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New Results of Stability Analysis for Singular Time-Delay Systems

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Abstract— **This paper establishes the equivalence among several stability criteria, and presents a simplified stability criterion for singular time-delay systems. Furthermore, by using a delay decomposition method, a new stability criterion which is much less conservative than the existing ones is obtained. A numerical example is given to illustrate the effectiveness and less conservatism of the new proposed stability criterion.**

I. INTRODUCTION

Over the past decades, much attention has been focused on the stability analysis and controller synthesis for singular linear time-delay systems due to the fact that the singular system model is a natural presentation of dynamic systems and it can describe a large class of systems than regular ones, such as large-scale systems, power systems and constrained control systems. Just like state-space time-delay systems, the results on stability analysis and stabilization for singular time-delay systems can be classied into two categories, that is, delay-independent criteria ([1],[2]) and delay- dependent ones ([3],[4]). Generally, the delay-dependent case is less conservative than delay-independent ones, especially when the delay is comparatively small.

Recently, there has been a growing interest in the study of stability analysis for singular systems with time-delay. By using various methods, many results have been reported in the literature (for example, [4], [5], [6], [7]). In this note, we will prove that the stability result proposed in [4] is equivalent to the ones in [5], [6], [7], and a simplified version of Theorem 1 in [4] will be derived. Furthermore, by using a delay composition method, a less conservative result will be presented.

Consider the following continuous-time singular system with a time-varying delay in the state [4]:

$$(\Sigma): \quad E\dot{x}(t) = Ax(t) + A_\tau x(t-\tau), \quad t > 0 \tag{1}$$

$$x(t) = \phi(t) \quad t \in [-\tau,\ 0], \tag{2}$$

where $x(t) \in R^n$ is the state, $\phi(t) \in \mathscr{C}_{n,\tau}$ is a compatible vector valued initial function. The matrix $E \in R^{n\times n}$ may be singular and $\mathrm{rank}E = p \leq n$. A, A_τ are constant matrices with appropriate dimensions. τ is an unknown but constant delay satisfying

$$0 \leq \tau \leq \tau_m. \tag{3}$$

Without loss of generality, the matrices E, A and A_τ are assumed to have the forms:

$$E = \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\ A_\tau = \begin{bmatrix} A_{\tau 11} & A_{\tau 12} \\ A_{\tau 21} & A_{\tau 22} \end{bmatrix}. \tag{4}$$

For system (Σ), [4] provided a stability criterion as follows.

Lemma 1. [4] The singular time-delay system (Σ) is regular, impulse free and asymptotically stable for any constant delay τ satisfying $0 \leq \tau \leq \tau_m$, if there exist matrices

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix},\ P_{11} > 0,\ Q > 0,\ Z = \begin{bmatrix} Z_{11} & Z_{12} \\ * & Z_{22} \end{bmatrix} > 0,$$

$$Y = \begin{bmatrix} Y_{11} & 0 \\ Y_{21} & 0 \end{bmatrix},\ W = \begin{bmatrix} W_{11} & 0 \\ W_{21} & 0 \end{bmatrix},\ Y_1 = \begin{bmatrix} Y_{11} \\ Y_{21} \end{bmatrix},$$

$$W_1 = \begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix},$$

with appropriate dimensions and $P_{11} \in R^{p\times p}$, $Z_{11} \in R^{p\times p}$, $Y_{11} \in R^{p\times p}$, $W_{11} \in R^{p\times p}$ satisfying the following LMI:

$$\Phi < 0, \tag{5}$$

where

$$\Phi = \begin{bmatrix} \Phi_1 & PA_\tau - Y + W^T + \tau_m A^T Z A_\tau & -\tau_m Y_1 \\ * & -Q - W - W^T + \tau_m A_\tau^T Z A_\tau & -\tau_m W_1 \\ * & * & -\tau_m Z_{11} \end{bmatrix},$$

$$\Phi_1 = PA + A^T P^T + Y + Y^T + Q + \tau_m A^T Z A.$$

For convenience of comparison, the stability criteria in [5], [6], [7] are listed as the following lemmas.

