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Finite-Time Control for Linear Discrete-Time Systems with Input Constraints

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Abstract— **Finite-time stabilization (FTS) and finite-time boundedness (FTB) control with input constraints are considered for linear discrete-time time-invariant systems. Design methods of state feedback and observer-based output feedback FTS/FTB controllers that satisfy input constraints are proposed based on reachable sets in finite-time period. Less conservative design method of controllers for the maximum magnitude of input signals is also given. Numerical examples are shown to illustrate the proposed design methods.**

I. INTRODUCTION

Finite-time stability (FTS) requires that the state of a system does not exceed a certain bound during a specified time interval for given bound on the initial state. While Lyapunov stability is used to deal with the behavior of a system within a sufficiently long (or infinite) time interval, FTS is used to deal with the behavior of a system within a finite (or very short) time interval. Therefore there are real applications such as operations of missiles and space vehicles from an initial point to a final point in a specified time interval. The concept of FTS is also extended to that of finite-time boundedness (FTB) by introducing an exogenous input and sufficient conditions for FTB are also given [1], [2]. Sufficient conditions for the existence of state feedback laws that guarantee FTB of a closed-loop system are given for linear continuous-time systems [1], [3] and for linear discrete-time systems [2]. Moreover sufficient conditions for the existence of output feedback controllers that guarantee FTS and FTB of a closed-loop system are given both for linear continuous-time and discrete-time systems [1]. In finite-time control problems, boundedness of the physical state of a system is of interest from the practical point of view and finite-time stabilization with observer-based output feedback controllers is considered for both linear continuous-time systems [2] and discrete-time systems [4].

In the above literatures on finite-time control, input signals could be larger as time has passed. Since trajectories do not always converge to the origin, input signals by state feedback laws could be larger and exceed a physical limitation on control. Similar situations may arise in the case of output feedback control. Input constraints in finite step are required to finite-time control from practical viewpoint. As far as Lyapunov stability, there are literatures on input constraint conditions using LMI [5], [6], [7]. However, any constraint conditions of finite-time period have not been discussed. In this paper, we give sufficient conditions for the existence of FTB (or FTS) controllers that satisfy a $\mathcal{N}$-step input constraint using reachable sets in finite-time steps. The obtained sufficient conditions are reduced to LMI conditions.

This paper is organized as follows. Section II gives preliminary results on FTB state feedback controller design without input constraints. An extension to observer-based output feedback controller design is also given. In section III, $\mathcal{N}$-step input constraint is defined and state feedback controller design with input constraint is discussed. Section IV discusses output feedback controller design with input constraint. Section V gives numerical examples. Finally, section VI concludes with remarks.

Notations: Let M_j be j-th row of a matrix M. $\mathbf{N}_0 = \{0, 1, 2, \ldots\}$.

II. PRELIMINARY RESULTS

Consider

$$x(k+1) = Ax(k) + B_1 w(k) + B_2 u(k), \quad x(0) = x_0, \quad (1)$$
$$w(k+1) = Sw(k), \quad w(0) = w_0, \quad (2)$$
$$y(k) = Cx(k) \quad (3)$$

where $x \in \mathbf{R}^n$ is the state, $w \in \mathbf{R}^{m_1}$ is the disturbance generated by the exosystem (2), $u \in \mathbf{R}^{m_2}$ is the control input, $y \in \mathbf{R}^p$ is the measurement and all matrices are of compatible dimensions. The following concepts are known.

Definition 2.1 ([2]): For given positive definite matrix Γ, $0 \le \delta_x < \varepsilon$ and $N \in \mathbf{N}_0$, if $x^T(k)\Gamma x(k) < \varepsilon$, $k = 1, \ldots, N$ whenever $x_0^T \Gamma x_0 \le \delta_x$, then the system $x(k+1) = Ax(k)$ with $x(0) = x_0$ is said to be finite-time stable (FTS) with respect to $(\delta_x, \varepsilon, \Gamma, N)$.

Definition 2.2 ([4]): For given positive definite matrix Γ, Π, $0 \le \delta_x < \varepsilon$, $0 \le \delta_w$ and $N \in \mathbf{N_0}$, if $x^T \Gamma x(k) < \varepsilon$, $k = 1, \ldots, N$ whenever $x_0^T \Gamma x_0 \le \delta_x$ and $w_0^T \Pi w_0 \le \delta_w$, then the system $x(k+1) = Ax(k) + B_1 w(k)$ and $w(k+1) = Sw(k)$ with $x(0) = x_0$ and $w(0) = w_0$ is said to finite-time bounded (FTB) with respect to $(\delta_x, \delta_w, \varepsilon, \Gamma, \Pi, N)$.

For the system (1) and (2), we also consider state feedback controllers

$$u(k) = Fx(k). \quad (4)$$

Then the closed-loop system (1), (2) and (4) is given by

$$x(k+1) = A_F x(k) + B_1 w(k), \ x(0) = x_0 \quad (5)$$

and (2) where $A_F = A + B_2 F$. Then we have the following result.

