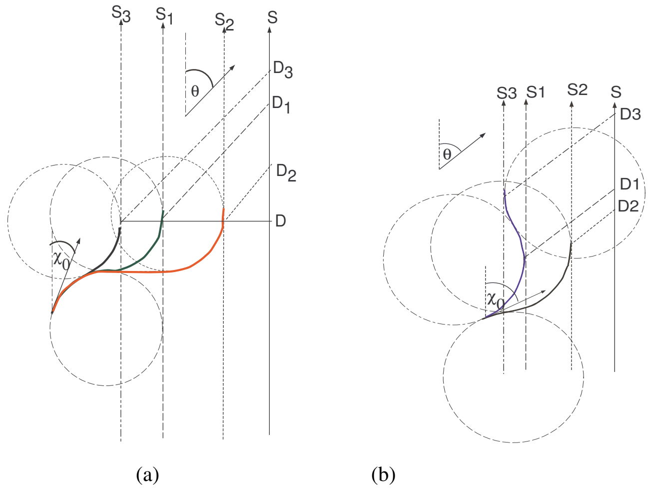


Fig. 9. (a) First quadrant: Wind case for RSL and RL^1 path (b) First quadrant: Wind case for RL^1, L and LR^1

after replacing x_0 by $x_0 + d_{12a} \sin\theta$, and d_{12a} is solution of the following equation,

$$\cos \frac{q d_{12a} + r\chi_0}{2r} - \frac{d_{12a}\sin\theta}{2r} - \cos\gamma_{1a} = 0 \quad (19)$$

b) if

$$-x_0 - r + r\cos\chi_0 = \{\chi_0 v_w r \sin\theta\}/v_a \quad (20)$$

then airpath is of L type and its length is $r\chi_0$.

c) if

$$-x_0 - r + r\cos\chi_0 < \{\chi_0 v_w r \sin\theta\}/v_a \quad (21)$$

then airpath is LR^1 and its length is, $(2\gamma_{w12a} + \chi_0)r$ Where, γ_{w12a} is defined in the same way as γ_{2a} after replacing x_0 by $x_0 + d_{13a}\sin\theta$, and d_{13a} is solution of the following equation,

$$\cos \frac{q d_{13a} - r\chi_0}{2r} + \frac{d_{13a}\sin\theta}{2r} - \cos\gamma_{2a} = 0 \quad (22)$$

IV. Simulation Results

Simulation results are given for the wind case with one value of orientation but for two different x_0 positions. We have used $r = 10, V_a = 20, y_0 = 0, \chi_0 = \pi/4, \chi_f = 0, x_f = 0$, and wind parameters are $V_w = 5, \theta = \pi/6$. RSL and RL^1 airpaths and corresponding ground paths to achieve the desired straight line that lies on Y axis are shown in Fig. 10 (a) and (b) for initial X position -50 and -15 respectively.

This is only a representative solution obtained from the analysis given in this paper. It is possible to obtain the solution to the optimal path profile for all orientations and positions of the MAV.

V. Conclusions and Future Work

The central idea developed in this paper is to direct an MAV to its optimal path that would enable it to fly along a given straight line. We have considered those cases when the MAV is placed anywhere with any orientation. The results obtained are general and the problem is solved completely in all its generality.

The ideas can be extended to terminal paths that are not necessarily straight lines (for example this can be circular).

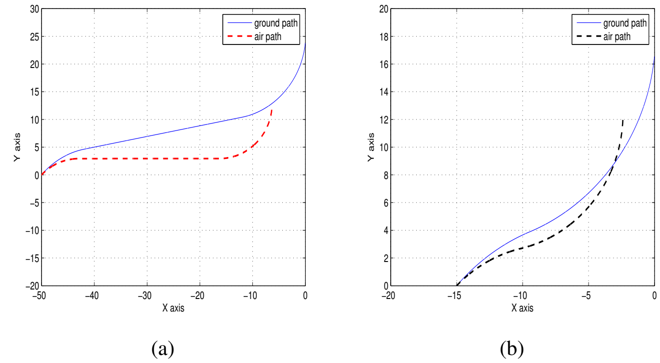


Fig. 10. First quadrant (a) RSL path (b) RL^1 path

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