Lemma 2. [5] Consider the descriptor system (Σ), for a given scalar $\tau_m > 0$, if there exist matrices $\tilde{P}_1 > 0$, $\tilde{P}_2$, $\tilde{P}_3$, $\tilde{Q} > 0$, $\tilde{R} > 0$, $\tilde{T}_i$ and $\tilde{S}_i$ of appropriate dimensions $(i = 1,\ 2,\ 3)$ such that

$$\Gamma < 0, \tag{6}$$

where

$$\Gamma = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \tau_m \tilde{T}_1 \\ * & \Gamma_{22} & \Gamma_{23} & \tau_m \tilde{T}_2 \\ * & * & \Gamma_{33} & \tau_m \tilde{T}_3 \\ * & * & * & -\tau_m \tilde{R} \end{bmatrix},$$

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$$
\begin{aligned}
\Gamma_{11} &= \tilde{Q} + \tilde{T}_1 E + E^T \tilde{T}_1^T - \tilde{S}_1 A - A^T \tilde{S}_1^T,\\
\Gamma_{12} &= -\tilde{T}_1 E + E^T \tilde{T}_2^T - \tilde{S}_1 A_\tau - A^T \tilde{S}_2^T,\\
\Gamma_{13} &= \tilde{P} + \tilde{S}_1 + E^T \tilde{T}_3^T - A^T \tilde{S}_3^T,\\
\Gamma_{22} &= -\tilde{Q} - \tilde{T}_2 E - E^T \tilde{T}_2^T - \tilde{S}_2 A_\tau - A_\tau^T \tilde{S}_2^T,\\
\Gamma_{23} &= \tilde{S}_2 - E^T \tilde{T}_3^T - A_\tau^T \tilde{S}_3^T,\\
\Gamma_{33} &= \tau_m \tilde{R} + \tilde{S}_3 + \tilde{S}_3^T,\\
P &= \begin{bmatrix} \tilde{P}_1 & \tilde{P}_2 \\ 0 & \tilde{P}_3 \end{bmatrix},
\end{aligned}
$$

then system (Σ) is E–exponentially stable.

Lemma 3. [6] Given a scalar $\tau_m > 0$. Then, for any delay $0 \le \tau \le \tau_m$, the singular delay system (Σ) is regular, impulse free and stable if there exist matrices $Q = Q^T > 0$, $Z = Z^T > 0$, P, Y and W, such that the following LMIs hold:

$$E^T P = P^T E \ge 0, \tag{7}$$

$$\Omega < 0, \tag{8}$$

where

$$
\begin{aligned}
\Omega &= \begin{bmatrix} \Omega_{11} & \Omega_{12} & \tau_m Y^T & \tau_m A^T Z \\ * & \Gamma_{22} & \tau_m W^T & \tau_m A_\tau^T Z \\ * & * & -\tau_m Z & 0 \\ * & * & * & -\tau_m Z \end{bmatrix},\\
\Omega_{11} &= P^T A + A^T P + Q - Y^T E - E^T Y,\\
\Omega_{12} &= P^T A_\tau + Y^T E - E^T W,\\
\Omega_{22} &= W^T E + E^T W - Q.
\end{aligned}
$$

Lemma 4. [7] Given a scalar $\tau_m > 0$. Then for any delay $0 < \tau \le \tau_m$, the singular delay system (Σ) is regular, impulse free and stable if there exist matrices $Q = Q^T > 0$, $Z = Z^T > 0$, and matrices P_1, P_2, P_3, X_{11}, X_{12}, X_{13}, X_{22}, X_{23}, X_{33}, Y_1, Y_2 and T_1, such that

$$E^T P_1 = P_1^T E \ge 0, \tag{9}$$

$$\Pi < 0, \tag{10}$$

$$X \ge 0, \tag{11}$$

where

$$
\begin{aligned}
\Pi &= \begin{bmatrix} \Pi_{11} & \Pi_{12} & -Y_1 E + P_2^T A_\tau + E^T T_1^T + \tau_m X_{13} \\ * & \Pi_{22} & -Y_2 E + P_3^T A_\tau + \tau_m X_{23} \\ * & * & -Q - T_1 E - E^T T_1^T + \tau_m X_{33} \end{bmatrix},\\
\Pi_{11} &= P_2^T A + A^T P_2 + Y_1 E + E^T Y_1^T + \tau_m X_{11} + Q,\\
\Pi_{12} &= P_1^T - P_2^T + A^T P_3 + E^T Y_2^T + \tau_m X_{12},\\
\Pi_{22} &= -P_3 - P_3^T + \tau_m X_{22} + \tau_m Z,\\
X &= \begin{bmatrix} X_{11} & X_{12} & X_{13} & Y_1 \\ * & X_{22} & X_{23} & Y_2 \\ * & * & X_{33} & T_1 \\ * & * & * & Z \end{bmatrix}.
\end{aligned}
$$

II. The Equivalence Among Several Stability Criteria

In this section, the equivalence among the existing stability criteria given in [4], [5], [6], [7] will be established.