Lemma 2.1 ([2]): The system (5) and (2) is FTB with respect to $(\delta_x, \ \delta_w, \ \varepsilon, \ \Gamma, \ \Pi, \ N)$ if there exist positive

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definite matrices $Q_1 \in \mathbf{R}^{n\times n}$, $Q_2 \in \mathbf{R}^{m_1\times m_1}$, a matrix $L \in \mathbf{R}^{m_2\times n}$ and a scalar $\gamma \geq 1$ such that

$$\begin{bmatrix} -Q_1 & AQ_1+B_2L & B_1 \\ (AQ_1+B_2L)^T & -\gamma Q_1 & 0 \\ B_1^T & 0 & S^TQ_2S-\gamma Q_2 \end{bmatrix} < 0, \quad (6)$$

$$\frac{\delta_x}{\lambda_{min}(\tilde{Q}_1)} + \lambda_{max}(\tilde{Q}_2)\delta_w < \frac{\varepsilon}{\gamma^N}\frac{1}{\lambda_{max}(\tilde{Q}_1)} \quad (7)$$

where $\tilde{Q}_1 = \Gamma^{\frac{1}{2}}Q_1\Gamma^{\frac{1}{2}}$ and $\tilde{Q}_2 = \Pi^{-\frac{1}{2}}Q_2\Pi^{-\frac{1}{2}}$. In this case the feedback gain F is given by $F = LQ_1^{-1}$.

Remark 2.1: If we set $S = 0$ and $\delta_w = 0$ in (6) and (7), then we can also derive sufficient conditions for FTS [2].

For the system (1)-(3), we consider output feedback controllers of the form

$$\hat{x}(k+1) = A\hat{x}(k) + B_2u(k) - K[y(k) - C\hat{x}(k)], \quad \hat{x}(0) = 0, \quad (8)$$
$$u(k) = F\hat{x}(k)$$

where F and K are matrices of compatible dimensions. In the finite-time control problems, boundedness of the physical state of the system is of interest, from practical point of view, we want to find a controller (4) such that the system

$$x(k+1) = A_Fx(k) + B_1w(k) - B_2Fe(k),$$
$$w(k+1) = Sw(k)$$

is FTB with respect to $(\delta_x, \delta_w, \varepsilon, \Gamma, \Pi, N)$ where $e = x - \hat{x}$. To find such observer-based output feedback controllers, we assume that a state feedback controller $u(k) = Fx(k)$, which makes the system (1) and (2) FTB with respect to $(\delta_x, \delta_w, \varepsilon, \Gamma, \Pi, N)$ (or the system (1) with $w \equiv 0$ FTS with respect to $(\delta_x, \varepsilon, \Gamma, N)$) exists and has been designed.

The closed-loop system (1)-(3) and (8) can be written as

$$\begin{aligned} x(k+1) &= A_Fx(k) + B_F(k)\tilde{w}(k),\ x(0) = x_0, \\ \tilde{w}(k+1) &= \tilde{S}\tilde{w}(k),\ \tilde{w}(0) = \begin{bmatrix} w_0^T & x_0^T \end{bmatrix}^T \end{aligned} \quad (9)$$

where

$$B_F = \begin{bmatrix} B_1 & -B_2F \end{bmatrix},\ \tilde{w} = \begin{bmatrix} w \\ e \end{bmatrix},\ \tilde{S} = \begin{bmatrix} S & 0 \\ B_1 & A_K \end{bmatrix}$$

and $A_K = A + KC$. For $e = 0$, the system (9) is FTB (or FTS) while the presence of a nonzero e may bring the norm of the state $x(k)$ outside the bound ε. Hence we want to design an observer gain K in (8) such that the FTB property of the system (5) and (2) is not lost in the presence of the estimation error. Note that the bound in the initial condition of the exosystem in (9) satisfies

$$w_0^T\Pi w_0 + x_0^T\Gamma x_0 \leq \delta_w + \delta_x.$$

Then if the system (9) is FTB with respect to $(\delta_x, \delta_w + \delta_x, \Gamma, \mathrm{diag}\{\Pi, \Gamma\}, N)$, then the closed-loop system (1)-(3) and (8) is FTB with respect to $(\delta_x, \delta_w, \Gamma, \Pi, N)$.

Lemma 2.2 ([4]): If there exist positive definite matrices P_1, $R \in \mathbf{R}^{n\times n}$, $P_2 \in \mathbf{R}^{m_1\times m_1}$, and a matrix $M \in \mathbf{R}^{n\times p}$, and a scalar $\gamma \geq 1$ such that

$$\begin{bmatrix} A_F^TP_1A_F - \gamma P_1 & A_F^TP_1B_1 & A_F^TP_1(-B_2F) & 0 \\ B_1^TP_1A_F & H_2 & B_1^TP_1(-B_2F) & B_1^TR \\ (-B_2F)^TP_1A_F & (-B_2F)^TP_1B_1 & H_3 & (RA+MC)^T \\ 0 & RB_1 & RA+MC & -R \end{bmatrix} < 0, \quad (10)$$

$$\left[\lambda_{max}(\tilde{P}_1) + \lambda_{max}(\tilde{R})\right]\delta_x + \lambda_{max}(\tilde{P}_2)\delta_w < \frac{\varepsilon}{\gamma^N}\lambda_{min}(\tilde{P}_1), \quad (11)$$

then an observer-based output feedback controller (8) makes the system (1)-(3) FTB with respect to $(\delta_x, \delta_w, \Gamma, \Pi, N)$ where

$$H_2 = B_1^TP_1B_1 + S^TP_2S - \gamma P_2,$$
$$H_3 = (-B_2F)^TP_1(-B_2F) - \gamma R,$$
$$\tilde{P}_1 = \Gamma^{-\frac{1}{2}}P_1\Gamma^{-\frac{1}{2}},\ \tilde{P}_2 = \Pi^{-\frac{1}{2}}P_2\Pi^{-\frac{1}{2}},\ \tilde{R} = \Gamma^{-\frac{1}{2}}R\Gamma^{-\frac{1}{2}}.$$

In this case the observer gain K is given by $K = R^{-1}M$.

Remark 2.2: If we set $S = 0$ and $\delta_w = 0$ in (10) and (11), then we can also derive sufficient conditions for the existence of observer-based output feedback FTS controllers [4].

III. State Feedback Controller Design with Input Constraints

Consider $\mathcal{N}$-step input constraints such that

$$|u_j(k)| \leq u_j^{\max},\ j = 1, \ldots, m_2,\ k = 0, 1, \ldots, \mathcal{N} \quad (12)$$

where $\mathcal{N} \in \mathbf{N}_0$ is a design parameter satisfying $0 \leq \mathcal{N} \leq N$. Then we want to design FTB state feedback controllers (4) satisfying (12) for the system (1) and (2). Using reachable sets of the state given by Lyapunov-like functions, we estimate the maximum magnitude of the input signals.

Theorem 3.1: There exist state feedback FTB controllers that satisfy (12) for the system (1) and (2) if there exist positive definite matrices $Q_1 \in \mathbf{R}^{n\times n}$, $Q_2 \in \mathbf{R}^{m_1\times m_1}$, a matrix $L \in \mathbf{R}^{m_2\times n}$, scalars $\gamma \geq 1$ and $\lambda_i > 0$, $i = 1, 2$, such that (6), (7) and

$$\begin{bmatrix} (u_j^{\max})^2/d_{\mathcal{N}} & L_j \\ L_j^T & Q_1 \end{bmatrix} \geq 0, \quad j = 1, \ldots, m_2 \quad (13)$$

where

$$d_{\mathcal{N}} = \begin{cases} \dfrac{\delta_x}{\lambda_{\min}(\tilde{Q}_1)}, & \mathcal{N} = 0, \\ \gamma^{\mathcal{N}}\left[\dfrac{\delta_x}{\lambda_{\min}(\tilde{Q}_1)} + \lambda_{\max}(\tilde{Q}_2)\delta_w\right], & \mathcal{N} \geq 1, \end{cases}$$

$\tilde{Q}_1 = \Gamma^{\frac{1}{2}}Q_1\Gamma^{\frac{1}{2}}$ and $\tilde{Q}_2 = \Pi^{-\frac{1}{2}}Q_2\Pi^{-\frac{1}{2}}$. In this case the feedback gain F is given by $F = LQ_1^{-1}$.

Proof: See the appendix. ■

To obtain more numerically tractable sufficient conditions for FTB with input constraints, we assume Q_1 and Q_2 in Theorem 3.1 satisfy

$$\lambda_1 I < \tilde{Q}_1 < I,\ \tilde{Q}_2 < \lambda_2 I \tag{14}$$

and

$$\begin{bmatrix} \frac{\varepsilon}{\gamma^N} - \lambda_2\delta_w & \delta^{\frac{1}{2}} \\ \delta^{\frac{1}{2}} & \lambda_1 \end{bmatrix} > 0. \tag{15}$$

Then using Shur complement formula for (15), we obtain

$$\frac{\delta_x}{\lambda_1} + \lambda_2\delta_w < \frac{\varepsilon}{\gamma^N}.$$

Hence

$$\begin{aligned} & \frac{\delta_x}{\lambda_{\min}(\tilde{Q}_1)} + \lambda_{\max}(\tilde{Q}_2)\delta_w \\ < \ & \frac{\delta_x}{\lambda_1} + \lambda_2\delta_w < \frac{\varepsilon}{\gamma^N} < \frac{\varepsilon}{\gamma^N}\frac{1}{\lambda_{\max}(\tilde{Q}_1)}, \end{aligned} \tag{16}$$

which satisfies (7). Using (16), we can take an upper bound on $d_{\mathcal{N}}$ such as

$$\bar{d}_{\mathcal{N}} := \varepsilon\gamma^{\mathcal{N}-N} > d_{\mathcal{N}}, \quad \mathcal{N} \geq 1$$

and

$$\bar{d}_0 := \frac{\delta_x}{\lambda_1} > \frac{\delta_x}{\lambda_{\min}(\tilde{Q}_1)} = d_0.$$

If we assume Q_1 and L satisfy

$$\begin{bmatrix} (u_j^{\max})^2/\bar{d}_{\mathcal{N}} & L_j \\ L_j^T & Q_1 \end{bmatrix} \geq 0, \quad j = 1, \ldots, m_2 \tag{17}$$

then (13) holds. In particular, for $\mathcal{N} = 0$, we may assume Q_1, L and λ_1 satisfy

$$\begin{bmatrix} \lambda_1(u_j^{\max})^2/\delta_x & L_j \\ L_j^T & Q_1 \end{bmatrix} \geq 0, \quad j = 1, \ldots, m_2.$$

Corollary 3.1: There exist state feedback FTB controllers that satisfy (12) for the system (1) and (2) if there exist positive definite matrices $Q_1 \in \mathbf{R}^{n\times n}$, $Q_2 \in \mathbf{R}^{m_1\times m_1}$, a matrix $L \in \mathbf{R}^{m_2\times n}$, scalars $\gamma \geq 1$ and $\lambda_i > 0$, $i = 1, 2$ such that (6), (14), (15) and (17) where

$$\bar{d}_{\mathcal{N}} = \begin{cases} \delta_x/\lambda_1, \ \mathcal{N} = 0, \\ \varepsilon\gamma^{\mathcal{N}-N}, \ \mathcal{N} \geq 1, \end{cases}$$

$\tilde{Q}_1 = \Gamma^{\frac{1}{2}}Q_1\Gamma^{\frac{1}{2}}$ and $\tilde{Q}_2 = \Pi^{-\frac{1}{2}}Q_2\Pi^{-\frac{1}{2}}$. In this case the feedback gain F is given by $F = LQ_1^{-1}$.