Now, we prove the equivalence among the stability conditions in Lemmas 1-4, and a new stability criterion which contains fewer decision variables is also derived.

Theorem 1. The following statements are equivalent:

i) inequality (5) is feasible.

ii) the following inequality is feasible:

$$\Psi < 0, \tag{12}$$

where

$$
\begin{aligned}
\Psi &= \begin{bmatrix} \Psi_1 & PA_\tau + \tau_m A^T Z A_\tau + \tau_m^{-1} H^T Z_{11} H \\ * & -Q + \tau_m A_\tau^T Z A_\tau - \tau_m^{-1} H^T Z_{11} H \end{bmatrix},\\
\Psi_1 &= PA + (PA)^T + Q + \tau_m A^T Z A - \tau_m^{-1} H^T Z_{11} H,\\
H &= \begin{bmatrix} I_p & 0 \end{bmatrix}.
\end{aligned}
$$

iii) inequality (6) is feasible.

iv) inequality (8) with (7) is feasible.

v) inequalities (10) and (11) with (9) are feasible.

Proof: i) $\Leftrightarrow$ ii):

Noticing that $Y = Y_1 H$ and $W = W_1 H$, pre- and post-multiplying

$$\begin{bmatrix} I & 0 & \tau_m^{-1} H^T \\ 0 & I & -\tau_m^{-1} H^T \\ 0 & 0 & I \end{bmatrix}$$

and its transpose on both sides of Φ in (5), and from the Schur complement, it follows that $\Phi < 0$ in Lemma 1 is equivalent to

$$\Psi + \begin{bmatrix} -\tau_m Y_1 - H^T Z_{11} \\ -\tau_m W_1 + H^T Z_{11} \end{bmatrix} (\tau_m Z_{11})^{-1} \begin{bmatrix} -\tau_m Y_1 - H^T Z_{11} \\ -\tau_m W_1 + H^T Z_{11} \end{bmatrix}^T < 0. \tag{13}$$

So, $\Psi < 0$ holds if $\Phi < 0$ holds.

Conversely, if $\Psi < 0$ holds, by letting

$$Y_1 = -\tau_m^{-1} H^T Z_{11}, \quad W_1 = \tau_m^{-1} H^T Z_{11},$$

it yields that $\Phi < 0$ also holds.

Thus, $\Psi < 0$ is equivalent to $\Phi < 0$.

ii) $\Leftrightarrow$ iii):

Pre- and post-multiplying

$$\begin{bmatrix} I & 0 & 0 & -\tau_m^{-1} E^T \\ 0 & I & 0 & \tau_m^{-1} E^T \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and its transpose on both sides of Γ in (6), it yields that

$$\begin{bmatrix} \Xi & \tilde{T} \\ * & -\tau_m \tilde{R} \end{bmatrix} < 0 \tag{14}$$

where

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} & \tilde{P}+\tilde{S}_1-A^T\tilde{S}_3^T \\ * & \Xi_{22} & \tilde{S}_2-A_\tau^T\tilde{S}_3^T \\ * & * & \tau_m\tilde{R}+\tilde{S}_3+\tilde{S}_3^T \end{bmatrix},$$

$$\Xi_{11} = \tilde{Q}-\tilde{S}_1A-(\tilde{S}_1A)^T-\tau_m^{-1}E^T\tilde{R}E,$$
$$\Xi_{12} = -\tilde{S}_1A_\tau-(\tilde{S}_2A)^T+\tau_m^{-1}E^T\tilde{R}E,$$
$$\Xi_{22} = -\tilde{Q}-\tilde{S}_2A_\tau-(\tilde{S}_2A_\tau)^T-\tau_m^{-1}E^T\tilde{R}E,$$
$$\tilde{T} = \begin{bmatrix} \tau_m\tilde{T}_1+E^T\tilde{R} \\ \tau_m\tilde{T}_2-E^T\tilde{R} \\ \tau_m\tilde{T}_3 \end{bmatrix}.$$

Similar to the proof of i) $\Leftrightarrow$ ii), it is clear that $\Gamma<0$ is feasible if and only if $\Xi<0$ is feasible.

Note that

$$\Xi = \bar{\Xi}+\tilde{S}\mathscr{A}+\mathscr{A}^T\tilde{S}^T, \tag{15}$$

where

$$\bar{\Xi} = \begin{bmatrix} \tilde{Q}-\tau_m^{-1}E^T\tilde{R}E & \tau_m^{-1}E^T\tilde{R}E & \tilde{P} \\ * & -\tilde{Q}-\tau_m^{-1}E^T\tilde{R}E & 0 \\ * & * & \tau_m\tilde{R} \end{bmatrix},$$
$$\tilde{S} = \begin{bmatrix} \tilde{S}_1^T & \tilde{S}_2^T & \tilde{S}_3^T \end{bmatrix}^T,$$
$$\mathscr{A} = \begin{bmatrix} -A & -A_\tau & I \end{bmatrix},$$

from the elimination lemma ([9], p. 22), it is known that $\Xi<0$ is equivalent to

$$\tilde{\Xi} := \mathscr{N}_{\mathscr{A}}^T\bar{\Xi}\mathscr{N}_{\mathscr{A}}<0, \tag{16}$$

where

$$\mathscr{N}_{\mathscr{A}} = \begin{bmatrix} I & 0 \\ 0 & I \\ A & A_\tau \end{bmatrix}.$$

After some manipulation, one can get

$$\tilde{\Xi} = \begin{bmatrix} \tilde{\Xi}_{11} & \tilde{P}A_\tau+\tau_m^{-1}E^T\tilde{R}E+\tau_mA^T\tilde{R}A_\tau^T \\ * & -\tilde{Q}-\tau_m^{-1}E^T\tilde{R}E+\tau_mA_\tau^T\tilde{R}A_\tau^T \end{bmatrix},$$

where

$$\tilde{\Xi}_{11} = \tilde{P}A+A^T\tilde{P}^T+\tilde{Q}-\tau_m^{-1}E^T\tilde{R}E+\tau_mA^T\tilde{R}A.$$

By letting $P=\tilde{P}$, $Q=\tilde{Q}$ and $Z=\tilde{R}$, it is easy to know that Ψ in (12) is the same as $\tilde{\Xi}$, so $\Psi<0$ if and only if $\tilde{\Xi}<0$.

Thus, from the above analysis, one can get that $\Psi<0$ if and only if $\Gamma<0$.

ii) $\Leftrightarrow$ iv): Similar to the proof of (i) $\Leftrightarrow$ ii)), the equivalence between ii) and iv) can be easily obtained, and omitted here.

ii) $\Leftrightarrow$ v): Similar to the proofs of (i) $\Leftrightarrow$ ii)) and Theorem 2 in [10], the equivalence between ii) and v) can also be derived, and omitted here.

This completes the proof. ■

Remark 1. By using a method given in [11] for eliminating redundant variables, Theorem 1 establishes the equivalence among several stability criteria reported in [4], [5], [6], [7]. Compared with Lemma 1 of [4], ii) of Theorem 1 involves less decision variables. Hence, from a mathematical point of view, ii) of Theorem 1 is more "powerful".

III. AN IMPROVED STABILITY CRITERION

In this section, an improved stability criterion will be proposed by using a delay decomposition method.

Theorem 2. The singular time-delay system (Σ) is regular, impulse free and asymptotically stable for a given positive integer N and any constant delay τ satisfying $0\le\tau\le\tau_m$, if there exist matrices

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix},\ P_{11}>0,\ Q_i>0,\ Z_i>0\ (i=1,2,\cdots,N), \tag{17}$$

with appropriate dimensions and $P_{11}\in R^{p\times p}$ satisfying the following LMI:

$$\Theta<0, \tag{18}$$

where

$$\Theta = \begin{bmatrix} \Theta_{1,1} & \Theta_{1,2} & 0 & \cdots & 0 & PA_\tau+\frac{\tau_m}{N}A^T\tilde{Z}A_\tau \\ * & \Theta_{2,2} & \Theta_{2,3} & \cdots & 0 & 0 \\ * & * & \Theta_{3,3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & \Theta_{N,N} & \frac{N}{\tau_m}E^TZ_NE \\ * & * & * & \cdots & * & \Theta_{N+1,N+1} \end{bmatrix},$$