Corollary 3.2: There exist state feedback FTS controllers that satisfy (12) for the system (1) with $w \equiv 0$ if there exist a positive definite matrix $Q \in \mathbf{R}^{n\times n}$, a matrix $L \in \mathbf{R}^{m_2\times n}$, scalars $\gamma \geq 1$ and $\lambda > 0$ such that

$$\begin{bmatrix} -Q & AQ + B_2L \\ (AQ + B_2L)^T & -\gamma Q \end{bmatrix} < 0,$$

$$\lambda I < \tilde{Q} < I, \quad 1 < \frac{1}{\gamma^N}\frac{\varepsilon}{\delta_x}\lambda,$$

$$\begin{bmatrix} \lambda(u_j^{\max})^2/(\gamma^{\mathcal{N}}\delta_x) & L_j \\ L_j^T & Q \end{bmatrix} \geq 0, \quad j = 1, \ldots, m_2$$

where $\tilde{Q} = \Gamma^{\frac{1}{2}}Q\Gamma^{\frac{1}{2}}$. In this case F is given by $F = LQ^{-1}$.

IV. Observer-Based Output Feedback Controller Design with Input Constraints

Here the input constraints are discussed for observer-based output feedback controller design. Due to the output feedback controllers (8), reachable sets of the state are not available to estimate the maximum magnitude of the inputs. In contrast with the state feedback case, we focus on searching reachable sets of the estimated state. We assume that a state feedback controller $u(k) = Fx(k)$, which makes the system (1) and (2) FTB with respect to $(\delta_x, \delta_w, \varepsilon, \Gamma, \Pi, N)$ (or the system (1) with $w \equiv 0$ FTS with respect to $(\delta_x, \varepsilon, \Gamma, N)$) and satisfies (12) exists and has been designed. Thanks to the observer-based output feedback controllers (8), it is not required to impose the 0-step input constraints because

$$u_j(0) = F_j\hat{x}(0) = 0, \ j = 1, \ldots, m.$$

Thus we may consider $\mathcal{N}$-step input constrains for $\mathcal{N} \geq 1$. The subsequent results represent design methods of observer-based output feedback FTB (FTS) controllers with $\mathcal{N}$-step input constrains for $\mathcal{N} \geq 1$.

Theorem 4.1: There exist output feedback FTB controllers (8) that satisfy (12) for the system (1)-(3) if there exist positive definite matrices P_1, $R \in \mathbf{R}^{n\times n}$, $P_2 \in \mathbf{R}^{m_1\times m_1}$, a matrix $M \in \mathbf{R}^{n\times p}$, scalars $\gamma \geq 1$ and $\mu > 1$ such that (10), (11),

$$\begin{bmatrix} (u_j^{\max})^2/(\mu d_{\mathcal{N}}) & F_j \\ F_j^T & R \end{bmatrix} \geq 0, \ j = 1, \ldots, m_2, \tag{18}$$

$$\mu P_1 \geq P_1 + R \tag{19}$$

where

$$d_{\mathcal{N}} = \gamma^{\mathcal{N}}\left\{\left[\lambda_{\max}(\tilde{P}_1) + \lambda_{\max}(\tilde{R})\right]\delta_x + \lambda_{\max}(\tilde{P}_2)\delta_w\right\},$$

$\tilde{P}_1 = \Gamma^{-\frac{1}{2}}P_1\Gamma^{-\frac{1}{2}}$, $\tilde{P}_2 = \Pi^{-\frac{1}{2}}P_2\Pi^{-\frac{1}{2}}$ and $\tilde{R} = \Gamma^{-\frac{1}{2}}R\Gamma^{-\frac{1}{2}}$. In this case the observer gain K is given by $K = R^{-1}M$.

Proof: See the appendix. ■

To obtain more numerically tractable sufficient conditions for FTB with input constraints, we assume P_1, P_2 and R in Theorem 4.1 satisfy

$$\lambda_1 I < \tilde{P}_1 < \lambda_2 I, \ 0 < \tilde{R} < \lambda_3 I, \ 0 < \tilde{P}_2 < \lambda_4 I, \tag{20}$$

$$(\lambda_2 + \lambda_3)\delta_x + \lambda_4\delta_w < \varepsilon\gamma^{-N}\lambda_1. \tag{21}$$

and

$$\mu\lambda_1 > \lambda_2 + \lambda_3. \tag{22}$$

Then we have

$$\begin{aligned} & \left[\lambda_{max}(\tilde{P}_1) + \lambda_{max}(\tilde{R})\right]\delta_x + \lambda_{max}(\tilde{P}_2)\delta_w \\ < \ & (\lambda_2 + \lambda_3)\delta_x + \lambda_4\delta_w < \varepsilon\gamma^{-N}\lambda_1 \\ < \ & \varepsilon\gamma^{-N}\lambda_{\min}(\tilde{P}_1) \end{aligned} \tag{23}$$