$$\Theta_{1,1} = PA+A^TP^T+Q_1+\frac{\tau_m}{N}A^T\tilde{Z}A-\frac{N}{\tau_m}E^TZ_1E,$$
$$\Theta_{1,2} = \frac{N}{\tau_m}E^TZ_1E,$$
$$\Theta_{2,3} = \frac{N}{\tau_m}E^TZ_2E,$$
$$\Theta_{i,i} = -Q_{i-1}+Q_i-\frac{N}{\tau_m}E^T(Z_{i-1}+Z_i)E \quad (i=2,\ 3,\ \cdots,\ N),$$
$$\Theta_{N+1,N+1} = -Q_N-\frac{N}{\tau_m}E^TZ_NE+\frac{\tau_m}{N}A_\tau^T\tilde{Z}A_\tau,$$
$$\tilde{Z} = \sum_{i=1}^{N}Z_i.$$

Proof: From (18), it follows that

$$PA+A^TP^T+Q_1-\frac{N}{\tau_m}E^TZ_1E<0 \tag{19}$$

holds, which implies that

$$P_{22}A_{22}+A_{22}^TP_{22}^T<0. \tag{20}$$

So, A_{22} is nonsingular. Pre- and post-multiplying $\begin{bmatrix} I & I & \cdots & I & I \end{bmatrix}$ and its transpose on the both sides of Θ in (18), it yields that

$$P(A+A_\tau)+(A+A_\tau)^TP^T-\frac{N}{\tau_m}\sum_{i=1}^{N}E^TZ_iE<0, \tag{21}$$

which implies that $A_{22}+A_{\tau22}$ is also nonsingular. Thus, the pairs $(E,\ A)$ and $(E,\ A+A_\tau)$ are regular and impulse free.

Construct the Lyapunov-Krasovskii functional for system (Σ) as

$$\begin{aligned} V(x_t) =& x^T(t)PEx(t) + \sum_{i=1}^{N}\Big(\int_{t-\tau_i}^{t-\tau_{i-1}} x^T(s)Q_i x(s)ds \\ &+ \int_{-\tau_i}^{-\tau_{i-1}}\int_{t+\theta}^{t} \dot{x}^T(s)E^T Z_i E\dot{x}(s)dsd\theta\Big), \end{aligned} \tag{22}$$

where $\tau_i = \frac{i}{N}\tau$ $(i = 0,\ 1,\ 2,\ \cdots,\ N)$.

Taking the time derivative of $V(x_t)$ along with the solution of (Σ) yields

$$\begin{aligned} \dot{V}(x_t) =& 2x^T(t)PE\dot{x}(t) + \sum_{i=1}^{N}\Big(x^T(t-\tau_{i-1})Q_i x(t-\tau_{i-1}) \\ & - x^T(t-\tau_i)Q_i x(t-\tau_i) + \frac{\tau}{N}\dot{x}^T(t)E^T Z_i E\dot{x}(t) \\ & - \int_{t-\tau_i}^{t-\tau_{i-1}} \dot{x}^T(s)E^T Z_i E\dot{x}(s)ds\Big) \\ \le& 2x^T(t)P[Ax(t) + A_\tau x(t-\tau)] \\ & + \sum_{i=1}^{N}\big(x^T(t-\tau_{i-1})Q_i x(t-\tau_{i-1}) - x^T(t-\tau_i)Q_i x(t-\tau_i)\big) \\ & + \sum_{i=1}^{N}\Big(\frac{\tau_m}{N}[Ax(t)+A_\tau x(t-\tau)]^T Z_i [Ax(t)+A_\tau x(t-\tau)]\Big) \\ & - \frac{N}{\tau_m}\sum_{i=1}^{N}\Big([x(t-\tau_{i-1}) - x(t-\tau_i)]^T E^T Z_i E \\ & \quad \times [x(t-\tau_{i-1}) - x(t-\tau_i)]\Big) \\ =& \xi^T(t)\Theta\xi(t), \end{aligned} \tag{23}$$

where

$$\xi(t) = [x^T(t) \quad x^T(t-\tau_1) \quad \cdots \quad x^T(t-\tau_{N-1}) \quad x^T(t-\tau)].$$

Therefore, by (18) it is easy to see that $\dot{V}(x_t) < 0$.