and

$$\mu\tilde{P}_1 > \mu\lambda_1 I > (\lambda_2 + \lambda_3)I > \tilde{P}_1 + \tilde{R}$$

which satisfy (11) and (19), respectively. Using (23), we can take upper bounds on $d_{\mathcal{N}}$ such as

$$\begin{aligned}\lambda_1 \bar{d}_{\mathcal{N}} &= \lambda_1 \varepsilon \gamma^{\mathcal{N}-N} = \gamma^{\mathcal{N}}(\varepsilon\gamma^{-N}\lambda_1) \\ &> \gamma^{\mathcal{N}} \left\{ \left[\lambda_{max}(\tilde{P}_1) + \lambda_{max}(\tilde{R}) \right] \delta_x + \lambda_{max}(\tilde{P}_2)\delta_w \right\} \\ &= d_{\mathcal{N}}\end{aligned}$$

where $\bar{d}_{\mathcal{N}} := \varepsilon\gamma^{\mathcal{N}-N}$. For this upper bound, if we assume R and β satisfy

$$\begin{bmatrix} \beta(u_j^{\max})^2/\bar{d}_{\mathcal{N}} & \beta F_j \\ \beta F_j^T & R \end{bmatrix} \geq 0, \; j = 1, \ldots, m_2, \tag{24}$$

then (18) holds where $\beta = \mu\lambda_1$.

Corollary 4.1: There exist output feedback FTB controllers (8) that satisfy (12) for the system (1)-(3) if there exist positive definite matrices P_1, $R \in \mathbf{R}^{n\times n}$, $P_2 \in \mathbf{R}^{m_1\times m_1}$, a matrix $M \in \mathbf{R}^{n\times p}$ and scalars $\gamma \geq 1$, $\beta > 0$ and $\lambda_i > 0$, $i = 1, 2, 3, 4$, such that (10), (20)-(22) and (24) where $\bar{d}_{\mathcal{N}} = \varepsilon\gamma^{\mathcal{N}-N}$, $\tilde{P}_1 = \Gamma^{-\frac{1}{2}} P_1 \Gamma^{-\frac{1}{2}}$, $\tilde{P}_2 = \Pi^{-\frac{1}{2}} P_2 \Pi^{-\frac{1}{2}}$ and $\tilde{R} = \Gamma^{-\frac{1}{2}} R \Gamma^{-\frac{1}{2}}$. In this case the observer gain K is given by $K = R^{-1}M$.

Remark 4.1: If the LMI problem in Corollary 4.2 has a feasible solution, then $\mu > 1$ holds since $\beta > \lambda_2 + \lambda_3 > \lambda_1$ from (20) and (22).

Corollary 4.2: There exist output feedback FTS controllers (8) that satisfy (12) for the system (1) with $w \equiv 0$ and (3) if there exist positive definite matrices P_1, $R \in \mathbf{R}^{n\times n}$, a matrix $M \in \mathbf{R}^{n\times p}$, scalars $\gamma \geq 1$, $\beta > 0$ and $\lambda_i > 0$, $i = 1, 2, 3$, such that

$$\begin{bmatrix} -\gamma P_1 & 0 & 0 & A_F^T P_1 \\ 0 & -\gamma R & H_{32}^T & (-P_1 B_2 F)^T \\ 0 & H_{32} & -R & 0 \\ P_1 A_F & -P_1 B_2 F & 0 & -P_1 \end{bmatrix} < 0$$

(20), (21) with $\delta_w \equiv 0$, (22) and (24) where $M_{32} = RA + MC$, $\bar{d}_{\mathcal{N}} = \varepsilon\gamma^{\mathcal{N}-N}$, $\tilde{P}_1 = \Gamma^{-\frac{1}{2}} P_1 \Gamma^{-\frac{1}{2}}$ and $\tilde{R} = \Gamma^{-\frac{1}{2}} R \Gamma^{-\frac{1}{2}}$. In this case the observer gain K is given by $K = R^{-1}M$.

V. Numerical Example

Consider the system (1)-(3) where

$$A = \begin{bmatrix} 1 & 2.1 \\ -1 & -1 \end{bmatrix}, \; B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; B_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$
$$S = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}, \; C = \begin{bmatrix} 2 & 1 \end{bmatrix}.$$

We shall design state feedback FTB controllers by Corollary 3.1 and output feedback FTB controllers by Corollary 4.2, respectively. To find feasible solutions of LMIs in the corollaries, we use YALMIP [8] and SeDuMi [9] on Matlab.

We first design state feedback controllers (4) which make the system FTB with respect to $(\delta_x, \delta_w, \varepsilon, \Gamma, \Pi, N) = (1, 1, 20, I_2, I_2, 5)$. We set $\gamma = 1.2$ in Corollary 3.1 and we design two cases $\mathcal{N} = 0$ and 5: We shall design state feedback FTB controllers with 0-step and 5-step input constraint. In the 0-step design with $u_{\max} = 2.0$, we obtain $F = \begin{bmatrix} 0.9299 & 1.4923 \end{bmatrix}$. In the 5-step design with $u_{\max} =$ 7.0, we obtain $F = \begin{bmatrix} 0.8548 & 1.3997 \end{bmatrix}$. The reachable sets and the input constraints of the both designs are shown in Fig. 1. We can see that the reachable sets are inside of the input constraints.

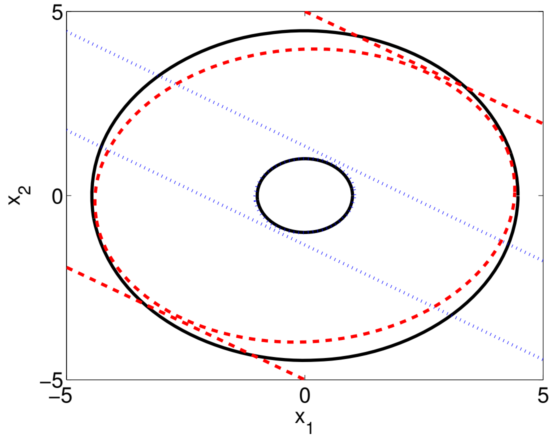

Fig. 1. Reachable sets and input constraints: The inner and outer solid ellipsoids represent $x^T\Gamma x \leq \delta_x$ and $x^T\Gamma x < \varepsilon$, respectively. The dotted ellipsoid and lines represent a reachable set $\mathcal{E}_0$ from (25) and an input constraint $|Fx| \leq 2.0$ by Corollary 3.1 at $\mathcal{N} = 0$. The dashed ellipsoid and lines represent $\mathcal{E}_5$ and $|Fx| \leq 7.0$ by Corollary 3.1 at $\mathcal{N} = 5$.

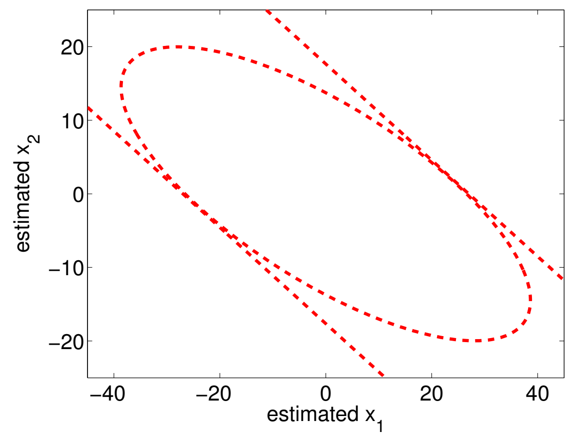

Fig. 2. Reachable sets of the estimated state and input constraints: The dashed ellipsoid and lines represent a reachable set $\mathcal{E}_2$ from (32) and an input constraint $|Fx| \leq 27.0$ by Corollary 4.2 at $\mathcal{N} = 2$.

We also design output feedback controllers (8) which makes the system FTB with respect to $(1, 1, 60, I_2, I_2, 2)$. Using Lemma 2.1 a state feedback gain F is given by $F = \begin{bmatrix} 1.0004 & 1.5300 \end{bmatrix}$ which make the system FTB with respect to $(1, 1, 60, I_2, I_2, 2)$. We set $\gamma = 1.4$ in Corollary 4.2 and we shall design output feedback FTB controllers with 2-step input constraints. In the 2-step design with $u_{\max} = 27.0$, we obtain $K = \begin{bmatrix} -0.7071 & 0.6389 \end{bmatrix}^T$ and $\mu = 22.5202$. The reachable set and the input constraints of the design are shown in Fig. 2. We can see that the reachable set of the estimated state is inside of the input constraints.

VI. Conclusions

We have proposed FTS/FTB control with input constraints for linear discrete-time time-invariant systems. In order to evaluate the maximum magnitude of control input signals, we estimate reachable sets of the state from the initial time to a required finite step imposing input constraints. On the basis of such knowledge, we have proposed state feedback FTS/FTB controller design methods which conditions are reduced to LMIs. Reachable sets of the estimated state are also analyzed to construct observer-based output feedback FTS/FTB controllers. Numerical examples have been shown to illustrate the proposed FTB state/output feedback design methods.

Appendix

Proof of Theorem 3.1

From (6) and (7), the system (5) and (2) is FTB with respect to $(\delta_x,\ \delta_w,\ \varepsilon,\ \Gamma,\ \Pi,\ N)$. Then for $V(x(k),w(k)) = x^T(k)Q_1^{-1}x(k) + w^T(k)Q_2w(k)$, $V(x(k+1),w(k+1)) < \gamma V(x(k),w(k))$ and $V(x(k),w(k)) < \gamma^k V(x(0),w(0))$ for $k \geq 1$. Then we have

$$x^T(k)Q_1^{-1}x(k) < \gamma^k \left[x_0^T Q_1^{-1}x_0 + w_0^T Q_2 w_0\right] \leq d_k$$

for $k \geq 1$. For $k = 0$, we have

$$x_0^T Q_1^{-1} x_0 \leq \frac{\delta_x}{\lambda_{\min}(\tilde{Q}_1)} = d_0.$$

Since the control input depends on the state, it is required to analyze reachable sets of the state for every time in $\{0, 1, \ldots, N\}$. Define reachable sets of the state by

$$\begin{aligned} &\mathcal{E}_k := \left\{\ z \in \mathbf{R}^n \mid z^T Q_1^{-1} z < d_k \right\},\ k \in \{1, \ldots, \mathcal{N}\}, \\ &\mathcal{E}_0 := \left\{\ z \in \mathbf{R}^n \mid z^T Q_1^{-1} z \leq d_0 \right\}. \end{aligned} \tag{25}$$