This completes the proof. ■

The following theorem shows the relationship between Theorem 2 and ii) of Theorem 1.

Theorem 3. Inequality (18) is feasible if inequality (12) is feasible.

Proof: If inequality (12) is feasible, then there exists a scalar $\varepsilon > 0$ such that

$$\tilde{\Psi} < 0, \tag{24}$$

where

$$\tilde{\Psi} = \begin{bmatrix} \tilde{\Psi}_1 & 0 & 0 & 0 & \cdots & PA_\tau + \tau_m A^T Z A_\tau + \tau_m^{-1}H^T Z_{11} H \\ * & -\varepsilon I & 0 & 0 & \cdots & 0 \\ * & * & -\varepsilon I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & -\varepsilon I & 0 \\ * & * & * & \cdots & * & -Q + \tau_m A_\tau^T Z A_\tau - \tau_m^{-1}H^T Z_{11} H \end{bmatrix},$$

$\tilde{\Psi}_1 = PA + (PA)^T + Q + (N-1)\varepsilon I + \tau_m A^T Z A - \tau_m^{-1}H^T Z_{11} H$, $H = \begin{bmatrix} I_p & 0 \end{bmatrix}$.

Letting $Z_i = Z$ $(i = 1,\ 2,\ \cdots,\ N)$, $Q_N = Q$, $Q_{N-1} = Q + \varepsilon\ I$, $\cdots$, $Q_1 = Q + (N-1)\varepsilon I$, and denoting $\Delta = \Theta - \tilde{\Psi}$, it yields that

$$\Delta = \begin{bmatrix} -\frac{N-1}{\tau_m}E^T ZE & \frac{N}{\tau_m}E^T ZE & 0 & \cdots & 0 & -\frac{1}{\tau_m}E^T ZE \\ * & -\frac{2N}{\tau_m}E^T ZE & \frac{N}{\tau_m}E^T ZE & \cdots & 0 & 0 \\ * & * & -\frac{2N}{\tau_m}E^T ZE & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \cdots & -\frac{2N}{\tau_m}E^T ZE & \frac{N}{\tau_m}E^T ZE \\ * & * & * & \cdots & * & -\frac{N-1}{\tau_m}E^T ZE \end{bmatrix}. \tag{25}$$

Next, we prove that $\Delta \le 0$ holds.

When $N = 1$, it is obvious that $\Delta = 0$, so $\Theta < 0$ is also feasible.

If $N = 2$, then Δ becomes to

$$\Lambda := \begin{bmatrix} -\frac{1}{\tau_m}E^T ZE & \frac{2}{\tau_m}E^T ZE & -\frac{1}{\tau_m}E^T ZE \\ * & -\frac{4}{\tau_m}E^T ZE & \frac{2}{\tau_m}E^T ZE \\ * & * & -\frac{1}{\tau_m}E^T ZE \end{bmatrix}. \tag{26}$$

Pre- and post-multiplying $\begin{bmatrix} I & I & I \\ 0 & I & 0 \\ 0 & \frac{1}{2}I & I \end{bmatrix}$ and its transpose on the both sides of Λ, it gets

$$\tilde{\Lambda} := \begin{bmatrix} 0 & 0 & 0 \\ * & -\frac{4}{\tau_m}E^T ZE & 0 \\ * & * & 0 \end{bmatrix}. \tag{27}$$

It is obvious that $\tilde{\Lambda} \le 0$, which implies that $\Delta \le 0$ holds.

For the case of $N > 2$, the proof is similar to that for $N = 2$, and omitted here.

The result is established. ■

TABLE I
COMPARISONS OF DELAY-DEPENDENT STABILITY CONDITIONS OF EXAMPLE 1

Methods	Maximum τ_m allowed	Number of variables
Theorem 1 [7]	1.1547	53
Theorem 1 [5]	1.1547	33
Theorem 3.5 [8]	1.1547	24
Theorem 1 [6]	1.1547	17
Theorem 1 [4]	1.1547	13
ii) of Theorem 1	1.1547	9
Theorem 2 $N=2$	1.1954	15
Theorem 2 $N=3$	1.2025	21
Theorem 2 $N=4$	1.2044	27
Theorem 2 $N=5$	1.2052	33

Remark 2. From Theorem 3, it is easy to see that Theorem 2 is less conservative than ii) of Theorem 1. As N increasing, the conservatism of Theorem 2 decreases. An example in the next section will verify this fact.