From $\gamma \geq 1$ and

$$x_0^T Q_1^{-1} x_0 \leq d_0 \leq \gamma^0 \left[\frac{1}{\lambda_{\min}(\tilde{Q}_1)}\delta_x + \lambda_{\max}(\tilde{Q}_2)\delta_w\right],$$

$d_k \leq d_{k+1}$ holds for $k \geq 0$. Then we can see the relation $\mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_{\mathcal{N}}$. Thus we obtain

$$\cup_{k=0,1,\ldots,\mathcal{N}} \mathcal{E}_k = \mathcal{E}_{\mathcal{N}}. \tag{26}$$

Using (26), for $\mathcal{N} \geq 1$, we have

$$\begin{aligned} & \max_{k\in\{0,1,\ldots,\mathcal{N}\}} |u_j(k)|^2 \\ = & \max_{k\in\{0,1,\ldots,\mathcal{N}\}} |(LQ_1^{-1}x(k))_j|^2 \\ \leq & \max_{z \in \cup_{k=0,1,\ldots,\mathcal{N}}\mathcal{E}_k} |(LQ_1^{-1}z)_j|^2 \\ = & \max_{z\in\mathcal{E}_{\mathcal{N}}} |(LQ_1^{-1}z)_j|^2 \\ \leq & \max_{\|(d_{\mathcal{N}}Q_1)^{-\frac{1}{2}}z\|_2=1} \left|\left(d_{\mathcal{N}}^{\frac{1}{2}} L Q_1^{-\frac{1}{2}} \left[(d_{\mathcal{N}}Q_1)^{-\frac{1}{2}} z\right]\right)_j\right|^2 \\ = & \max_{\|(d_{\mathcal{N}}Q_1)^{-\frac{1}{2}}z\|_2=1} \left|\left(d_{\mathcal{N}}^{\frac{1}{2}} L Q_1^{-\frac{1}{2}}\right)_j \left[(d_{\mathcal{N}}Q_1)^{-\frac{1}{2}} z\right]\right|^2 \\ \leq & \max_{\|(d_{\mathcal{N}}Q_1)^{-\frac{1}{2}}z\|_2=1} \left\|\left[\left(d_{\mathcal{N}}^{\frac{1}{2}} L Q_1^{-\frac{1}{2}}\right)_j\right]^T\right\|_2^2 \left\|(d_{\mathcal{N}}Q_1)^{-\frac{1}{2}} z\right\|_2^2 \\ = & \left\|\left[\left(d_{\mathcal{N}}^{\frac{1}{2}} L Q_1^{-\frac{1}{2}}\right)_j\right]^T\right\|_2^2 \\ = & \left\|\left[d_{\mathcal{N}}^{\frac{1}{2}} L_j Q_1^{-\frac{1}{2}}\right]^T\right\|_2^2 \\ = & d_{\mathcal{N}} L_j Q_1^{-1} L_j^T. \end{aligned}$$

Applying Schur complement formula to (13), we obtain

$$\max_{k\in\{0,1,\ldots,\mathcal{N}\}} |u_j(k)|^2 \leq d_{\mathcal{N}} L_j Q_1^{-1} L_j^T \leq (u_j^{\max})^2$$

for $j = 1, \ldots, m_2$. For $\mathcal{N} = 0$, we have

$$|u_j(0)|^2 = |(F\hat{x}(0))_j|^2 \leq \max_{z\in\mathcal{E}_0} |(Fz)_j|^2 \leq d_0 L_j Q_1^{-1} L_j^T$$

for $j = 1, \ldots, m_2$. Using Schur complement formula to (13) again, we obtain

$$|u_j(0)|^2 \leq d_0 L_j Q_1^{-1} L_j^T \leq (u_j^{\max})^2$$

and we have the assertion.

Proof of Theorem 4.1

From (10) and (11), the system (9) is FTB with respect to $(\delta_x, \delta_w + \delta_x, \varepsilon, \Gamma, \mathrm{diag}\{\Pi, \Gamma\}, T)$. Then for

$$\begin{aligned} & V(x(k), w(k), e(k)) \\ & = x^T(k)P_1x(k) + w^T(k)P_2w(k) + e^T(k)Re(k), \end{aligned}$$

we have

$$V(x(k+1), w(k+1), e(k+1)) < \gamma V(x(k), w(k), e(k))$$

and

$$V(x(k), w(k), e(k)) < \gamma^k V(x(0), w(0), e(0)),\ k \geq 1.$$

Then we have

$$\begin{aligned} & x^T(k)P_1x(k) + w^T(k)P_2w(k) \\ & \qquad + (x(k) - \hat{x}(k))^T R (x(k) - \hat{x}(k)) \\ < \ & \gamma^k \left[x_0^T P_1 x_0 + w_0^T P_2 w_0 + x_0^T R x_0\right] \leq d_k \end{aligned} \tag{27}$$

for $k \geq 1$. Since the control input depends on the estimated state, we need to analyze a reachable set of the estimated state for every step in $\{0, 1, \ldots, \mathcal{N}\}$. For $k \in \{1, \ldots, \mathcal{N}\}$, using (32), we may consider an optimization problem as follows:

$$\begin{aligned} \min_{x,\hat{x},w} \quad & -\hat{x}^T R \hat{x} \\ \text{s.t.} \quad & f(x,\hat{x},w) := x^T P_1 x + w^T P_2 w \\ & \qquad + (x - \hat{x})^T R (x - \hat{x}) - d'_k \leq 0 \end{aligned} \tag{28}$$

where $d'_k < d_k$. If we have a solution of the problem (28), then we can obtain an upper bound on $\hat{x}(k)^T R \hat{x}(k)$ for $k \in \{1, \ldots, \mathcal{N}\}$. It is known that strong duality holds for nonconvex quadratic optimization problems with signal quadratic constraint and their Lagrange dual problems [10] under Slater's constraint qualification. In this case, the constraint in (28) is a convex set, so that there exists a $(x, \hat{x}, w)$ with $f(x, \hat{x}, w) < 0$. Thus the constraint satisfies the qualification. The Lagrangian of (28) is

$$L(x, \hat{x}, w, \mu) = -\hat{x}^T R \hat{x} + \mu f(x, \hat{x}, w),\ \mu \geq 0$$

and the dual function is

$$\begin{aligned} g(\mu) &= \inf_{x=x^*,\hat{x}=\hat{x}^*,w=w^*} L(x, \hat{x}, w, \mu) \\ &= \begin{cases} -\mu d_k,\ (19), \\ -\infty,\ \text{otherwise} \end{cases} \end{aligned}$$

where (19) and $\mu > 1$ make

$$\frac{\partial^2}{\partial p^2} L(x, \hat{x}, w, \mu) = \begin{bmatrix} 2\mu(P_1 + R) & -2\mu R & 0 \\ -2\mu R & 2(\mu - 1)R & 0 \\ 0 & 0 & 2\mu P_2 \end{bmatrix}$$

positive semidefinite. Also, the points

$$p^* = \begin{bmatrix} (x^*)^T & (\hat{x}^*)^T & (w^*)^T \end{bmatrix}^T,$$

which achieve the infimum of L, satisfy

$$P_1 x^* + R(x^* - \hat{x}^*) = 0, \quad (29)$$

$$\hat{x}^* + \mu(x^* - \hat{x}^*) = 0, \quad (30)$$

$$w^* = 0 \quad (31)$$

from $\partial L / \partial p = 0$. Using (29)-(31), we have

$$\begin{aligned} & L(x^*, \hat{x}^*, w^*, \mu) \\ &= -(\hat{x}^*)^T R \hat{x}^* + \mu (x^*)^T P_1 x^* + \mu (w^*)^T P_2 w^* \\ &\quad + \mu (x^* - \hat{x}^*)^T R (x^* - \hat{x}^*) - \mu d'_k \\ &= -(\hat{x}^*)^T R \hat{x}^* + \mu (x^*)^T P_1 x^* - (x^* - \hat{x}^*)^T R \hat{x}^* - \mu d'_k \\ &= \mu (x^*)^T P_1 x^* - (x^*)^T R \hat{x}^* - \mu d'_k \\ &= -\mu d'_k. \end{aligned}$$

Then the Lagrange dual problem of (28) is

$$\max_{\mu = \mu^*} \ -\mu d'_k \ \text{ s.t. } \ (19).$$

Since the optimal value of the dual problem is $-\mu^* d'_k$, it is also the optimal value of the problem (28). Thus we obtain

$$\hat{x}^T(k) R \hat{x}(k) \le \mu^* d'_k < \mu^* d_k, \ k = 1, \ldots, \mathcal{N}.$$

Define reachable sets of the estimated state by

$$\mathcal{E}_k := \left\{ \ z \in \mathbf{R}^n \mid z^T R z < \mu^* d_k \right\}, \ k \in \{1, \ldots, \mathcal{N}\}. \quad (32)$$

Since $\gamma \ge 1$, $d_k \le d_{k+1}$ holds for $k \ge 1$. Then we have

$$\mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_{\mathcal{N}}$$

and

$$\cup_{k=1,\ldots,\mathcal{N}} \mathcal{E}_k = \mathcal{E}_{\mathcal{N}}.$$

For $\mathcal{N} \ge 1$ we have

$$\begin{aligned} \max_{k \in \{0,1,\ldots,\mathcal{N}\}} |u_j(k)|^2 &= \max_{k \in \{0,1,\ldots,\mathcal{N}\}} |(Fx(k))_j|^2 \\ &\le \max_{z \in \cup_{k=1,\ldots,\mathcal{N}} \mathcal{E}_k} |(Fz)_j|^2 \\ &= \max_{z \in \mathcal{E}_{\mathcal{N}}} |(Fz)_j|^2 \\ &= \max_{\|(\mu^* d_{\mathcal{N}})^{-\frac{1}{2}} R^{\frac{1}{2}} z\|_2 = 1} |(Fz)_j|^2 \\ &\le \mu^* d_{\mathcal{N}} F_j R^{-1} F_j^T. \end{aligned}$$

Applying Schur complement formula to (18), we obtain

$$\max_{k \in \{0,1,\ldots,\mathcal{N}\}} |u_j(k)|^2 \le \mu^* d_{\mathcal{N}} F_j R^{-1} F_j^T \le (u_j^{\max})^2$$

for $j = 1, \ldots, m_2$. Hence we have the assertion.

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