IV. EXAMPLE

Example 1. [4] Consider a singular delay system which is in the form of (1) with

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.5 & 0 \\ -1 & -1 \end{bmatrix}, \quad A_\tau = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Table 1 lists the comparison of the calculating results obtained by the stability criteria in [4], [5], [6], [7], [8] and this note.

It is worth pointing out that the maximum τ_m obtained by Theorem 3.5 in [8] should be 1.1547, and not 1.1612 as given in [4].

Certainly, the maximum τ_m obtained by Theorem 1 in [4] should be 1.1547, and not 1.2011 as listed in [4].

From Table 1, it is clear that Theorem 1 in [4] may not be less conservative than Theorem 3.5 in [8]. Fortunately, Example 2 in [6] showed that the calculating results obtained by Theorem 1 in [6] may be less conservative than the ones obtained by Theorem 3.5 in [8], and no theoretical proof had been provided in [6].

Summarily, ii) of Theorem 1 in this note contains the fewest variables and Theorem 2 in this note is less conservative than those in [4], [5], [6], [7].

V. CONCLUSION

This note presents some comments and further results concerning delay-dependent stability analysis for singular systems with state delay. A technique for eliminating redundant variables is developed. By making use of a delay decomposition method, a result which is much less conservative than previous relevant ones is obtained, which has been shown by a numerical example. As a future work, we will extend the delay decomposition method to the systems with time-varying delays. In addition, it should be noticed that some difficulties for solving the resulting LMIs with large dimensions will be encountered as N increases. So, how to overcome this shortage is also an important task.

REFERENCES

[1] S. Xu, P. Van Dooren, R. Stefan, and J. Lam, Robust stability and stabilization for singular systems with state delay and parameter uncertainty, *IEEE Transactions on Automatic Control*, vol. 47, no. 7, pp. 1122-1128, 2002.

[2] J. Feng, S. Zhu, and Z. Cheng, Guaranteed cost control of linear uncertain singu- lar time-delay systems, *in Proceedings of the 41st IEEE Conference Decision Control*, Las Vegas, NV, Dec. 2002, pp. 1802-1807.

[3] D. Yue and Q.-L. Han, Delay-dependent robust H1 controller design for uncer- tain descriptor systems with time-varying discrete and distributed delays, *IEE Proceedings of Control Theory & Applications*, vol. 152, no. 6, pp. 628-638, 2005.

[4] S. Zhu, C. Zhang, Z. Cheng, and J. Feng, "Delay-dependent robust stability criteria for two classes of uncertain singular time-delay systems", *IEEE transactions on Automatic Control*, vol. 52, no. 5, pp. 880-885, 2007.

[5] D. Yue, Q. L. Han, "A delay-dependent stability criterion of neutral systems and its application to a partial element equivalent circuit model", *IEEE Transactions on Circuits and Systems–II: Express Briefs*, vol. 51, no. 12, pp. 685-689, 2004.

[6] S. Xu, J. Lam, Y. Zou, "An improved characterization of bounded realness for singular delay systems and its applications", *International Journal of Robust and Nonliear Control*, vol. 18, no. 3, pp. 263-277, 2008.

[7] Z. Wu and W. Zhou, "Delay-dependent robust H_∞ control for uncertain singular time-delay systems", *IET Control Theory & Applications*, vol. 1, no. 5, pp. 1234-1241, 2007.

[8] E. Fridman, and U. Shaked, "H_∞-control of linear state-delay descriptor systems: an LMI approach", *Linear Algebra and its Applications*, vol. 351-352, pp. 271-302, 2002.

[9] S. Boyd, L. El. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequality in Systems and Control Theory*. Philadelphia, PA: SIAM, 1994.

[10] S. Xu, J. Lam, and Y. Zou, "Simplified descriptor system approach to delay-dependent stability and performance analyses for time-delay systems", *IEE Proceedings of Control Theory & applications*, vol. 152, no. 2, pp. 147-151, 2005.

[11] X. L. Zhu, G. H. Yang, Jensen integral inequality approach to stability analysis of continuous-time systems with time-varying delay, *IET Control Theory & Applications*, vol. 2, no. 6, pp.524-534, 2008.